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ON A MOVING GRID NUMERICAL SCHEME FOR HAMILTON–JACOBI EQUATIONS

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1. Introduction

Analysis by the method of characteristics shows that if f and u_0 are smooth and u_0 has compact support, then the Hamilton-Jacobi equation

$$\text{(H-J)} \qquad \begin{aligned} &u_t + f(u_x) = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

has a unique C^1 solution u on some maximal time interval $0 \le t < T$ for which $\lim_{t \to T} u(x,t)$ exists uniformly; but this limiting function is not continuously differentiable. Thus u_x becomes discontinuous at $t = T$. Crandall and Lions [2] showed both existence and uniqueness of the generalized solutions that satisfy so called "viscosity" condition. They also showed that viscosity solutions of (H-J) are stable in L^∞ with respect to perturbation in the initial data, and consequently that the space of Lipschitz continuous functions forms a regularity space for (H-J). In one space dimension, Hong [3] has recently shown that if f is approximated in $L^\infty(\mathbb{R})$ to order $(2^n)^{-3}$ by a C^1 piecewise quadratic function f_n and if u_0 is approximated in L^∞ by a continuous, piecewise quadratic polynomial w_0 with 2^n free knots, then the solution $w(\cdot,t)$ of

$$\begin{aligned} &w_t + f_n(w_x) = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ &w(x,0) = w_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

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is again continuous, piecewise quadratic for all time and has no more than $C2^n$ pieces for some C. As a result, $u(\cdot,t)$ can be approximated with an error not exceeding the error of approximation of u_0 plus $O((2^n)^{-3})$.

In this paper, we construct a moving grid numerical scheme using the method of the characteristics that the continuous, piecewise quadratic solution $w(x,t)$ of Hamilton-Jacobi equations in one space dimension may be approximated in $L^\infty(\mathbb{R})$ to within $O(N^{-3})$ by a continuous, piecewise quadratic polynomial $w_0(x)$ with $O(N)$ meshpoints.

To show this, we use the simple relationship between the equations of (H-J) and hyperbolic single conservation laws. This relationship is very simple: if w is the viscosity solution of (H-J) then $v = w_x$ is the entropy solution of scalar conservation laws

$$
\begin{aligned}
v_t + f(v)_x &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
v(x,0) = v_0(x) &= w_0'(x), \quad x \in \mathbb{R}.
\end{aligned}
$$

Therefore, if one calculates shocks of single conservation laws according to the Rankine-Hugoniot jump condition and the entropy condition, then one can calculate the viscosity solution $u(x,t)$ by integrating $v(x,t)$ with property that $w(x,t)$ is continuous.

2. Stability of (H-J)

THEOREM 2.1. *Suppose that f and g are Lipschitz continuous and $f(0) = g(0) = 0$. If u_0 and w_0 are bounded and Lipschitz continuous, and u and w are the viscosity solutions of*

$$
\begin{aligned}
u_t + f(u_x) &= 0 \quad x \in \mathbb{R}, \quad t > 0, \\
u(x,0) &= u_0(x) \quad x \in \mathbb{R},
\end{aligned}
$$

and

$$
\begin{aligned}
w_t + g(w_x) &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
w(x,0) &= w_0(x), \quad x \in \mathbb{R},
\end{aligned}
$$

then for any $t > 0$,

$$
\text{(H)} \qquad \|u(\cdot,t) - w(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \|u_0 - w_0\|_{L^\infty(\mathbb{R})} + t\|f - g\|_{L^\infty(\mathbb{R})}.
$$

Proof. See Hong [4].

3. A Construction of a Perturbed Equation

Suppose that $f \in W^{3,\infty}(\mathbb{R})$ is strictly convex and $f(0) = 0$. Then there is a C^1 piecewise, quadratic approximation f_N to f with knots at the point j/N for $j = 0, 1, \ldots, N$, that is defined by $f_N'(j/N) = f'(j/N)$, $f_N(0) = 0$; f_N is strictly convex. Moreover $\|f - f_N\|_{L^\infty(\mathbb{R})} \leq C\|f^{(3)}\|_{L^\infty(\mathbb{R})}N^{-3}$ where $C = (9 + \frac{4}{\pi})\sqrt{\frac{4}{\pi}}(\frac{e}{4})^3$; see [1] and [7].

