A NOTE ON PATH-CONNECTED ORTHOMODULAR LATTICES

Eunsoon Park

1. Preliminaries

An orthomodular lattice (abbreviated by OML) is an ortholattice $L$ which satisfies the orthomodular law: if $x \leq y$, then $y = x \lor (x' \land y)$ [5]. A Boolean algebra $B$ is an ortholattice satisfying the distributive law: $x \lor (y \land z) = (x \lor y) \land (x \lor z)$ $\forall x, y, z \in B$.

A subalgebra of an OML $L$ is a nonempty subset $M$ of $L$ which is closed under the operations $\lor$, $\land$ and $\prime$. We write $M \leq L$ if $M$ is a subalgebra of $L$. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then the relative interval sublattice $M[a, b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with the relative orthocomplementation $^c$ on $M[a, b]$ given by $c^c = (a \lor c') \land b = a \lor (c' \land b)$ $\forall c \in M[a, b]$. In particular, $L[a, b]$ will be denoted by $[a, b]$ if there is no ambiguity.

The commutator of $a$ and $b$ of an OML $L$ is denoted by $a \ast b$, and is defined by $a \ast b = (a \lor b) \land (a \lor b') \land (a' \lor b) \land (a' \lor b')$. The set of all commutators of $L$ is denoted by $\text{Com}L$ and $L$ is said to be commutator-finite if $|\text{Com}L|$ is finite. For elements $a, b$ of an OML, we say $a$ commutes with $b$, in symbols $aC b$, if $a \ast b = 0$. If $M$ is a subset of an OML $L$, the set $C(M) = \{x \in L \mid xCm \forall m \in M\}$ is called the commutant of $M$ in $L$ and the set $\text{Con}(M) = C(M) \cap M$ is called the center of $M$. The set $C(L)$ is called the center of $L$ and then $C(L) = \bigcap \{C(a) \mid a \in L\}$. An OML $L$ is called irreducible if $C(L) = \{0, 1\}$, and $L$ is called reducible if it is not irreducible.

A block of an OML $L$ is a maximal Boolean subalgebra of $L$. The set of all blocks of $L$ is denoted by $\mathcal{A}_L$. Note that $\bigcup \mathcal{A}_L = L$ and $\bigcap \mathcal{A}_L = C(L)$. An OML $L$ is said to be block-finite if $|\mathcal{A}_L|$ is finite.

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For any $e$ in an OML $L$, the subalgebra $S_e = [0, e'] \cup [e, 1]$ is called the \textit{(principal) section generated by $e$}. Note that for $A, B \in A_L$, if $e \in (A \cap B)$ and $A \cap B = S_e \cap (A \cup B)$, then $A \cap B = S_e \cap A = S_e \cap B$.

**Definition 1.1.** For blocks $A, B$ of an OML $L$ define $A^\omega B$ if and only if $A \cap B = S_e \cap (A \cup B)$ for some $e \in A \cap B$; $A \sim B$ if and only if $A \neq B$ and $A \cup B \leq L$; $A \approx B$ if and only if $A \sim B$ and $A \cap B \neq C(L)$.

A \textit{path} in $L$ is a finite sequence $B_0, B_1, ..., B_n (n \geq 0)$ in $A_L$ satisfying $B_i \sim B_{i+1}$ whenever $0 \leq i < n$. The path is said to join the blocks $B_0$ and $B_n$. The number $n$ is said to be the \textit{length} of the path. A path is said to be proper if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. A path is called to be \textit{strictly proper} if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$ [1].

Let $A, B$ be two blocks of an OML $L$. If $A \sim B$ holds, then there exists a unique element $e \in A \cap B$ satisfying $A \cap B = (A \cup B) \cap S_e$ [1]. Using this element $e$, we say that $A$ and $B$ are \textit{linked at $e$ (strongly linked at $e$)} if $A \sim B (A \approx B)$, and use the notation $A \sim_e B (A \approx_e B)$. This element $e$ is called a \textit{vertex} of $L$ and it is the commutator of any $x \in A \setminus B$ and $y \in B \setminus A$ [1]. The set of all vertices of $L$ is denoted by $V_L$ and $L$ is said to be \textit{vertex-finite} if $|V_L|$ is finite.

Note that $A \approx B$ implies $A \sim B$, and $A \sim B$ implies $A^\omega B$. Some authors, for example Greechie, use the phrase \textit{"A and B meet in the section $S_e$"} to describe $A^\omega B$ [3].

