J. Korean Math. Soc. **33** (1996), No. 2, pp. 217–225

A NOTE ON PATH–CONNECTED ORTHOMODULAR LATTICES

EUNSOON PARK

1. Preliminaries

An *orthomodular lattice* (abbreviated by OML) is an ortholattice L which satisfies *the orthomodular law*: if $x \leq y$, then $y = x \vee (x' \wedge y)$ [5]. A *Boolean algebra* B is an ortholattice satisfying the *distributive law*: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in B$.

A *subalgebra* of an OML L is a nonempty subset M of L which is closed under the operations $\vee$, $\wedge$ and $'$. We write $M \leq L$ if M is a subalgebra of L. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then the *relative interval sublattice* $M[a,b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with the *relative orthocomplementation* $\sharp$ on $M[a,b]$ given by $c^{\sharp} = (a \vee c') \wedge b = a \vee (c' \wedge b) \quad \forall c \in M[a,b]$. In particular, $L[a,b]$ will be denoted by $[a,b]$ if there is no ambiguity.

The *commutator of a and b* of an OML L is denoted by $a * b$, and is defined by $a * b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. The set of all commutators of L is denoted by $ComL$ and L is said to be *commutator-finite* if $|ComL|$ is finite. For elements a, b of an OML, we say *a commutes with b*, in symbols $a\,\mathbf{C}\,b$, if $a * b = 0$. If M is a subset of an OML L, the set $\mathbf{C}(M) = \{x \in L \mid x\,\mathbf{C}\,m \quad \forall m \in M\}$ is called the *commutant* of M in L and the set $\mathbf{C}en(M) = \mathbf{C}(M) \cap M$ is called the *center* of M. The set $\mathbf{C}(L)$ is called the center of L and then $\mathbf{C}(L) = \bigcap \{\mathbf{C}(a) \mid a \in L\}$. An OML L is called *irreducible* if $\mathbf{C}(L) = \{0, 1\}$, and L is called *reducible* if it is not irreducible.

A *block* of an OML L is a maximal Boolean subalgebra of L. The set of all blocks of L is denoted by $\mathcal{A}_L$. Note that $\bigcup \mathcal{A}_L = L$ and $\bigcap \mathcal{A}_L = \mathbf{C}(L)$. An OML L is said to be *block-finite* if $|\mathcal{A}_L|$ is finite.

Received July 2, 1994.
1991 AMS Subject Classification: 06C15
Key words: Orthomodular lattice, Path-connected, commutator, Vertex.

For any e in an OML L, the subalgebra $S_e = [0, e'] \cup [e, 1]$ is called the *(principal) section generated by* e. Note that for $A, B \in \mathcal{A}_L$, if $e \in (A \cap B)$ and $A \cap B = S_e \cap (A \cup B)$, then $A \cap B = S_e \cap A = S_e \cap B$.

DEFINITION 1.1. For blocks A, B of an OML L define $A \overset{wk}{\sim} B$ if and only if $A \cap B = S_e \cap (A \cup B)$ for some $e \in A \cap B$; $A \sim B$ if and only if $A \neq B$ and $A \cup B \leq L$; $A \approx B$ if and only if $A \sim B$ and $A \cap B \neq \mathbf{C}(L)$.

A *path* in L is a finite sequence $B_0, B_1, \ldots, B_n$ $(n \geq 0)$ in $\mathcal{A}_L$ satisfying $B_i \sim B_{i+1}$ whenever $0 \leq i < n$. The path is said to *join* the blocks B_0 and B_n. The number n is said to be the *length* of the path. A path is said to be *proper* if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. A path is called to be *strictly proper* if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$ [1].

Let A, B be two blocks of an OML L. If $A \sim B$ holds, then there exists a unique element $e \in A \cap B$ satisfying $A \cap B = (A \cup B) \cap S_e$ [1]. Using this element e, we say that A and B are *linked at* e (*strongly linked at* e) if $A \sim B$ $(A \approx B)$, and use the notation $A \sim_e B$ $(A \approx_e B)$. This element e is called a *vertex* of L and it is the commutator of any $x \in A \setminus B$ and $y \in B \setminus A$ [1]. The set of all vertices of L is denoted by V_L and L is said to be *vertex-finite* if $|V_L|$ is finite.

Note that $A \approx B$ implies $A \sim B$, and $A \sim B$ implies $A \overset{wk}{\sim} B$. Some authors, for example Greechie, use the phrase "*A and B meet in the section S_e*" to describe $A \overset{wk}{\sim} B$ [3].