We suppose that a Lipschitz continuous function u_0 having the support in $[0, 1]$ has no polynomial piece of degree ≤ 1 and that the range of u_0 is in $[0, 1]$. We also assume that $u_0^{(i)}$ is in $BV(\mathbb{R})$ for $i = 0, 1$ and that u_0'' is also in $BV(\mathbb{R})$ outside a finite set of points $\{d_i\}_{i=1}^k$ where u_0' is discontinuous.

Now construct a continuous, piecewise quadratic approximation w_0 to u_0.

(1) choose $\tau_i = i\frac{1}{N}$ for $i = 0, 1, \ldots, N$ as initial meshpoints.

(2) If $u_0'(x)$ is discontinuous at d_i, then choose $d_i - \frac{1}{2N^3}$ and $d_i + \frac{1}{2N^3}$, and add these two points to the set of initial meshpoints.

(3) Insert least number of meshpoints in the interval where $u_0'(x)$ is smooth so that

$$|\tau_{i+1} - \tau_i| \int_{\tau_i}^{\tau_{i+1}} |u_0^{(3)}|\, dx \leq \frac{1}{N^2}|u_0''|_{BV(\mathbb{R}-\{d_i\})}.$$

(4) Construct a continuous, piecewise quadratic function $w_0(x)$ as follows. Let $I_i = [\tau_i, \tau_{i+1}]$. Set $w_0(\tau_i) = u_0(\tau_i)$ and $w_0(\tau_{i+1}) = u_0(\tau_{i+1})$. Now we need one more information to determine an unknown coefficient for $w_0(x)$. Let $M_i = \sup_{I_i} u_0'(x)$ and $m_i = \inf_{I_i} u_0'(x)$. Then choose any number for w_0' satisfying $m_i \leq \min_{I_i} w_0'(x) \leq \max_{I_i} w_0'(x) \leq M_i$.

(5) Insert new meshpoints τ_i satisfying $w_0'(\tau_i) = \frac{j}{N}$ for all $j = 0, \ldots, N$.

From these two approximations f_N and w_0, we have the following properties.

LEMMA 3.1. $|w_0'(x)|_{BV(\mathbb{R})} \leq 3|u_0'(x)|_{BV(\mathbb{R})}$.

Proof. Consider one interval $I_i = [\tau_i, \tau_{i+1}]$. Let $\Delta_i = |I_i|$, $M_i = \sup_{I_i} u_0'$, and $m_i = \inf_{I_i} u_0'$. Let s_i be the slope of w_0' on $[\tau_i, \tau_{i+1}]$.

Then since $|w_0'|_{BV(I_i)} = |s_i|\Delta_i \leq M_i - m_i \leq |u_0'|_{BV(I_i)}$,

$$\sum_i |w_0'|_{BV(I_i)} \leq \sum_i |u_0'|_{BV(I_i)}$$
$$\leq |u_0'(x)|_{BV(\mathbb{R})}.$$

We now measure the jump $|w_0'(\tau_i^+) - w_0'(\tau_i^-)|$. Since

$$|w_0'(\tau_i^+) - w_0'(\tau_i^-)| \leq (M_i - m_i) + (M_{i+1} - m_{i+1})$$
$$\leq |u_0'|_{BV(I_i)} + |u_0'|_{BV(I_{i+1})},$$

$\sum_i |w_0'(\tau_i^+) - w_0'(\tau_i^-)| \leq \sum_i(|u_0'|_{BV(I_i)} + |u_0'|_{BV(I_{i+1})}) \leq 2|u_0'(x)|_{BV(\mathbb{R})}$. Hence $|w_0'(x)|_{BV(\mathbb{R})} \leq 3|u_0'(x)|_{BV(\mathbb{R})}$.