**Definition 1.2.** Let $L$ be an OML, and $A, B \in A_L$. We will say that $A$ and $B$ are \textit{path-connected in $L$}, \textit{strictly path-connected in $L$} if $A$ and $B$ are joined by a proper path, a strictly proper path, respectively. We will say $A$ and $B$ are \textit{nonpath-connected} if there is no proper path joining $A$ and $B$, and $L$ is called \textit{nonpath-connected} if there exist two blocks which are nonpath-connected. An OML $L$ is called \textit{path-connected in $L$}, \textit{strictly path-connected in $L$} if any two blocks in $L$ are joined by a proper path, a strictly proper path, respectively. An OML $L$ is called \textit{relatively path-connected} iff each $[0, x]$ is path-connected for all $x \in L$.

The following lemma and propositions are well known.
**Lemma 1.3 [Bruns].** If \( L_1, L_2 \) are OMLs, \( L = L_1 \times L_2 \), \( A, B \in A_{L_1} \) and \( C, D \in A_{L_2} \), then \( A \times C \sim B \times D \) holds in \( L \) if and only if either \( A = B \) and \( C \sim D \) or \( A \sim B \) and \( C = D \). If \( A \) and \( B \) are linked at \( a \) then \( A \times C \) and \( B \times C \) are linked at \((a, 0)\). If \( C \) and \( D \) are linked at \( c \) then \( A \times C \) and \( A \times D \) are linked at \((0, c)\) [1].

**Proposition 1.4.** Every finite direct product of path-connected OMLs is path-connected [7].

**Proposition 1.5.** Every infinite direct product of path-connected OMLs containing infinitely many non-Boolean factors is nonpath-connected [6].

**Proposition 1.6.** Let \( L \) be an OML. Then the following are equivalent:

1. \( L \) is relatively path-connected;
2. \( C(x) \) is path-connected \( \forall x \in L \);
3. \( S_x \) is path-connected \( \forall x \in L \) [7].

### 2. Path-connected Orthomodular Lattices

Recall that the set of all vertices of an OML \( L \) is denoted by \( V_L \) and \( L \) is said to be vertex-finite if \( |V_i| \) is finite. Then \( V_L \subset \text{Com} L \). We define \( V_L^y = \{ \beta \in V_L | \beta \leq y \} \), for \( y \in L \). Note that \( 0 \in (\text{Com} L) \setminus V_L \).

**Lemma 2.1.** If \( C_0 \sim v_1 \ C_1 \sim v_2 \ C_2 \sim v_3 \ \ldots \sim v_{n-1} \ C_{n-1} \sim v_n \ C_n \) is a proper path in \( L \), then \( \bigwedge_{i=1}^{n} v_i \in \bigcap_{i=0}^{n} C_i \).

**Proof.** By induction on the length \( n \) of a proper path: If \( n = 1 \), then we have \( C_0 \sim v_1 \ C_1 \) and hence \( v_1 \in C_0 \cap C_1 \). Let \( n > 1 \) and assume the conclusion is true for all proper paths with length less than or equal to \( n-1 \). Let \( C_0 \sim v_1 \ C_1 \sim v_2 \ C_2 \sim v_3 \ \ldots \sim v_{n-1} \ C_{n-1} \sim v_n \ C_n \) be a path with length \( n \) in \( L \). We claim that \( \bigwedge_{i=1}^{n} v_i \in \bigcap_{i=0}^{n} C_i \). For \( 1 \leq k \leq n-1 \), \( \bigwedge_{i=1}^{k} v_i \in \bigcap_{i=0}^{k} C_i \) and \( \bigwedge_{i=k+1}^{n} v_i \in \bigcap_{i=k}^{n} C_i \) by induction hypothesis since the paths \( C_0 \sim v_1 \ C_1 \sim v_2 \ C_2 \sim v_3 \ \ldots \sim v_{n-1} \ C_{n-1} \sim v_n \ C_n \) and \( C_k \sim v_{k+1} \ C_{k+1} \sim v_{k+2} \ \ldots \sim v_n \ C_n \) are proper. Thus \( \bigwedge_{i=1}^{n} v_i \in C_k \) \((1 \leq k \leq n-1)\) since \( \bigwedge_{i=1}^{n} v_i \in C_k \) and \( \bigwedge_{i=k+1}^{n} v_i \in C_k \). Similarly, \( \bigwedge_{i=1}^{n} v_i \in C_0 \) since \( \bigwedge_{i=2}^{n} v_i \in C_1 \) by induction hypothesis. \( v_1 \in C_0 \), \( C_0[v_1, 1] = C_1[v_1, 1] \).
and \( \bigvee_{i=1}^{n} v_i \geq v_1 \). Similarly, \( \bigvee_{i=1}^{n} v_i \in C_n \) since \( \bigvee_{i=1}^{n-1} v_i \in C_{n-1} \) by induction hypothesis, \( v_n \in C_n \), \( C_{n-1}[v_n, 1] = C_n[v_n, 1] \) and \( \bigvee_{i=1}^{n} v_i \geq v_n \). Thus \( \bigvee_{i=1}^{n} v_i \in \bigcap_{i=0}^{n} C_i \).

