EXISTENCE OF SOLUTIONS FOR
P-LAPLACIAN TYPE EQUATIONS

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1. Introduction

In this paper, we shall show the existence of solutions of the following nonlinear partial differential equation

\[
\begin{align*}
\begin{cases}
\text{div}A(-\nabla u) = f(x, u, \nabla u) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\end{align*}
\]

where \( f(x, u, \nabla u) = -u|\nabla u|^{p-2} + h, \quad p \geq 2, \quad h \in L^\infty \). Also, we will deal, via mountain pass theorem, with the problem of existence of solutions for a quasilinear elliptic equation

\[
\begin{align*}
\begin{cases}
\text{div}A(-\nabla u) = g(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\end{align*}
\]

where \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function with primitive \( G(x, u) = \int_0^u g(x, v)dv \) which satisfies the following assumptions:

\[(g1) \limsup_{u \to 0} \frac{g(x, u)}{|u|^{p-1}} = 0;\]

\[(g2) \exists s < p^* = \frac{np}{n-p}, C : |g(x, u)| \leq C(1 + |u|^{s-1});\]

\[(g3) \exists t > p, R_o : 0 < tG(x, u) \leq g(x, u)u, \quad \text{if } |u| \geq R_o.\]

An easy example of such \( g \) is \( g(x, u) = |u|^{s-2}u \) with \( p < s < p^* \). Here, \( \text{div}A(-\nabla u) \) is the \( p \)-Laplacian type operator, which was introduced in

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\(^1\)Partially supported by BSRI 93-107 and by GARC.

\(^2\)After the work of this paper was completed, the first author passed away. In rememberance, the second author would like to dedicate this paper to him.
[2], defined as follows: Let $\alpha : \mathbb{R}^n \to [0, \infty)$ a convex function of class $C^1(\mathbb{R}^n - \{0\})$ satisfying
\begin{equation}
(1.1) \quad \alpha(t\xi) = t\alpha(\xi) \quad \text{for} \quad t > 0 \quad \text{and} \quad \xi \in \mathbb{R}^n.
\end{equation}
Define $A(0) = 0$ and $A(\xi) = \alpha(\xi)^{p-1} \nabla \alpha(\xi)$ for $\xi \in \mathbb{R}^n - \{0\}$, and for a fixed $p > 1$. Then $A : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous homogeneous mapping of degree $p-1$. We also assume that $A$ satisfies the following condition: There exist positive constants $\Gamma$ and $\gamma$ such that
\begin{align}
(1.2) \quad (A(\xi) - A(\eta)) \cdot (\xi - \eta) &\geq \gamma(|\xi| + |\eta|)^{p-1}|\xi - \eta|^2 \\
(1.3) \quad |A(\xi) - A(\eta)| &\leq \Gamma(|\xi| + |\eta|)^{p-2}|\xi - \eta|
\end{align}
for all $\xi, \eta \in \mathbb{R}^n$. Note that if $\alpha \in C^2(\mathbb{R}^n - \{0\})$ satisfies (1.1) and if there exists $\sigma > 0$ such that
\begin{equation}
(1.4) \quad \sum_{i,j=1}^{n} \frac{\partial^2 \alpha(\xi)}{\partial \xi_i \partial \xi_j} \eta_i \eta_j \geq \sigma |\eta|^2 \quad \text{whenever} \quad \alpha(\xi) = 1 \quad \text{and} \quad \nabla \alpha(\xi) \cdot \eta = 0,
\end{equation}
then $A$ satisfies (1.2) and (1.3). Note also that if $p \geq 2$ then (1.2) implies that
\begin{equation}
(1.5) \quad (A(\xi) - A(\eta)) \cdot (\xi - \eta) \geq \gamma|\xi - \eta|^p
\end{equation}
for all $\xi, \eta \in \mathbb{R}^n$. By (1.1) – (1.3), we have
\begin{equation}
(1.6) \quad A(\xi) \cdot \xi = \alpha(\xi)^p
\end{equation}
\begin{equation}
(1.7) \quad \gamma|\xi|^p \leq A(\xi) \cdot \xi \leq \Gamma|\xi|^p
\end{equation}
for all $\xi \in \mathbb{R}^n$. Let $\Omega$ be an open set in $\mathbb{R}^n$. By the $p$-Laplacian type operator we mean the operator $A : u \mapsto \text{div}A(-\nabla u)$ that assigns
\begin{equation}
\text{div}A(-\nabla u) \in W_{\text{loc}}^{-1, q}(\Omega)
\end{equation}
to each $u \in W_{\text{loc}}^{1, p}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Thus if $f \in W_{\text{loc}}^{-1, q}(\Omega)$ is given, we mean by a solution of the problem
\begin{equation}
\text{div}A(-\nabla u) = f \quad \text{in} \quad \Omega
\end{equation}
a function $u \in W_{\text{loc}}^{1, p}(\Omega)$ such that
\begin{equation}
- \int A(-\nabla u) \cdot \nabla \phi = < f, \phi >
\end{equation}
for all $\phi \in C_0^1(\Omega)$. 
EXAMPLES. We give a few examples of \( \alpha \) satisfying (1.1) – (1.3).