DEFINITION 1.2. Let L be an OML, and $A, B \in \mathcal{A}_L$. We will say that A and B are *path-connected in L*, *strictly path-connected in L* if A and B are joined by a proper path, a strictly proper path, respectively. We will say A and B are *nonpath-connected* if there is no proper path joining A and B, and L is called *nonpath-connected* if there exist two blocks which are nonpath-connected. An OML L is called *path-connected in L*, *strictly path-connected in L* if any two blocks in L are joined by a proper path, a strictly proper path, respectively. An OML L is called *relatively path-connected* iff each $[0, x]$ is path-connected for all $x \in L$.

The following lemma and propositions are well known.

LEMMA 1.3 [BRUNS]. *If L_1, L_2 are OMLs, $L = L_1 \times L_2$, $A, B \in \mathcal{A}_{L_1}$ and $C, D \in \mathcal{A}_{L_2}$, then $A \times C \sim B \times D$ holds in L if and only if either $A = B$ and $C \sim D$ or $A \sim B$ and $C = D$. If A and B are linked at a then $A \times C$ and $B \times C$ are linked at $(a, 0)$. If C and D are linked at c then $A \times C$ and $A \times D$ are linked at $(0, c)$ [1].*

PROPOSITION 1.4. *Every finite direct product of path-connected OMLs is path-connected* [7].

PROPOSITION 1.5. *Every infinite direct product of path-connected OMLs containing infinitely many non-Boolean factors is nonpath-connected* [6].

PROPOSITION 1.6. *Let L be an OML. Then the following are equivalent:*

(1) *L is relatively path-connected;*
(2) *$\mathbf{C}(x)$ is path-connected $\forall x \in L$;*
(3) *S_x is path-connected $\forall x \in L$* [7].

2. Path-connected Orthomodular Lattices

Recall that the set of all vertices of an OML L is denoted by V_L and L is said to be vertex-finite if $|V_L|$ is finite. Then $V_L \subset ComL$. We define $V_L^y = \{\beta \in V_L \mid \beta \leq y\}$, for $y \in L$. Note that $0 \in (ComL) \setminus V_L$.

LEMMA 2.1. *If $C_0 \sim_{v_1} C_1 \sim_{v_2} C_2 \sim_{v_3} \cdots \sim_{v_{n-1}} C_{n-1} \sim_{v_n} C_n$ is a proper path in L, then $\bigvee_{i=1}^n v_i \in \bigcap_{i=0}^n C_i$.*

Proof. By induction on the length n of a proper path: If $n = 1$, then we have $C_0 \sim_{v_1} C_1$ and hence $v_1 \in C_0 \cap C_1$. Let $n > 1$ and assume the conclusion is true for all proper paths with length less than or equal to $n-1$. Let $C_0 \sim_{v_1} C_1 \sim_{v_2} C_2 \sim_{v_3} \cdots \sim_{v_{n-1}} C_{n-1} \sim_{v_n} C_n$ be a path with length n in L. We claim that $\bigvee_{i=1}^n v_i \in \bigcap_{i=0}^n C_i$. For $1 \leq k \leq n-1$, $\bigvee_{i=1}^k v_i \in \bigcap_{i=0}^k C_i$ and $\bigvee_{i=k+1}^n v_i \in \bigcap_{i=k}^n C_i$ by induction hypothesis since the paths $C_0 \sim_{v_1} C_1 \sim_{v_2} C_2 \sim_{v_3} \cdots \sim_{v_k} C_k$ and $C_k \sim_{v_{k+1}} C_{k+1} \sim_{v_{k+2}} \cdots \sim_{v_n} C_n$ are proper Thus $\bigvee_{i=1}^n v_i \in C_k$ $(1 \leq k \leq n-1)$ since $\bigvee_{i=1}^k v_i \in C_k$ and $\bigvee_{i=k+1}^n v_i \in C_k$. Similarly, $\bigvee_{i=1}^n v_i \in C_0$ since $\bigvee_{i=2}^n v_i \in C_1$ by induction hypothesis, $v_1 \in C_0$, $C_0[v_1, 1] = C_1[v_1, 1]$

and $\bigvee_{i=1}^{n} v_i \geq v_1$. Similarly, $\bigvee_{i=1}^{n} v_i \in C_n$ since $\bigvee_{i=1}^{n-1} v_i \in C_{n-1}$ by induction hypothesis, $v_n \in C_n$, $C_{n-1}[v_n, 1] = C_n[v_n, 1]$ and $\bigvee_{i=1}^{n} v_i \geq v_n$. Thus $\bigvee_{i=1}^{n} v_i \in \bigcap_{i=0}^{n} C_i$. ∎

We have the following corollary.