We have the following corollary.

**Corollary 2.2.** If \( L \) is a path-connected OML such that \( \bigvee V_L \) exists, then \( \bigvee V_L \in C(L) \).

**Proof.** Let \( A \) be a block of \( L \). Then \( A \) is path-connected with all blocks of \( L \). Thus each vertex of \( L \) belongs to at least one proper path from \( A \). For each path \( \pi \) from \( A \) to another block \( B \) of \( L \), let \( c_\pi = \bigvee \{ v | v \text{ is a vertex in } \pi \} \). By Lemma (2.1), \( \bigvee c_\pi \in A \). Thus \( \bigvee V_L = \bigvee \{ c_\pi | \pi \text{ is a path from } A \text{ to another block of } L \} \in A \) since \( A \) is subcomplete, \( \bigvee c_\pi \in A \) and \( \bigvee V_L \) exists by the given hypothesis.

**Proposition 2.3.** Let \( L \) be an OML and let \( y \in L \). Then \( V_{[0,y]} = V^y_L \).

**Proof.** Let \( v \in V_{[0,y]} \). Then there exist distinct blocks \( A, B \in \mathcal{A}_{[0,y]} \) with \( A \sim_v B \) in \([0,y]\). In particular, \( v \leq y \). Let \( D \in \mathcal{A}_{[0,y']} \). Then \( A \oplus D \sim_{v \oplus 0} B \oplus D \) by Lemma (1.3) and hence \( v \in V^y_L \) so that \( V_{[0,y]} \subseteq V^y_L \). To show the reverse inclusion, let \( v \in V^y_L \). Then there exist distinct blocks \( E, F \in \mathcal{A}_L \) such that \( E \sim_v F \) in \( L \) and \( v \leq y \). In particular, \( E[0,v] \sim_v F[0,v] \) in \( L[0,v] \). Let \( G \in \mathcal{A}_{L[0,v']} \) such that \( y' \in G \). Then \( (E[0,v] \oplus G) \sim_{v \oplus 0} (F[0,v] \oplus G) \) by Lemma (1.3) in \( C(v) \) and, therefore, in \( L \). In particular, \( y' \in E[0,v] \oplus G \in \mathcal{A}_L \) and \( y' \in F[0,v] \oplus G \in \mathcal{A}_L \) since \( y' \in G \) so that \( y \in E[0,v] \oplus G \in \mathcal{A}_L \) and \( y \in F[0,v] \oplus G \in \mathcal{A}_L \). Therefore \( v \in V_{[0,y]} \).

We need the following theorem to prove Theorem (2.5).

**Theorem 2.4 [Greechie & Herman].** Let \( L \) be an OML. Then the set \( \mathcal{C}A(L) \) of all central Abelian elements of \( L \) is the set of orthocomplements of the upper bounds for the set \( ComL \), and \( \mathcal{C}A(L) \) exists if and only if \( \bigvee ComL \) exists. If \( h = \bigvee ComL \) exists, then \( \mathcal{C}A(L) = [0, h'] \) and \([0,h]\) contains no nonzero elements which are central Abelian elements of \([0,h]\) (and, therefore, of \( L \)) [4].