1. \( \alpha(\xi) = |\xi| = \left( \sum_{i=1}^{n} |\xi_i|^2 \right)^{\frac{1}{2}}, A(\xi) = |\xi|^{p-2}\xi \).

2. \( \alpha(\xi) = \left( \sum_{i=1}^{n} |\xi_i|^p \right)^{\frac{1}{p}}, A(\xi) = (|\xi_1|^{p-2}\xi_1, \cdots, |\xi_n|^{p-2}\xi_n). \)

3. \( \alpha = \alpha_1 + \alpha_2 \) with \( \alpha_1 \) and \( \alpha_2 \) satisfying (1.1) – (1.3).

4. The function \( \alpha : \mathbb{R}^n \to \mathbb{R} \) defined implicitly by \( \varphi(\xi/\alpha(\xi)) = 1 \) where \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is a function of class \( C^2(\mathbb{R}^n) \) satisfying that there exists \( \sigma > 0 \) such that

\[
\sum_{i=1}^{n} \frac{\partial^2 \varphi(\xi)}{\partial \xi_i \partial \xi_j} \eta_i \eta_j \geq \sigma |\eta|^2 \quad \text{for all} \quad \xi, \eta \in \mathbb{R}^n
\]

and that \( \{ \xi \in \mathbb{R}^n | \varphi(\xi) < 1 \} \) is a bounded neighborhood of the origin in \( \mathbb{R}^n \).

When \( p = 2 \) and \( \alpha(\xi) = |\xi| \), \( \text{div} A(-\nabla u) = -\Delta u \). In this case, the convergence and existence results have been obtained. See for instance Struwe[9], Boccardo-Murat-Puel[3], Rabinowitz[8]. Also, note that the properties of solutions of \( p \)-Laplacian type operator have studied generally by Baek[2]. In this paper, we extend the results in Laplacian case to generalized versions in \( p \)-Laplacian type operator.

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2. Convergence Results For Nonlinear Elliptic Equations

In this section, we shall prove the existence of the solution of a nonlinear boundary value problem of the type

\[
\text{div} A(-\nabla u) + u|\nabla u|^{p-2} = h \quad \text{in} \quad \Omega
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega
\]

by an approximation method. At first we state a lemma for the basic property of \( \alpha \) and prove a theorem stating that under certain compensated condition the gradients of approximate solutions converge as follows.
Lemma 2.1. Let $L^p(\mu)^n$ be the Banach space of all $\mathbb{R}^n$-valued $\mu$-measurable functions $X$ with finite $L^p$-norm $||X||_{L^p(\mu)^n} = \left( \int |X|^p d\mu \right)^{1/p}$. If $X_j$ is a sequence in $L^p(\mu)^n$ with the weak limit $X$ such that $\int \alpha(X_j)^p d\mu \to \int \alpha(X)^p d\mu$, then $X_j \to X$ strongly in $L^p(\mu)^n$.

Proof. The proof of the lemma is in [2]. But for the sake of completeness, it is presented here. If $X = 0$, then $\gamma \int ||X_j||^p d\mu \leq \int \alpha(X_j)^p d\mu \to 0$. Assume $X \neq 0$. Put $Y_j = (X + X_j)/2$ and $Z_j = (X - X_j)/2$. By weak lower semicontinuity, we have

$$\liminf_{j \to \infty} \int \alpha(X + X_j/2)^p d\mu \geq \int \alpha(X)^p d\mu.$$ 

$$\alpha(X)^p + \alpha(X_j)^p - 2\alpha(Y_j)^p$$

$$= p \int_0^1 (A(Y_j + tZ_j) - A(Y_j - tZ_j)) \cdot Z_j dt$$

$$\geq C_1 |Z_j|^p \quad \text{if} \quad p \geq 2$$

$$\geq C_2 (|X| + |X_j|)^{p-2} |Z_j|^2 \quad \text{if} \quad 1 < p < 2$$

If $1 < p < 2$, by Hölder inequality

$$\int (|X| + |X_j|)^{p-2} |Z_j|^2 d\mu \geq \left( \int (|X| + |X_j|)^p d\mu \right)^{p-2/p} \left( \int |Z_j|^p d\mu \right)^{2/p}$$

Since $\int \alpha(X)^p + \alpha(X_j)^p - 2\alpha(X + X_j/2)^p d\mu$ goes to zero, we obtain $\int |Z_j|^p d\mu \to 0$ as desired.