LEMMA 3.2. *Suppose that s is a piecewise linear interpolation to f at $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$, and the t_i's are chosen so that*

$$(t_{i+1} - t_i)\int_{t_i}^{t_{i+1}} |f''|dx \leq \frac{1}{N^2}|f'|_{BV([0,1])} \text{ for all } i,$$

then $\|f - s\|_\infty \leq \frac{1}{4N^2}|f'|_{BV([0,1])}$.

Proof. See [3].

LEMMA 3.3. *If $u_0'(x)$ has k number of discontinuities, then there are at most $(6|u_0'|_{BV(\mathbb{R})} + 6)N + 6k - 1$ meshpoints.*

Proof. Step (1) and step (2) give at most $2k + N + 1$ meshpoints. By de Boor [1], step (3) gives at most N new meshpoints. We now count the number of meshpoints τ_i inserted by step (5). If call it ψ_i and order them, then each interval $[\psi_i, \psi_{i+1}]$ may or may not contain previously inserted meshpoints. If $[\psi_i, \psi_{i+1}]$ does not contain any old meshpoint generated by (1), (2) and (3), then ψ_i and ψ_{i+1} are two adjacient points constructed by (5). Therefore $|w_0'(\psi_i) - u_0'(\psi_{i+1})| = \frac{j+1}{N} - \frac{j}{N}$ or $\frac{j}{N} - \frac{j-1}{N} = \frac{1}{N}$ for some j. So the total number of meshpoints in this case is no more than $2|w_0'|_{BV(\mathbb{R})}N \leq 6|u_0'|_{BV(\mathbb{R})}N$. If $[\psi_i, \psi_{i+1}]$ does contain any old meshpoint, then, since the number of old meshpoints is $2k + 2N + 1$, at most $(2k + 2N - 1)$ number of the interval $[\psi_i, \psi_{i+1}]$ may contain old meshpoints. So the total number of meshpoints of this kind is no more than $2(2k + 2N - 1) = 4k + 4N - 2$. Therefore step (5) adds no more than $(6|u_0'|_{BV(\mathbb{R})} + 4)N + 4k - 2$ number of new meshpoints. So we complete the proof.

THEOREM 3.4.

$$\|u_0 - w_0\|_{L^\infty(\mathbb{R})} \leq \frac{1}{N^3}(|u_0'|_{BV(\mathbb{R})} + \frac{1}{4}|u_0''|_{BV(\mathbb{R}-\{d_i\})}).$$

Proof. Let B be the union of $I_i = [\tau_i, \tau_{i+1}]$ containing a point at which u_0' is discontinuous. Let $A = [0,1] - B$. Then

$$\begin{aligned}
\|u_0 - w_0\|_{L^\infty(\mathbb{R})} &\leq \sup_B |u_0 - w_0| + \sup_A |u_0 - w_0| \\
&= \max_B \sup_{I_i} |u_0(\tau_i) - w_0(\tau_i) + \int_{\tau_i}^{x} (u_0'(s) - w_0'(s))\, ds| \\
&\quad + \max_A \sup_{I_i} |u_0(\tau_i) - w_0(\tau_i) + \int_{\tau_i}^{x} (u_0'(s) - w_0'(s))\, ds| \\
&\leq \max_B \sup_i \int_{I_i} |u_0'(s) - w_0'(s)|\, ds \\
&\quad + \max_A \sup_i |\tau_{i+1} - \tau_i| \|u_0'(s) - w_0'(s)\|_{L^\infty(I_i)} \\
&\leq \max_B \sup_i \frac{1}{N^3} |u_0'|_{BV(I_i)} \\
&\quad + \frac{1}{N} \|u_0' - w_0'\|_{L^\infty(A)} \\
&\leq \frac{1}{N^3} |u_0'|_{BV(\mathbb{R})} + \frac{1}{N}(\frac{1}{4N^2} |u_0'' - u_0''|_{BV(\mathbb{R}-\{d_i\})}) \\
&= \frac{1}{N^3}(|u_0'|_{BV(\mathbb{R})} + \frac{1}{4}|u_0''|_{BV(\mathbb{R}-\{d_i\})}).
\end{aligned}$$