**Theorem 2.5.** Let \( L \) be a relatively path-connected vertex-finite OML and \( \alpha \in Com L \). Then \( \alpha = \bigvee V^\alpha_L \).
Proof. Let $\alpha \in Com L$ and consider $L[0, \alpha]$. Then $L[0, \alpha]$ has no nontrivial Boolean factor by Theorem (2.4) since $\bigvee Com L[0, \alpha] = \alpha$. And $\bigvee V_L^\alpha$ exists since $L$ is vertex-finite. Let $v = \bigvee V_L^\alpha$. Then $v \in Cen (L[0, \alpha])$ by Corollary (2.2). Thus $L[0, \alpha] = L[0, v] \oplus L[0, v' \land \alpha]$. We claim that $L[0, v' \land \alpha]$ is a Boolean algebra. Suppose that $L[0, v' \land \alpha]$ is non-Boolean. Then there exists a commutator $0 \neq \beta \in Com L[0, v' \land \alpha]$. Thus there exist at least two distinct path-connected blocks $A, B$ in $L[0, v' \land \alpha]$ since $L[0, v' \land \alpha]$ is path-connected by hypothesis. Therefore there exists at least one vertex $w$ in $L[0, v' \land \alpha]$ and hence in $L$ by Proposition (2.3); then $w \leq v \land v' = 0$ so that $w = 0$, a contradiction. Thus $L[0, v' \land \alpha]$ is Boolean. Moreover $L[0, v' \land \alpha]$ is a trivial Boolean factor since $L[0, \alpha]$ has no nontrivial Boolean factor. Thus $\alpha = v$. This completes the proof.

Since each commutator-finite OML is a relatively path-connected vertex-finite OML [2, 6], the following two corollaries immediately follow from Theorem (2.5).

**Corollary 2.6.** $L$ is a relatively path-connected vertex-finite OML if and only if $L$ is commutator-finite.

Two elements $a, b$ of an OML $L$ are said to be $p$-ideal in $L$ is a lattice ideal which is closed under perspectivity.

**Corollary 2.7.** Every irreducible commutator-finite OML is simple [4].

Proof. The conclusion follows since each commutator-finite OML is a vertex-finite relatively path-connected OML and each irreducible path-connected OML such that no proper $p$-ideal of $L$ contains infinitely many vertices is simple [7].

Now the following two corollaries hold.

**Corollary 2.8.** If $L$ is a commutator-finite OML and $\alpha \in Com L$, then $\alpha = \bigvee V_L^\alpha$.

**Corollary 2.9.** If $L$ is a commutator-finite OML, then $\bigvee Com L = \bigvee V_L$. 
The following propositions (2.10) and (2.11) give us some properties of path-connected OMLs, but it is not known whether there is an OML for which the conclusion of (2.10) fails.

**Proposition 2.10.** Let \( L \) be a path-connected OML, and \( x \in L \setminus C(L) \). Then there exist two blocks \( B, C \in \mathcal{A}_L \) such that \( x \in B \setminus C \) and \( B \cup C \leq L \).

**Proof.** \( \mathcal{A}_{C(x)} \) is properly contained in \( \mathcal{A}_L \) since \( x \notin C(L) \). Thus there exist two blocks \( D, E \) such that \( D \in \mathcal{A}_{C(x)} \) and \( E \in \mathcal{A}_L \setminus \mathcal{A}_{C(x)} \). There exists a proper path \( \{(B_j)\}_{j=0}^n \) from \( D = B_0 \) to \( E = B_n \) since \( L \) is path-connected. Let \( k \) be the minimal index such that \( B_k \notin \mathcal{A}_{C(x)} \). Then \( B_{k-1} \in \mathcal{A}_{C(x)} \). Let \( B_{k-1} = B \) and \( B_k = C \). Then \( x \in B \setminus C \) and \( B \sim C \). This completes the proof.

**Proposition 2.11.** Let \( L \) be a path-connected OML, and \( A, B \in \mathcal{A}_L \) with \( A \neq B \). If \( A \cap B \neq C(L) \), then \( A \) and \( B \) are strictly path-connected.

**Proof.** If one of the proper paths from \( A \) to \( B \) has length \( n \geq 2 \), then that path is a strictly proper path by the definition. Otherwise, every path from \( A \) to \( B \) has length 1 and so is a strictly proper path since \( A \cap B \neq C(L) \).

Let \( L \) be an OML, and \( A, B \in \mathcal{A}_L \). We define \( A \equiv B \) if and only if \( A \) and \( B \) are strictly path-connected. Then \( \equiv \) is an equivalence relation in \( \mathcal{A}_L \).