Theorem 2.2. Suppose $\{u_m\} \in H^{1,p}_0(\Omega)$ is a sequence of solutions to elliptic equation

$$\text{div} A(-\nabla u_m) = f_m \quad \text{in} \quad \Omega$$

$$u_m = 0 \quad \text{on} \quad \partial\Omega$$

in a smooth bounded domain $\Omega$ in $\mathbb{R}^n$. Let $q$ be such that

$$\begin{cases} 
q > \frac{p^*}{p^* - 1} & \text{if} \quad 1 < p < n \\
q > 1 & \text{if} \quad p = n \\
q = 1 & \text{if} \quad p > n
\end{cases}$$
where \( p^* = \frac{np}{n-p} \). Suppose \( u_m \rightharpoonup u \) weakly in \( H^1_{0, p}(\Omega) \) while \( \{ f_m \} \) is bounded in \( L^q(\Omega) \). Then there is a subsequence such that \( \nabla u_m \to \nabla u \) in \( L^p(\Omega) \) and \( \nabla u_m \to \nabla u \) pointwise almost everywhere.

**Proof.** By weak lower semicontinuity

\[
\liminf_{m \to \infty} \int \alpha(-\nabla u_m)^p dx \geq \int \alpha(-\nabla u)^p dx.
\]

We want to show that \( \limsup_{m \to \infty} \int \alpha(-\nabla u_m)^p dx \leq \int \alpha(-\nabla u)^p dx \).

Note that

\[
\alpha(-\nabla u)^p - \alpha(-\nabla u_m)^p - pA(-\nabla u_m) \cdot (-\nabla u + \nabla u_m)
\]

\[
= p \int_0^1 (A((-\nabla u_m) + t(-\nabla u + \nabla u_m)) - A(-\nabla u_m)) \cdot (-\nabla u + \nabla u_m) dt
\]

\[
\geq \gamma p \int_0^1 (| - \nabla u_m + t(-\nabla u + \nabla u_m)| + | - \nabla u_m|)^{p-2} \cdot (-\nabla u + \nabla u_m)^2 \geq 0
\]

By the uniform boundedness of \( \{ f_m \} \) and the Rellich-Kondrakov theorem

\[- \int A(-\nabla u_m) \cdot (-\nabla u + \nabla u_m) = \int f_m(-u + u_m) \to 0 \quad \text{as} \quad m \to \infty.\]

Therefore \( \int \alpha(-\nabla u_m)^p dx \to \int \alpha(-\nabla u)^p \). By Lemma 2.1, \( \nabla u_m \to \nabla u \) in \( L^p(\Omega) \) and \( \nabla u_m \to \nabla u \) pointwise almost everywhere.

We obtain the following theorem stating that a weak limit of approximate solutions is a solution of the given equation in case the operator is monotone and continuous in \( \mathbb{R}^n \).

**Theorem 2.3.** Let \( \{ u_m \} \) and \( \{ f_m \} \) be as in Theorem 2.2 and if \( f_m \rightharpoonup f \) weakly in \( L^q(\Omega) \). Take a subsequence of \( \{ u_m \} \), still called \( \{ u_m \} \) as in Theorem 2.2, then \( u \) is a weak solution of

\[
\begin{align*}
\text{div} A(-\nabla u) &= f & \text{in} & \Omega \\
u &= 0 & \text{on} & \partial \Omega.
\end{align*}
\]

**Proof.** Since \( A \) is monotone,

\[
0 \leq \int (A(-\nabla v) - A(-\nabla u_m)) \cdot (-\nabla v + \nabla u_m) dx
\]
for all \( v \in H^1_0(\Omega) \). Furthermore, the identity

\[ -\int A(-\nabla u_m) \cdot (-\nabla v + \nabla u_m)dx = \int f_m(-v + u_m)dx \]

holds. Now pass to the limit to get

\[ 0 \leq \int A(-\nabla v) \cdot (-\nabla v + \nabla u) + f(-v + u)dx. \]

Fix \( \lambda > 0 \), \( w \in H^1_0(\Omega) \), and set \( v = u + \lambda w \). Upon cancelling \( \lambda \), we have

\[ 0 \geq -\int A(-\nabla u - \lambda \nabla w) \cdot \nabla w - fwdx. \]

Then send \( \lambda \) to zero to deduce

\[ 0 \leq -\int A(-\nabla u) \cdot \nabla w - fwdx. \]

Replacing \( w \) by \(-w\), we obtain

\[ 0 = -\int A(-\nabla u) \cdot \nabla w - fwdx \]

for each \( w \in H^1_0(\Omega) \).

To get the existence and uniqueness of solutions, we shall use the following theorem, due to Struwe[9], giving sufficient conditions for a functional to be bounded from below and to attain its infimum.

**Theorem 2.4.** Suppose \( V \) is a reflexive Banach space, and let \( M \) be its weakly closed subset. Suppose \( E : M \to \mathbb{R} \cup \{+\infty\} \) is coercive on \( M \) with respect to \( V \), and (sequentially) weakly lower semicontinuous on \( M \) with respect to \( V \). Then \( E \) is bounded from below on \( M \) and attains its infimum in \( M \).