COROLLARY 2.2. *If L is a path-connected OML such that $\bigvee V_L$ exists, then $\bigvee V_L \in \mathbf{C}(L)$.*

Proof. Let A be a block of L. Then A is path-connected with all blocks of L. Thus each vertex of L belongs to at least one proper path from A. For each path π from A to another block B of L, let $c_\pi = \bigvee\{v | v \text{ is a vertex in } \pi\}$. By Lemma (2.1), $\bigvee c_\pi \in A$. Thus $\bigvee V_L = \bigvee\{c_\pi \mid \pi \text{ is a path from } A \text{ to another block of } L\} \in A$ since A is subcomplete, $\bigvee c_\pi \in A$ and $\bigvee V_L$ exists by the given hypothesis. ∎

PROPOSITION 2.3. *Let L be an OML and let $y \in L$. Then $V_{[0,y]} = V_L^y$.*

Proof. Let $v \in V_{[0,y]}$. Then there exist distinct blocks $A, B \in \mathcal{A}_{[0,y]}$ with $A \sim_v B$ in $[0, y]$. In particular, $v \leq y$. Let $D \in \mathcal{A}_{[0,y']}$. Then $A \oplus D \sim_{v\oplus 0} B \oplus D$ by Lemma (1.3) and hence $v \in V_L^y$ so that $V_{[0,y]} \subseteq V_L^y$. To show the reverse inclusion, let $v \in V_L^y$. Then there exist distinct blocks $E, F \in \mathcal{A}_L$ such that $E \sim_v F$ in L and $v \leq y$. In particular, $E[0, v] \sim_v F[0, v]$ in $L[0, v]$. Let $G \in \mathcal{A}_{L[0,v']}$ such that $y' \in G$. Then $(E[0, v] \oplus G) \sim_{v\oplus 0} (F[0, v] \oplus G)$ by Lemma (1.3) in $\mathbf{C}(v)$ and, therefore, in L. In particular, $y' \in E[0, v] \oplus G \in \mathcal{A}_L$ and $y' \in F[0, v] \oplus G \in \mathcal{A}_L$ since $y' \in G$ so that $y \in E[0, v] \oplus G \in \mathcal{A}_L$ and $y \in F[0, v] \oplus G \in \mathcal{A}_L$. Therefore $v \in V_{[0,y]}$. ∎

We need the following theorem to prove Theorem (2.5).

THEOREM 2.4 [GREECHIE & HERMAN]. *Let L be an OML. Then the set $\mathbf{CA}(L)$ of all central Abelian elements of L is the set of orthocomplements of the upper bounds for the set $ComL$, and $\mathbf{CA}(L)$ exists if and only if $\bigvee ComL$ exists. If $h = \bigvee ComL$ exists, then $\mathbf{CA}(L) = [0, h']$ and $[0, h]$ contains no nonzero elements which are central Abelian elements of $[0, h]$ (and, therefore, of L) [4].*

THEOREM 2.5. *Let L be a relatively path-connected vertex-finite OML and $\alpha \in Com\ L$. Then $\alpha = \bigvee V_L^\alpha$.*

Proof. Let $\alpha \in Com\ L$ and consider $L[0,\alpha]$. Then $L[0,\alpha]$ has no nontrivial Boolean factor by Theorem (2.4) since $\bigvee Com\ L[0,\alpha] = \alpha$. And $\bigvee V_L^\alpha$ exists since L is vertex-finite. Let $v = \bigvee V_L^\alpha$. Then $v \in$Cen $(L[0,\alpha])$ by Corollary (2.2). Thus $L[0,\alpha] = L[0,v] \oplus L[0,v' \wedge \alpha]$. We claim that $L[0,v' \wedge \alpha]$ is a Boolean algebra. Suppose that $L[0,v' \wedge \alpha]$ is non-Boolean. Then there exists a commutator $0 \neq \beta \in Com\ L[0,v' \wedge \alpha]$. Thus there exist at least two distinct path-connected blocks A, B in $L[0,v' \wedge \alpha]$ since $L[0,v' \wedge \alpha]$ is path-connected by hypothesis. Therefore there exists at least one vertex w in $L[0,v' \wedge \alpha]$ and hence in L by Proposition (2.3); then $w \leq v \wedge v' = 0$ so that $w = 0$, a contradiction. Thus $L[0,v' \wedge \alpha]$ is Boolean. Moreover $L[0,v' \wedge \alpha]$ is a trivial Boolean factor since $L[0,\alpha]$ has no nontrivial Boolean factor. Thus $\alpha = v$. This completes the proof. ∎

Since each commutator-finite OML is a relatively path-connected vertex-finite OML [2, 6], the following two corollaries immediately follow from Theorem (2.5).