Bruns and Greechie have proved the following lemma for an OML \( L \) under the conditions that \( L \) is a path-connected OML without non-trivial Boolean factor [2]. We improve the lemma with no restriction except for the path-connectedness.

**Lemma 2.12.** Let \( L \) be a path connected OML, and \( (\mathcal{B}_i)_{i \in I} \) be the equivalence classes of \( \mathcal{A}_L \) modulo \( \equiv \). Then each \( \bigcup \mathcal{B}_i \) \( i \in I \) is a subalgebra of \( L \) with \( \mathcal{A}_{\bigcup \mathcal{B}_i} = \mathcal{B}_i \).

**Proof.** To prove that \( \bigcup \mathcal{B}_i \) \( i \in I \) are subalgebras, it is sufficient to show that \( a, b \in \bigcup \mathcal{B}_i \) implies \( a \lor b \in \bigcup \mathcal{B}_i \). If \( a \in C(L) \), then this is immediate. Thus we may assume \( a \notin C(L) \). There exist \( A \in \mathcal{B}_i \) and \( B \in \mathcal{A}_L \) such that \( a \in A \) and \( a, a \lor b \in B \). Then \( A \) and \( B \) are strictly path-connected by Proposition (2.11) since \( A \cap B \neq C(L) \).
Thus $A \equiv B$, that is $B \in B_i$. Thus $a \vee b \in \bigcup B_i$. Therefore $\bigcup B_i$ is a subalgebra, and $A_{\cup B_i} = B_i$ since each block belongs to one and only one of equivalence classes.

We do not know whether each path-connected OML $L$ has a maximal Boolean factor, but we know if $L$ is path-connected and not strictly path-connected then $L$ has a maximal Boolean factor as in Corollary (2.16).

**Lemma 2.13.** If there exists a block $A$ of an OML $L$ such that $A \subseteq C(L)$, then $L$ is Boolean.

**Proof.** If $A \subseteq C(L)$, then $L = CC(L) \subseteq C(A) = A \subseteq L$. Hence $L = A$ so that $L$ is a Boolean algebra and, therefore, $L = C(L)$.

**Lemma 2.14.** If $L$ is an OML with $C(L) = A \cap B = S_r \cap A$ for some $r \in A$ and $A, B \in A_L$, then $L = L_0 \oplus L_1$ where $L_0$ is a Boolean algebra and $L_1$ is an irreducible OML.

**Proof.** Let $L$ be an OML with $A, B \in A_L$ such that $C(L) = A \cap B = S_r \cap A$ for some $r \in A$. Then $C(L) = A \cap B = A[0, x'] \oplus \{0, x\}$. If $y \in A[0, x']$, then $y \in S_r \cap A = C(L) = \bigcap A_L$. Hence $A[0, x'] \subset (\bigcap A_L) \cap [0, x'] = \bigcap A[0, x] = Cen[0, x']$. Therefore $A[0, x') = [0, x']$ by Lemma (2.13) since $A[0, x']$ is a block of $[0, x']$. Hence $L = L[0, x'] \oplus L[0, x] = A[0, x'] \oplus L[0, x]$ since $x \in C(L)$ and $L[0, x'] = A[0, x']$. Furthermore, $Cen(L[0, x]) = \{0, x\}$ since $A[0, x], B[0, x] \in A_L[0, x]$ and $A[0, x] \cap B[0, x] = \{0, x\}$. Thus $L[0, x']$ is irreducible. Let $L_0 = A[0, x'] = L[0, x']$ and $L_1 = L[0, x]$. Then $L = L_0 \oplus L_1$ satisfies the requirements of the lemma.

We have the following two corollaries.

**Corollary 2.15.** If $L$ is an OML with $A \cup B \leq L$ and $A \cap B = C(L)$ for some $A, B \in A_L$, then $L = L_0 \oplus L_1$ where $L_0$ is a Boolean algebra and $L_1$ is an irreducible OML.

**Proof.** We may assume that $L$ is not a Boolean algebra. Then $A \neq B$ and there exists a unique element $x \in A \cap B = C(L)$ satisfying $A \cap B = S_r \cap (A \cup B) = S_r \cap A$. Thus the assertion holds by Lemma (2.14).
COROLLARY 2.16. If an OML $L$ is path-connected but not strictly path-connected, then $L = L_0 \oplus L_1$ where $L_0$ is a Boolean algebra and $L_1$ is an irreducible path-connected OML which is not strictly path-connected.