**Proof.** Refer to Theorem 1.2 in [9].
Theorem 2.5. Let Ω be a bounded domain in \( \mathbb{R}^n \) and \( f \in H^{-1,q}(\Omega) \) be given. Then there exists a weak solution \( u \in H^{1,p}_0(\Omega) \) of the boundary value problem

\[
\begin{align*}
\text{div} A(-\nabla u) &= f & \text{in} & \Omega \\
u &= 0 & \text{on} & \partial \Omega.
\end{align*}
\]

(2.1)

Proof. Set the corresponding functional

\[
E(u) = \frac{1}{p} \int_{\Omega} \alpha(-\nabla u)^p \, dx - \int_{\Omega} f u \, dx
\]

on the Banach space \( H^{1,p}_0(\Omega) \); that is, problem (2.1) is of variational form. Note that \( H^{1,p}_0(\Omega) \) is reflexive. Moreover, \( E \) is coercive. In fact,

\[
E(u) \geq \frac{1}{p} \gamma \|u\|^p_{H^{1,p}_0} - \|f\|_{H^{-1,q}} \|u\|_{H^{1,p}_0} \geq \frac{\gamma}{p} (\|u\|^p_{H^{1,p}_0} - c \|u\|_{H^{1,p}_0})
\]

\[
\geq C_1 \|u\|^p_{H^{1,p}_0} - C_2.
\]

Finally, \( E \) is weakly lower semicontinuous: It suffices to show that

\[
\int f u_m \, dx \to \int f u \, dx.
\]

for \( u_m \to u \) weakly in \( H^{1,p}_0(\Omega) \). This follows from the very definition of weak convergence, since \( f \in H^{-1,q}(\Omega) \). Hence Theorem 2.4 implies that there is a minimizer \( u \in H^{1,p}_0 \).

Remark. In the same way, a result like Theorem 2.5 is obtained for \( f = f(x, u, \nabla u) \) with \( |f(x, u, \nabla u)| \leq C \).

Remark. Our operator is strictly monotone in the sense that

\[
\int (A(-\nabla u) - A(-\nabla v)) \cdot (-\nabla u + \nabla v) \, dx
\]

\[
\geq \gamma \int (|\nabla u| + |\nabla v|)^{p-2} |\nabla u + \nabla v|^2 \, dx.
\]
Now it is bigger than $\gamma \int | - \nabla u + \nabla v|^p dx$ when $p \geq 2$. If $1 < p < 2$, we have

$$\int (| - \nabla u| + | - \nabla v|)^{p-2} | - \nabla u + \nabla v|^{p} dx$$

$$\geq \left( \int (| - \nabla u| + | - \nabla v|)^p dx \right)^{\frac{p-2}{p}} \left( \int | - \nabla u + \nabla v|^p dx \right)^{\frac{2}{p}}$$

by Hölder inequality. So, in particular, the solution $u$ is unique.

We close this section by proving the existence of a solution of the following equation as a way of illustrating the use of results we obtained:

**Theorem 2.6.** Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^n$. Suppose $p \geq 2$ and $h \in L^\infty(\Omega)$. Then the following equation

$$\begin{cases} 
\text{div}A(-\nabla u) + u|\nabla u|^{p-2} = h & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega 
\end{cases}$$

(2.2)

has a solution in $H^{1,p}_0(\Omega)$.

**Proof.** Set the nonlinear term $g(u, \nabla u) = u|\nabla u|^{p-2}$ and approximate $g$ by functions

$$g_\epsilon(u, \nabla u) = \frac{g(u, \nabla u)}{1 + \epsilon|g(u, \nabla u)|}, \quad \epsilon > 0$$

satisfying $|g_\epsilon| \leq \frac{1}{\epsilon}$ and $g_\epsilon(u, \nabla u)u \geq 0$.

Now, since $g_\epsilon$ is uniformly bounded, the map $H^{1,p}_0 \ni u \mapsto g_\epsilon(u, \nabla u) \in H^{-1,q}$ is compact and bounded for any $\epsilon > 0$. Denote $F_\epsilon(u) = A(u) + g_\epsilon(u, \nabla u) = \text{div}A(-\nabla u) + g_\epsilon(u, \nabla u)$. The remark after Theorem 2.5 indicates that there is a solution $u_\epsilon \in H^{1,p}_0(\Omega)$ of the equation $F_\epsilon u_\epsilon = h$.