Here the first part of the third inequality is because the area of the set of points that are greater than u_0' but less than w_0' plus those points that are less than u_0' but greater than w_0' is obviously less than or equal to $|\tau_{i+1} - \tau_i| = \frac{1}{N^3}$ times $|u_0'|_{BV(I_i)}$ because w_0' is in between $\sup_{I_i} u_0'$ and $\inf_{I_i} u_0'$, and the second part of the fourth inequality is valid because of (3) in the construction of $w_0(x)$.

4. A Moving Grid Numerical Scheme for Perturbed Equation

Let $v_0(x) = w_0'(x)$. Then $v_0(x)$ is discontinuous piecewise linear having support in $[0,1]$.

By the method of characteristics, we calculate the solution of the perturbed single conservation laws

$$v_t + f_N(v)_x = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$v(x,0) = v_0(x), \quad x \in \mathbb{R}.$$

Lax [6] showed the general theory of hyperbolic conservation laws. If v is continuous on $[\tau_i(t), \tau_{i+1}(t)]$, then v is linear on $[\tau_i(t), \tau_{i+1}(t)]$ since f_N is quadratic. So it is enough to find the nodal values using the Rankine-Hugoniot jump condition and the entropy condition. That is to determine the evolution of shocks. Let $v_l^i(t) = \lim_{x\to\tau_i(t)^-} v(x,t)$ and $v_r^i(t) = \lim_{x\to\tau_i(t)^+} v(x,t)$. Since f_N is strictly convex and v_0 is continuous, the following two inequalities (the entropy condition) trivially hold. Let $v(x,t)$ be discontinuous at $\tau_i(t)$. Then

$$(1) \quad \frac{f_N(v_l^i(t)) - f_N(v_r^i(t))}{v_l^i(t) - v_r^i(t)} \geq \frac{f_N(\eta) - f_N(v_r^i(t))}{\eta - v_r^i(t)} \quad \text{for } \eta \in [v_r^i(t), v_l^i(t)]$$

if $v_l^i(t) > v_r^i(t)$, or

$$(2) \quad \frac{f_N(v_l^i(t)) - f_N(v_r^i(t))}{v_l^i(t) - v_r^i(t)} \geq \frac{f_N(\eta) - f_N(v_l^i(t))}{\eta - v_l^i(t)} \quad \text{for } \eta \in [v_l^i(t), v_r^i(t)]$$

if $v_l^i(t) < v_r^i(t)$. Therefore no meshpoints are generated during the evolution of v because f_N is strictly convex and v_0 is continuous. The meshpoint moves along the characterics and the solution v is constant along the trajectory $\tau_i = \tau_i(t)$ which propagates with speed, due to Rankine-Hugoniot jump condition,

$$(3) \qquad \frac{d\tau_i}{dt} = \begin{cases} f_N'(v_l^i(t)), & \text{if } v_l^i(t) = v_r^i(t), \\ \frac{f_N(v_l^i(t)) - f_N(v_r^i(t))}{v_l^i(t) - v_r^i(t)}, & \text{if } v_l^i(t) \neq v_r^i(t). \end{cases}$$

So if v is continuous on $[\tau_i(t), \tau_{i+1}(t)]$, $\frac{d\tau_i}{dt}$ is constant and $\frac{dv_l^i(t)}{dt} = \frac{dv_r^i(t)}{dt} = 0$ because