Proof. Let $L$ be a path-connected but not strictly path-connected OML. Then there exist two distinct blocks $A, B \in \mathcal{A}_L$ with $A \cap B = C(L)$ and $A \cup B \leq L$. This completes the proof by Corollary (2.15).

If $L$ is a path-connected OML with a maximal Boolean factor, then the following structure theorem holds.

THEOREM 2.17. If $L$ is a path-connected OML with a maximal Boolean factor $L_0$, then $L = L_0 \oplus L_1 \oplus L_2 \oplus \ldots \oplus L_n (n \geq 0)$, where $L_i (1 \leq i \leq n)$ are irreducible non-Boolean path-connected OMLs.

Proof. We may assume that $L$ is non-Boolean. Thus $L = L_0 \oplus L_s$ where $L_s$ is a path-connected OML which has no nontrivial Boolean factor. If $L_s$ is irreducible, then there is nothing to prove. Thus we may assume that $L_s$ is reducible. Then $L_s$ has only finitely many irreducible non-Boolean path-connected factors, otherwise $L_s$ and therefore $L$ would not be path-connected by Proposition (1.5).

It is well known that if $L$ be a path-connected OML with a trivial Boolean factor and $(\mathcal{B}_i)_{i \in I}$ are the equivalence classes of $\mathcal{A}_L$ modulo $\equiv$, then either $L$ is strictly path-connected, or $L$ is the horizontal sum of the family $\{\bigcup \mathcal{B}_i | i \in I\}$ of subalgebras [2]. We improve on this result as in Theorem (2.18).

THEOREM 2.18. Let $L$ be a path-connected OML, and $(\mathcal{B}_i)_{i \in I}$ be the equivalence classes of $\mathcal{A}_L$ modulo $\equiv$. Then $L$ is either strictly path-connected or the weak horizontal sum of the family $\{\bigcup \mathcal{B}_i | i \in I\}$ of subalgebras.

Proof. Let $L$ be a path-connected OML. We may assume $L$ is not strictly path-connected. Therefore $L = L_0 \oplus L_1$ where $L_0$ is a Boolean algebra and $L_1$ is an irreducible path-connected OML by Corollary (2.16). Thus it is sufficient to prove that $L_1$ is the horizontal sum of the family $\{\bigcup \mathcal{B}_i | i \in I\}$ of subalgebras of $L_1$ where $\mathcal{B}_i (i \in I)$ are the equivalence classes of $\mathcal{A}_{L_1}$ modulo $\equiv$. Then each $\bigcup \mathcal{B}_i (i \in I)$ is a subalgebra of $L_1$ with $\mathcal{A}_{\bigcup \mathcal{B}_i} = \mathcal{B}_i$ by Lemma (2.12). Moreover
(\bigcup B_i) \cap (\bigcup B_j) = \{0,1\} \text{ if } i \neq j, \text{ otherwise there exist two blocks } A \in \bigcup B_i, C \in \bigcup B_j (i \neq j) \text{ such that } A \cap C \neq \{0,1\} \text{ and hence } A \text{ and } C \text{ are strictly path-connected by Proposition (2.11) contradicting } A \neq C. \text{ This completes the proof. }

It is not known whether each block of a non-Boolean OML has a nonzero commutator, but the following partial answer for the path-connected OMLs as in Corollary (2.19) is known.

**Corollary 2.19.** If \( L \) is a non-Boolean path-connected OML, then every block \( A \) of \( L \) which is not a horizontal summand of \( L \) contains a vertex \( v \neq 0,1 \).

**Proof.** By Theorem (2.18), \( A \) is either strictly path-connected with each block of \( L \) or \( A \) is belong to one and only one strictly path-connected subalgebra \( \bigcup B_i \) of \( L \). If \( A \) is strictly path-connected with each block of \( L \), then there exist another block \( C \) such that \( A \approx_v C \) since \( L \) is non-Boolean. Thus \( v \neq \{0,1\} \) since \( A \cap C \neq C(L) \). Similarily, if \( A \) belongs to a strictly path-connected subalgebra \( \bigcup B_i \) of \( L \), then the desired conclusion follows by applying the above argument to \( \bigcup B_i \).

**References**


Department of Mathematics
Soongsil University
Seoul 156-743, Korea