In addition, we have

$$\gamma \|u_\epsilon\|_{H^{1,p}_0}^p \leq \int \alpha(-\nabla u_\epsilon)^p dx \leq < u_\epsilon, F_\epsilon u_\epsilon >= < u_\epsilon, h >$$

$$\leq \|u_\epsilon\|_{H^{1,p}_0} \|h\|_{H^{-1,q}}.$$
so \( \{u_\epsilon\} \) is uniformly bounded in \( H^1_0, p(\Omega) \). We also deduce the uniform \( L^q \)-bound of \( g_\epsilon (u_\epsilon, \nabla u_\epsilon) \) by letting \( q = \frac{\delta p^*}{p^* - 1} \) where \( \delta = \frac{p(p^* - 1)}{p + p^*(p - 2)} > 1 \). In fact,

\[
\| g_\epsilon (u_\epsilon, \nabla u_\epsilon) \|_{L^q} \leq \int |u_\epsilon| |\nabla u_\epsilon|^{p-2} |q| dx \\
\leq (\int |u_\epsilon|^{p^*} dx)^{1-r} (\int |\nabla u_\epsilon|^p)^r \leq C
\]

where \( r = \frac{\delta p^*(p-2)}{p(p^* - 1)} \). We may assume that the sequence \( \{u_m = u_{\epsilon_m}\} \) weakly converges in \( H^1_0, p(\Omega) \) to a limit \( u \in H^1_0, p(\Omega) \). By Theorem 2.2, moreover, we may assume \( u_m \) converges strongly in \( H^1_0, p(\Omega) \) and \( u_m \) and \( \nabla u_m \) converge pointwise almost everywhere. Finally, Theorem 2.3 implies that \( u \) weakly solves (2.2) as desired.

**Remark.** In case of \( f(x, u, \nabla u) = -|\nabla u|^{p-1} + h \) with \( p \geq 1 \) and \( h \in L^\infty \), we can prove the existence of solution in the same way as in Theorem 2.6.

### 3. Existence Results For Quasilinear Elliptic Problem

In this section we deal with the existence of solutions of the quasilinear elliptic equation

\[
\begin{align*}
\text{div} A(-\nabla u) &= g(x, u) & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega
\end{align*}
\]

assuming Conditions (g1)-(g3).

Let \( V \) be a Banach space. Recall that an operator \( \mathcal{F} : V \to V^* \) is said to be pseudo-monotone if

1. \( \mathcal{F} \) is bounded
2. \( u_j \rightharpoonup u \) in \( V \) and \( \limsup_{j \to \infty} (\mathcal{F}(u_j), u_j - u) \leq 0 \) imply

\[
\text{(3.1)} \quad \liminf_{j \to \infty} (\mathcal{F}(u_j), u_j - v) \geq (\mathcal{F}(u), u - v) \quad \forall v \in V.
\]

The following lemma, whose proof is given below, is taken from [7].
Lemma 3.1. A pseudo-monotone operator $\mathcal{F}$ has the following property:
If $u_j \rightharpoonup u$ in $V$, $\mathcal{F}(u_j) \rightharpoonup \chi$ in $V^*$ and $\limsup_{j \to \infty} (\mathcal{F}(u_j), u_j) \leq (\chi, u)$, then $\chi = \mathcal{F}(u)$.

Theorem 3.2. The $p$-Laplacian type operator $A : H_0^{1,p} \to H^{-1,q}$ given by $A(u) = \text{div}A(-\nabla u)$ is pseudo-monotone. Thus $A$ has the property as in Lemma 3.1.

Proof. First, note that

$$||A(v)||_{H^{-1,q}} = \sup_{||\varphi||_{H_0^{1,p}}=1} \int A(-\nabla v) \nabla \varphi dx \leq \sup \int \Gamma | - \nabla v|^{p-1} |\nabla \varphi| dx \leq \sup \Gamma \left( \int | - \nabla v|^p dx \right)^{p-1} \left( \int |\nabla \varphi|^p dx \right)^{1 \over p} = \Gamma ||v||_{H_0^{1,p}}^{p-1}$$

imply the boundedness of $A$. Next, if $u_j$ satisfy the hypotheses of (2) above, then

(3.2) \hspace{1cm} (A(u_j), u_j - u) \to 0

In fact, since $A$ is monotone and $u_j - u \rightharpoonup 0$ in $H_0^{1,p}(\Omega)$,

$(A(u_j), u_j - u) \geq (A(u), u_j - u) \to 0$

Suppose $w = (1 - \epsilon)u + \epsilon v$, $\epsilon \in (0,1)$; we have

$(A(u_j) - A(w), u_j - w) \geq 0$

Therefore

$\epsilon(A(u_j), u - v) \geq -(A(u_j), u_j - u) + (A(w), u_j - u) - \epsilon(A(w), v - u)$.

By (3.2),

$$\epsilon \liminf_{j \to \infty} (A(u_j), u - v) \geq -\epsilon(A(w), v - u).$$
dividing by $\epsilon$ and using (3.2) again, we have

$$\liminf_{j \to \infty} (A(u_j), u_j - v) \geq (A(w), u - v).$$

Passing $\epsilon \to 0$ in this equation, we deduce (3.1) as desired.