COROLLARY 2.6. *L is a relatively path-connected vertex-finite OML if and only if L is commutator-finite.*

Two elements a, b of an OML L are said to *perspective* if there exists $z \in L$ such that $a \vee z = b \vee z = 1$ and $a \wedge z = b \wedge z = 0$. A *p-ideal* in L is a lattice ideal which is closed under perspectivity.

COROLLARY 2.7. *Every irreducible commutator-finite OML is simple [4].*

Proof. The conclusion follows since each commutator-finite OML is a vertex-finite relatively path-connected OML and each irreducible path-connected OML such that no proper p-ideal of L contains infinitely many vertices is simple [7].

Now the following two corollaries hold.

COROLLARY 2.8. *If L is a commutator-finite OML and $\alpha \in Com\ L$, then $\alpha = \bigvee V_L^\alpha$.*

COROLLARY 2.9. *If L is a commutator-finite OML, then $\bigvee Com\ L = \bigvee V_L$.*

The following propositions (2.10) and (2.11) give us some properties of path-connected OMLs, but it is not known whether there is an OML for which the conclusion of (2.10) fails.

PROPOSITION 2.10. *Let L be a path-connected OML, and $x \in L \setminus \mathbf{C}(L)$. Then there exist two blocks $B, C \in \mathcal{A}_L$ such that $x \in B \setminus C$ and $B \cup C \leq L$.*

Proof. $\mathcal{A}_{\mathbf{C}(x)}$ is properly contained in $\mathcal{A}_L$ since $x \notin \mathbf{C}(L)$. Thus there exist two blocks D, E such that $D \in \mathcal{A}_{\mathbf{C}(x)}$ and $E \in \mathcal{A}_L \setminus \mathcal{A}_{\mathbf{C}(x)}$. There exists a proper path $\{(B_j)\}_{j=0}^{n}$ from $D = B_0$ to $E = B_n$ since L is path-connected. Let k be the minimal index such that $B_k \notin \mathcal{A}_{\mathbf{C}(x)}$. Then $B_{k-1} \in \mathcal{A}_{\mathbf{C}(x)}$. Let $B_{k-1} = B$ and $B_k = C$. Then $x \in B \setminus C$ and $B \sim C$. This completes the proof. ∎

PROPOSITION 2.11. *Let L be a path-connected OML, and $A, B \in \mathcal{A}_L$ with $A \neq B$. If $A \cap B \neq \mathbf{C}(L)$, then A and B are strictly path-connected.*

Proof. If one of the proper paths from A to B has length $n \geq 2$, then that path is a strictly proper path by the definition. Otherwise, every path from A to B has length 1 and so is a strictly proper path since $A \cap B \neq \mathbf{C}(L)$. ∎

Let L be an OML, and $A, B \in \mathcal{A}_L$. We define $A \equiv B$ if and only if A and B are strictly path-connected. Then $\equiv$ is an equivalence relation in $\mathcal{A}_L$.

Bruns and Greechie have proved the following lemma for an OML L under the conditions that L is a path-connected OML without non-trivial Boolean factor [2]. We improve the lemma with no restriction except for the path-connectedness.

LEMMA 2.12. *Let L be a path connected OML, and $(\mathcal{B}_i)(i \in I)$ be the equivalence classes of $\mathcal{A}_L$ modulo $\equiv$. Then each $\bigcup \mathcal{B}_i (i \in I)$ is a subalgebra of L with $\mathcal{A}_{\cup \mathcal{B}_i} = \mathcal{B}_i$.*

Proof. To prove that $\bigcup \mathcal{B}_i (i \in I)$ are subalgebras, it is sufficient to show that $a, b \in \bigcup \mathcal{B}_i$ implies $a \vee b \in \bigcup \mathcal{B}_i$. If $a \in \mathbf{C}(L)$, then this is immediate. Thus we may assume $a \notin \mathbf{C}(L)$. There exist $A \in \mathcal{B}_i$ and $B \in \mathcal{A}_L$ such that $a \in A$ and $a, a \vee b \in B$. Then A and B are strictly path-connected by Proposition (2.11) since $A \cap B \neq \mathbf{C}(L)$.