(4)
$$\frac{dv_l^i(t)}{dt} = \begin{cases} 0, & \text{if } v_l^i(t) = v_r^i(t), \\ \frac{v_l^i(t) - v_r^{i-1}(t)}{\tau_i(t) - \tau_{i-1}(t)} \left(\frac{f_N(v_l^i(t)) - f_N(v_r^i(t))}{v_l^i(t) - v_r^i(t)} - f_N'(v_l^i(t)) \right), & \text{otherwise,} \end{cases}$$

and

(5)

$$\frac{dv_r^i(t)}{dt} = \begin{cases} 0, & \text{if } v_l^i(t) = v_r^i(t), \\ \frac{v_l^{i+1}(t)-v_r^i(t)}{\tau_{i+1}(t)-\tau_i(t)}\left(f_N'(v_r^i(t)) - \frac{f_N(v_l^i(t))-f_N(v_r^i(t))}{v_l^i(t)-v_r^i(t)}\right), & \text{otherwise.} \end{cases}$$

(4) and (5) are directly from $\frac{dv(\tau_i(t),t)}{dt} = \frac{\partial v(\tau_i(t),t)}{\partial t} + \frac{\partial v(\tau_i(t),t)}{\partial \tau_i(t)}\frac{d\tau_i(t)}{dt}$.

To determine where the shocks occur, we solve the above equations between shock interaction times. Therefore it is equivalent to find the trajectory of meshpoint τ_i. We use the fact that the conservation of mass holds near the discontinuity to find the shock trajectory as follows. Let, to the right of the shock, (respectively, to the left of the shock) $f_N^r(v) = a_r v^2 + 2b_r v + c_r$ and let v have the initial slope s_r. ($f_N^l(v) = a_l v^2 + 2b_l v + c_l$ and let v have the initial slope s_l.) Suppose that τ_i is the initial shock point.

We use the fact that mass is conserved near the discontinuity to find the shock trajectory. Let the triangle ABC having the vertices $A(x,t)$, $B(x_l,0)$ and $C(x_r,0)$, where $x_l < \tau_i < x_r$. Let the characteristic line from B to A (recpectively, from C to A) be

(6)
$$\begin{aligned} &x = x_l + tf_N^l(v_l(x_l)) \text{ where } v_l = b_{10} + b_{11}x, \\ &(x = x_r + tf_N^r(v_r(x_r)) \text{ where } v_l = b_{20} + b_{21}x \text{ respectively.}) \end{aligned}$$

where v_l and v_r are polynomial pieces to the left and right hand sides of τ_i respectively.

Suppose that the shock curve joining the point $(\tau_i, 0)$ and $A(x,t)$ is inside the triangle ABC, then v must solve the equation weakly on the triangle ABC;

(7)
$$\begin{aligned} 0 = &\int_{\Delta ABC} [v_t + f_N(v)_x]\, dxdt \\ =&t(-f_N^l(v_l(x_l)) + f_N^{l\,\prime}(v_l(x_l))v_l(x_l)) \\ &+ t(f_N^r(v_r(x_r)) - f_N^{r\,\prime}(v_r(x_r))v_r(x_r)) \\ &- \int_{x_l}^{\tau_i} v_l(x)\, dx - \int_{\tau_i}^{x_r} v_r(x)\, dx. \end{aligned}$$

The second equality simplly follows if one uses the divergence theorem. One can plug in the formulas for $f_N^l, f_N^r, x_l, x_r, v_l$ and v_r. Then one can get rid of x_l and x_r from the system (6) and (7) through a process called "elimination". We just use built-in eliminate commands in Macsyma or Maple. Finally x and t satisfy the following polynomial equation.