**Proof of Lemma 3.1.** We shall still use the same notations as in Theorem 3.2. Suppose $u_j \rightharpoonup u$ in $H^{1,p}_0(\Omega)$, $A(u_j) \rightharpoonup \chi$ in $H^{-1,q}(\Omega)$ and

$$\limsup_{j \to \infty} (A(u_j), u_j) \leq (\chi, u).$$

Then,

$$\limsup_{j \to \infty} (A(u_j), u_j - u) \leq 0$$

and by (3.1),

$$(A(u), u - v) \leq \liminf_{j \to \infty} (A(u_j), u_j - v) \leq (\chi, u - v) \quad \forall v \in H^{1,p}_0(\Omega).$$

Therefore $\chi = A(u)$.

To obtain the result we want, we shall use the famous mountain pass lemma, see Ambrosetti and Rabinowitz [1].

**Theorem 3.3.** Suppose $E \in C^1(V)$ satisfies ($P$.-S.). Suppose

1. $E(0) = 0$;
2. $\exists \rho > 0, \alpha > 0 : ||u|| = \rho \Rightarrow E(u) \geq \alpha$;
3. $\exists u_1 \in V : ||u_1|| \geq \rho$ and $E(u_1) \leq \alpha$.

Define

$$P = \{ p \in C^0([0, 1]; V) ; p(0) = 0, p(1) = u_1 \}.$$

Then

$$\beta = \inf_{p \in P} \sup_{u \in p} E(u)$$

is a critical value.

**Remark.** The conclusion of Theorem 3.3 remains valid at level $\beta$ under the weaker assumption, which we call ($P$.-S.)$\beta$ condition, that ($P$.-S.)-sequences $\{u_m\}$ for $E$ such that $E(u_m) \to \beta$ are relatively compact.
THEOREM 3.4. Let $\Omega$ be a smooth, bounded domain in $\mathbb{R}^n$, $n > p$ and let $q : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with primitive $G(x, v) = \int_0^u g(x, v)dv$. Suppose the following conditions hold:

1. $\limsup_{u \to 0} \frac{g(x, u)}{|u|^{p-1}} = 0$ uniformly in $x \in \Omega$;
2. $\exists s < p^* = \frac{np}{n-p}, C : |g(x, u)| \leq C(1 + |u|^{s-1})$, for almost every $x \in \Omega, u \in \mathbb{R}$;
3. $\exists t > p, R_o : 0 < tG(x, u) \leq g(x, u)u$ for almost every $x \in \Omega$, if $|u| \geq R_o$.

Then the problem

$$
(3.3) \quad \begin{cases}
\text{div}A(-\nabla u) = q(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

admits non-trivial solutions $u^+ \geq 0 \geq u^-$. 

REMARK. A similar result in a Laplacian case was proved by Struwe [9; p.102].

Proof. The problem (3.3) corresponds to the Euler-Lagrange equation of the functional

$$
E(u) = \frac{1}{p} \int_\Omega \alpha(-\nabla u)^p dx - \int_\Omega G(x, u)dx
$$

on the space $H^{1,p}_0(\Omega)$. Note that

$$
||\text{div}A(-\nabla u) - \text{div}A(-\nabla v)||_{H^{-1,q}} = \sup_{||\varphi||_{H^{1,p}_0} = 1} \left| \int_\Omega (A(-\nabla u) + A(-\nabla v)) \cdot \nabla \varphi dx \right|
$$

$$
\leq \sup_\Omega \int_\Omega |A(-\nabla u) + A(-\nabla v)||\nabla \varphi| dx
$$

$$
\leq \sup_\Omega \left( \int_\Omega |A(-\nabla u) + A(-\nabla v)|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left( \int_\Omega |\nabla \varphi|^p dx \right)^{\frac{1}{p}}
$$

$$
\leq \Gamma \left[ \int_\Omega \left( |\nabla u| + |\nabla v| \right)^{p-2} \left| -\nabla u + \nabla v \right|^{\frac{p}{p-1}} dx \right]^{\frac{p-1}{p}}.
$$
Now it is less than $\Gamma(\int_\Omega |-\nabla u + \nabla v|^p dx)^{\frac{p-1}{p}}$ when $1 < p \leq 2$. If $p \geq 2$, we have
\[
\int_\Omega ((| - \nabla u| + | - \nabla v|)^p - | - \nabla u + \nabla v|^{\frac{p}{p-1}}) dx
\]
\[
\leq (\int_\Omega (| - \nabla u| + | - \nabla v|)^p dx)^{\frac{p-2}{p-1}} (\int_\Omega | - \nabla u + \nabla v|^p dx)^{\frac{1}{p-1}}
\]
by Hölder inequality. Therefore if $u \to v$ in $H^{1,p}_0(\Omega)$, then $\text{div} A(-\nabla u) \to \text{div} A(-\nabla v)$ in $H^{-1,q}(\Omega)$. This fact and assumption (2) imply that $E$ is of class $C^1$.