Thus $A \equiv B$, that is $B \in \mathcal{B}_i$. Thus $a \vee b \in \bigcup \mathcal{B}_i$. Therefore $\bigcup \mathcal{B}_i$ is a subalgebra, and $\mathcal{A}_{\cup \mathcal{B}_i} = \mathcal{B}_i$ since each block belongs to one and only one of equivalence classes. ∎

We do not know whether each path-connected OML L has a maximal Boolean factor, but we know if L is path-connected and not strictly path-connected then L has a maximal Boolean factor as in Corollary (2.16).

LEMMA 2.13. *If there exists a block A of an OML L such that $A \subseteq \mathbf{C}(L)$, then L is Boolean.*

Proof. If $A \subseteq \mathbf{C}(L)$, then $L = \mathbf{CC}(L) \subseteq \mathbf{C}(A) = A \subseteq L$. Hence $L = A$ so that L is a Boolean algebra and, therefore, $L = \mathbf{C}(L)$. ∎

LEMMA 2.14. *If L is an OML with $\mathbf{C}(L) = A \cap B = S_x \cap A$ for some $x \in A$ and $A, B \in \mathcal{A}_L$, then $L = L_0 \oplus L_1$ where L_0 is a Boolean algebra and L_1 is an irreducible OML.*

Proof. Let L be an OML with $A, B \in \mathcal{A}_L$ such that $\mathbf{C}(L) = A \cap B = S_x \cap A$ for some $x \in A$. Then $\mathbf{C}(L) = A \cap B = A[0, x'] \oplus \{0, x\}$. If $y \in A[0, x']$, then $y \in S_x \cap A = \mathbf{C}(L) = \bigcap \mathcal{A}_L$. Hence $A[0, x'] \subset (\bigcap \mathcal{A}_L) \cap [0, x'] = \bigcap \mathcal{A}_{[0, x']} = \mathbf{C}en[0, x']$. Therefore $A[0, x'] = [0, x']$ by Lemma (2.13) since $A[0, x']$ is a block of $[0, x']$. Hence $L = L[0, x'] \oplus L[0, x] = A[0, x'] \oplus L[0, x]$ since $x \in \mathbf{C}(L)$ and $L[0, x'] = A[0, x']$. Furthermore, $\mathbf{C}en(L[0, x]) = \{0, x\}$ since $A[0, x], B[0, x] \in \mathcal{A}_{L[0, x]}$ and $A[0, x] \cap B[0, x] = \{0, x\}$. Thus $L[0, x]$ is irreducible. Let $L_0 = A[0, x'] = L[0, x']$ and $L_1 = L[0, x]$. Then $L = L_0 \oplus L_1$ satisfies the requirements of the lemma. ∎

We have the following two corollaries.

COROLLARY 2.15. *If L is an OML with $A \cup B \leq L$ and $A \cap B = \mathbf{C}(L)$ for some $A, B \in \mathcal{A}_L$, then $L = L_0 \oplus L_1$ where L_0 is a Boolean algebra and L_1 is an irreducible OML.*

Proof. We may assume that L is not a Boolean algebra. Then $A \neq B$ and there exists a unique element $x \in A \cap B = \mathbf{C}(L)$ satisfying $A \cap B = S_x \cap (A \cup B) = S_x \cap A$. Thus the assertion holds by Lemma (2.14). ∎

COROLLARY 2.16. *If an OML L is path-connected but not strictly path-connected, then $L = L_0 \oplus L_1$ where L_0 is a Boolean algebra and L_1 is an irreducible path-connected OML which is not strictly path-connected.*

Proof. Let L be a path-connected but not strictly path-connected OML. Then there exist two distinct blocks $A, B \in \mathcal{A}_L$ with $A \cap B = \mathbf{C}(L)$ and $A \cup B \leq L$. This completes the proof by Corollary (2.15). ∎

If L is a path-connected OML with a maximal Boolean factor, then the following structure theorem holds.