$$
\begin{aligned}
0 =&(2b_{11}a_{12}t+1)(2b_{21}a_{22}t+1)\\
&(2b_{11}b_{21}a_{22}tx^2 - 2b_{11}b_{21}a_{12}tx^2 - b_{21}x^2 + b_{11}x^2\\
&- 4a_{11}b_{11}b_{21}a_{22}t^2x + 4a_{21}b_{11}b_{21}a_{12}t^2x + 4b_{10}b_{21}a_{22}tx\\
&- 4b_{20}b_{11}a_{12}tx + 2a_{21}b_{21}tx - 2a_{11}b_{11}tx - 2b_{20}x + 2b_{10}x\\
&+ 8a_{20}b_{11}b_{21}a_{12}a_{22}t^3 - 8a_{10}b_{11}b_{21}a_{12}a_{22}t^3 + 2a_{11}^2b_{11}b_{21}a_{22}t^3\\
&- 2a_{21}^2b_{11}b_{21}a_{12}t^3 - 4b_{10}^2b_{21}a_{12}a_{22}t^2 + 4b_{20}^2b_{11}a_{12}a_{22}t^2\\
&- 4b_{10}a_{11}b_{21}a_{22}t^2 + 4a_{20}b_{21}a_{22}t^2 - a_{10}b_{21}a_{22}t^2\\
&+ 4b_{20}a_{21}b_{11}a_{12}t^2 + 4a_{20}b_{11}a_{12}t^2 - 4a_{10}b_{11}a_{12}t^2 - a_{21}^2b_{21}t^2\\
&+ a_{11}^2b_{11}t^2 + 2b_{20}^2a_{22}t - 2b_{10}^2a_{12}t + 2b_{20}a_{21}t\\
&- 2b_{10}a_{11}t + 1a_{20}t - 2a_{10}t).
\end{aligned}
$$

Let the last factor be $P(x,t)=0$. Then P has total degree three, and quadratic in x. So we can write

$$P(x,t) = P_1(t)x^2 + P_2(t)x + P_3(t),$$

where $P_i(t)$ has the degree i in t, and $x(t)$ can be calculated by the quadratic formula. The behavior is rather complicated. Newton claimed to classify all cubic curves, and he drew many useful diagrams. This is contained in his collected mathematical works published by Cambridge University Press. To find which branch we go on, we start with the fact that we normalized the shock curve to begin at the point $(0,0)$, so that tells you the sign of the discriminant. We also need to know which branch (plus or minus) to choose. However it is sufficient to see which branch goes through $(0,0)$ again according to Newton's mathematical works.

Once all $\tau_i(t)$ have been found, we compute the viscosity solution $w(x,t)$ of

$$
\text{(P)} \qquad
\begin{aligned}
&w_t + f_N(w_x) = 0, \quad x\in\mathbb{R}, \quad t>0.\\
&w(x,0) = w_0(x), \quad x\in\mathbb{R},
\end{aligned}
$$

as follows; We know that $w(x,t)$ and $v(x,t)$ have the same compact support; see [5]. To the left side of $\tau_0(t)$, $w(x,t) = 0$ because $v(x,t) = 0$. Because $w(x,t) = \int_{\tau_i(t)}^{x} v(s,t)\,ds$ and $w(x,t)$ is continuous at all $\tau_i(t)$, the constant of each quadratic piece of $w(x,t)$ can obtained successively from $\tau_0(t)$.

THEOREM 4.1. *Let $f \in W^{3,\infty}(\mathbb{R})$ be strictly convex. Let a Lipschitz continuous function u_0 have the support in $[0,1]$ having the range in $[0,1]$. Suppose that $u_0^{(i)}$ is in $BV(\mathbb{R})$ for $i = 0,1$ with $u_0'' \in BV(\mathbb{R})$ outside a finite set of points $\{d_i\}_{i=1}^k$ where u_0' is discontinuous. Assume that $u(x,t)$ is the viscosity solution of* (H-J)*. If the approximate solution $w(x,t)$ of* (P) *obtained by above numerical scheme, then*

$$\|u(\cdot,t) - w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{N^3}\left[|u_0'|_{BV(\mathbb{R})} + \frac{1}{4}|u_0''|_{BV(\mathbb{R}-\{d_i\})} + Ct\|f^{(3)}\|_{L^\infty(\mathbb{R})}\right]$$

where $C = (9 + \frac{4}{\pi})\sqrt{\frac{4}{\pi}}(\frac{e}{4})^3$.

Proof. It directly follows from Theorem 2.1, Lemma 3.3 and Theorem 3.4.

This shows that this scheme achieves $O(N^{-3})$ approximation w because w_0 has $O(N)$ meshpoints.

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