To see that $E$ satisfies $(P.-S.)_\beta$, we claim that
\[
\|u_m\|_{H^{1,p}_0} \leq C
\]
for a sequence $\{u_m\}$ in $H^{1,p}_0$ such that $E(u_m) \to \beta$ and $DE(u_m) \to 0$ in $H^{-1,q}$. We obtain
\[
C + o(1)\|u_m\|_{H^{1,p}_0} \geq tE(u_m) - \langle u_m, DE(u_m) \rangle
\]
\[
= t\left(\frac{1}{p} \int \alpha(-\nabla u_m)^p dx - \int G(x, u_m)dx \right) - \int \alpha(-\nabla u_m)^p dx
\]
\[
+ \int g(x, u_m) u_m dx
\]
\[
= \frac{t-p}{p} \int \alpha(-\nabla u_m)^p dx + \int (g(x, u_m) u_m - tG(x, u_m)) dx
\]
\[
\geq \frac{t-p}{p} \gamma\|u_m\|_{H^{1,p}_0}^p + \mathcal{L}^n(\Omega) \inf_{x \in \Omega, v \in R} (g(x, v)v - tG(x, v))
\]
where $o(1) \to 0$ as $m \to \infty$.

Thus we may assume that $u_m \to u$ weakly in $H^{1,p}_0(\Omega)$. Since the map $u \mapsto g(\cdot, u) : H^{1,p}_0(\Omega) \xrightarrow{\text{cpt}} L^p(\Omega) \xrightarrow{\text{cpt}} L^{\frac{p}{p-1}}(\Omega) \xrightarrow{\text{cpt}} H^{-1,q}(\Omega)$ is compact, we also may assume that
\[
u_m \to u \quad \text{weakly in} \quad H^{1,p}_0(\Omega)
\]
\[
u_m \to u \quad \text{in} \quad L^p(\Omega)
\]
\[
u_m \to u \quad \text{a.e.} \quad x \in \Omega
\]
\[
g(\cdot, u_m) \to g(\cdot, u) \quad \text{in} \quad H^{-1,q}(\Omega)
\]
\[
\text{div} A(-\nabla u_m) \to \chi \quad \text{weakly in} \quad H^{-1,q}(\Omega).
\]
Since \( \text{div} A(-\nabla u_m) - g(x, u_m) = \zeta_m \) where \( \zeta_m \to 0 \) in \( H^{-1,q}(\Omega) \), then for any \( \varphi \in H^{1,p}_0(\Omega) \)

\[
< \text{div} A(-\nabla u_m), \varphi > - < g(x, u_m), \varphi > = < \zeta_m, \varphi > .
\]

Passing to the limit \( m \to \infty \), we have \( \chi = g(x, u) \). Also,

\[
\limsup_{m \to \infty} < \text{div} A(-\nabla u_m), u_m > \\
\leq \limsup_{m \to \infty} \int_{\Omega} g(x, u_m)u_m \, dx + o(1) ||u_m||_{H^{1,p}_0}
\]

\[
= \int_{\Omega} g(x, u)u \, dx = < \chi, u >
\]

Thus Theorem 3.2 implies \( \chi = \text{div} A(-\nabla u) \). Moreover, since

\[
||\text{div} A(-\nabla u_m) - \text{div} A(-\nabla u)||_{H^{-1,q}} \\
\leq ||\text{div} A(-\nabla u_m) - g(x, u_m)||_{H^{-1,q}} + ||g(x, u_m) - \chi||_{H^{-1,q}},
\]

div \( A(-\nabla u_m) \to \text{div} A(-\nabla u) \) in \( H^{-1,q}(\Omega) \). So, \( E \) satisfies (P.-S.)\( \beta \).

From assumption (1), for any \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( |u| < \delta \) implies \( \frac{g(x, u)}{|u|^{p-1}} < \epsilon \). Then

\[
G(x, u) = \int_{0}^{u} g(x, v) \, dv \leq \frac{\epsilon}{p} |u|^p
\]

if \( |u| < \delta \). Also by (2) we obtain

\[
G(x, u) \leq C(\epsilon)|u|^s
\]

for some constant \( C(\epsilon) \), if \( |u| \geq \delta \). Thus

\[
G(x, u) \leq \epsilon |u|^p + C(\epsilon)|u|^s
\]

for all \( u \in R \) and almost every \( x \in \Omega \). It follows that

\[
E(u) \geq \frac{1}{p} \int_{\Omega} \alpha(-\nabla u)^p \, dx - \epsilon \int_{\Omega} |u|^p \, dx - C(\epsilon) \int_{\Omega} |u|^s \, dx
\]

\[
\geq \frac{1}{p} \gamma ||u||_{H^s_0}^p - \frac{\Gamma^p}{\lambda_1} ||u||_{H^s_0}^p - C(\epsilon)||u||_{H^s_0}^s
\]