THEOREM 2.17. *If L is a path-connected OML with a maximal Boolean factor L_0, then $L = L_0 \oplus L_1 \oplus L_2 \oplus \ldots \oplus L_n (n \geq 0)$, where $L_i\,(1 \leq i \leq n)$ are irreducible non-Boolean path-connected OMLs.*

Proof. We may assume that L is non-Boolean. Thus $L = L_0 \oplus L_s$ where L_s is a path-connected OML which has no nontrivial Boolean factor. If L_s is irreducible, then there is nothing to prove. Thus we may assume that L_s is reducible. Then L_s has only finitely many irreducible non-Boolean path-connected factors, otherwise L_s and thefore L would not be path-connected by Proposition (1.5). ∎

It is well known that if L be a path-connected OML with a trivial Boolean factor and $(\mathcal{B})_{i \in I}$ are the equivalence classes of $\mathcal{A}_L$ modulo $\equiv$, then either L is strictly path-connected, or L is the horizontal sum of the family $\{\bigcup \mathcal{B}_i | i \in I\}$ of subalgebras [2]. We improve on this result as in Theorem (2.18).

THEOREM 2.18. *Let L be a path-connected OML, and $(\mathcal{B}_i)_{i \in I}$ be the equivalence classes of $\mathcal{A}_L$ modulo $\equiv$. Then L is either strictly path-connected or the weak horizontal sum of the family $\{\bigcup \mathcal{B}_i | i \in I\}$ of subalgebras.*

Proof. Let L be a path-connected OML. We may assume L is not strictly path-connected. Therefore $L = L_0 \oplus L_1$ where L_0 is a Boolean algebra and L_1 is an irreducible path-connected OML by Corollary (2.16). Thus it is sufficient to prove that L_1 is the horizontal sum of the family $\{\bigcup \mathcal{B}_i | i \in I\}$ of subalgebras of L_1 where $\mathcal{B}_i$ $(i \in I)$ are the equivalence classes of $\mathcal{A}_{L_1}$ modulo $\equiv$. Then each $\bigcup \mathcal{B}_i$ $(i \in I)$ is a subalgebra of L_1 with $\mathcal{A}_{\cup \mathcal{B}_i} = \mathcal{B}_i$ by Lemma (2.12). Moreover

$(\bigcup \mathcal{B}_i) \cap (\bigcup \mathcal{B}_j) = \{0, 1\}$ if $i \neq j$, otherwise there exist two blocks $A \in \bigcup \mathcal{B}_i$, $C \in \bigcup \mathcal{B}_j$ $(i \neq j)$ such that $A \cap C \neq \{0, 1\}$ and hence A and C are strictly path-connected by Proposition (2.11) contradicting $A \not\approx C$. This completes the proof. ∎

It is not known whether each block of a nonBoolean OML has a nonzero commutator, but the following partial answer for the path-connected OMLs as in Corollary (2.19) is known.

COROLLARY 2.19. *If L is a non-Boolean path-connected OML, then every block A of L which is not a horizontal summand of L contains a vertex $v \neq 0, 1$.*

Proof. By Theorem (2.18), A is either strictly path-connected with each block of L or A is belong to one and only one strictly path-connected subalgebra $\bigcup B_i$ of L. If A is strictly path-connected with each block of L, then there exist another block C such that $A \approx_v C$ since L is non-Boolean. Thus $v \neq \{0, 1\}$ since $A \cap C \neq \mathbf{C}(L)$. Similary, if A belongs to a strictly path-connected subalgebra $\bigcup B_i$ of L, then the desired conclusion follows by applying the above argument to $\bigcup B_i$.

References

1. Bruns, G., *Block-finite Orthomodular Lattices*, Can. J. Math. **31** (1979), 961-985.
2. Bruns, G. and Greechie, R., *Blocks and Commutators in Orthomodular Lattices*, Algebra Universalis **27** (1990), 1-9.
3. Greechie, R., *On the Structure of Orthomodular Lattices Satisfying the Chain Condition*, J. of Combinatorial Theory **4** (1968), 210-218.
4. Greechie, R. and Herman, L., *Commutator-finite Orthomodular Lattices*, Order **1** (1985), 277-284.
5. Kalmbach, G., *Orthomodular Lattices*, Academic Press Inc. (London) Ltd., 1983.
6. Park, E., *Path-connected Orthomodular Lattices*, Kansas State University Ph. D. Thesis, 1989.
7. Park, E., *A Note on Relatively Path-connected Orthomodular Lattices*, Bull. of Korean Math. Soc. **31** (1994), 61-72

Department of Mathematics
Soongsil University
Seoul 156-743, Korea