\[
= (\frac{\gamma}{p} - \frac{\Gamma^p}{\lambda_1} - C(\epsilon)||u||_{H^s_0}^{s-p}) ||u||_{H^s_0}^p \geq \alpha > 0
\]
if \( ||u||_{H^{1,p}_o} = \rho \) is sufficiently small. Here, we have used the fact that

\[
\lambda_1 \leq \frac{\int \alpha (-\nabla u)^p dx}{\int |u|^p dx} \leq \frac{\Gamma ||u||^p_{H^{1,p}_o}}{||u||^p_{L^p}}
\]

and the fact that \( H^{1,p}_o(\Omega) \hookrightarrow L^s(\Omega) \).

Observe that \( E(0) = 0 \). Finally, (3) can be restated in the form

\[
u|u|^t \frac{d}{du}(|u|^{-t}G(x,u)) \geq 0 \quad \text{for } |u| \geq R_o
\]

Upon integration, we have

\[
G(x,u) \geq \gamma_o(x)|u|^t
\]

with \( \gamma_o(x) = R_o^{-t} \text{min}\{G(x,R_o), G(x,-R_o)\} > 0 \), if \( |u| \geq R_o \). Hence,

\[
E(\lambda u) = \frac{\lambda^p}{p} \int_{\Omega} \alpha (-\nabla u)^p dx - \int_{\Omega} G(x,\lambda u) dx
\]

\[
\leq \frac{\Gamma}{p} \lambda^p ||u||^p_{H^{1,p}_o} - \lambda^t \int_{x \in \Omega, |u| \geq R_o} \gamma_o(x)|u|^t dx
\]

\[
+ \mathcal{L}^n(\Omega) \inf_{x \in \Omega, |v| \leq R_o} |G(x,v)| \to -\infty \quad \text{as } \lambda \to \infty.
\]

We may let \( u_1 = \lambda u \) for fixed \( u \neq 0 \) and sufficiently large \( \lambda > 0 \). Therefore we obtain, from Theorem 3.3 the existence of a nontrivial solution to (3.3).

In order to obtain a solution \( u^+ \geq 0 \), we may truncate \( g \) below \( u = 0 \), replacing \( g \) by

\[
g^+(x,u) = \begin{cases} 
g(x,u) & \text{if } u \geq 0 \\
0 & \text{if } u \leq 0
\end{cases}
\]

with primitive \( G^+(x,u) = \int_0^u g^+(x,v) dv \). Note that (1), (2) remain valid for \( g^+ \) while (3) will hold for \( u > R_o \), almost everywhere in \( \Omega \). Moreover for \( u \leq 0 \) all terms in (3) vanish. Denote

\[
E^+(u) = \frac{1}{p} \int_{\Omega} \alpha (-\nabla u)^p dx - \int_{\Omega} G^+(x,u) dx.
\]
Our previous reasoning then yields a nontrivial critical point $u^+$ of $E^+$ which weakly solves the equation

$$\text{div}A(-\nabla u^+) = g^+(x, u^+) \quad \text{in} \quad \Omega.$$ 

Rewriting it as

$$\text{div}A(-\nabla u^+) + N(g^+(x, u^+)) = P(g^+(a, u^+))$$

where $P(a) = \max(a, 0)$ and $N(a) = \max(-a, 0)$, we have

$$-\int A(-\nabla u^+) \cdot \nabla \varphi + \int N(g^+(x, u^+))\varphi \geq 0$$

for all $\varphi \in H^1_0(\Omega)$ with $\varphi \geq 0$. Substituting $\varphi = \nu(u^+)$, we deduce that

$$\int_{\{u^+ < 0\}} A(-\nabla u^+) \cdot (-\nabla u^+) - \int_{\{u^+ < 0\}} N(g^+(x, u^+))(-u^+) \leq 0$$

while the left hand side is not less than a positive constant multiple of $\int_{\{u^+ < 0\}} |-\nabla u^+|^p$. Therefore $N(u^+) = 0$, that is, $u^+ \geq 0$ a.e. in $\Omega$. Hence we conclude that $u^+$ is a weak solution of the original equation (3.3). Similarly, we can show that $u^- \leq 0$ is also a weak solution of (3.3) as desired.

**Remark.** We note that if $u \in H^1_0(\Omega)$ weakly solves (3.3) with $g$ satisfying the hypotheses of Theorem 3.4, then $u$ weakly solves the equation

$$\text{div}A(-\nabla u) = a(x)(1 + |u|^{p-1})$$

with

$$a(x) = \frac{g(x, u(x))}{1 + |u|^{p-1}} \in L^\infty(\Omega)$$

We can then deduce that $u \in L^q$ for any $q < \infty$, therefore $u \in C^{1,\alpha}$ with some $\alpha > 0$; see [2], [6], [10].
References


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