J. Korean Math. Soc. **33** (1996), No. 2, pp. 291–307

EXISTENCE OF SOLUTIONS FOR P–LAPLACIAN TYPE EQUATIONS

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1. Introduction

In this paper, we shall show the existence of solutions of the following nonlinear partial differential equation

$$\begin{cases} \operatorname{div} A(-\nabla u) = f(x,u,\nabla u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $f(x,u,\nabla u) = -u|\nabla u|^{p-2} + h, \quad p \geq 2, \quad h \in L^\infty$. Also, we will deal, via mountain pass theorem, with the problem of existence of solutions for a quasilinear elliptic equation

$$\begin{cases} \operatorname{div} A(-\nabla u) = g(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with primitive $G(x,u) = \int_0^u g(x,v)dv$ which satisfies the following assumptions:

(g1) $\limsup_{u\to 0} \frac{g(x,u)}{|u|^{p-1}} = 0$;

(g2) $\exists s < p^* = \frac{np}{n-p}, C : |g(x,u)| \leq C(1+|u|^{s-1})$;

(g3) $\exists t > p, R_o : 0 < tG(x,u) \leq g(x,u)u, \quad \text{if} \quad |u| \geq R_o$.

An easy example of such g is $g(x,u) = |u|^{s-2}u$ with $p < s < p^*$. Here, $\operatorname{div} A(-\nabla u)$ is the p-Laplacian type operator, which was introduced in

Received December 10, 1994.

1991 AMS Subject Classification: 35J20, 35J60.

Key words: p-Laplacian, approximate solution, weak limit, pseudo-monotone.

[1]Partially supported by BSRI 93-107 and by GARC.

[2]After the work of this paper was completed, the first author passed away. In rememberance, the second author would like to dedicate this paper to him.

[2], defined as follows: Let $\alpha : \mathbb{R}^n \to [0,\infty)$ a convex function of class $C^1(\mathbb{R}^n - \{0\})$ satisfying

$$\alpha(t\xi) = t\alpha(\xi) \quad \text{for} \quad t > 0 \quad \text{and} \quad \xi \in \mathbb{R}^n. \tag{1.1}$$

Define $A(0) = 0$ and $A(\xi) = \alpha(\xi)^{p-1}\nabla\alpha(\xi)$ for $\xi \in \mathbb{R}^n - \{0\}$, and for a fixed $p > 1$. Then $A : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous homogeneous mapping of degree $p-1$. We also assume that A satisfies the following condition: There exist positive constants Γ and γ such that

$$(A(\xi) - A(\eta)) \cdot (\xi - \eta) \geq \gamma(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \tag{1.2}$$

$$|A(\xi) - A(\eta)| \leq \Gamma(|\xi| + |\eta|)^{p-2}|\xi - \eta| \tag{1.3}$$

for all $\xi, \eta \in \mathbb{R}^n$. Note that if $\alpha \in C^2(\mathbb{R}^n - \{0\})$ satisfies (1.1) and if there exists $\sigma > 0$ such that

$$\sum_{i,j=1}^{n} \frac{\partial^2\alpha(\xi)}{\partial\xi_i\partial\xi_j}\eta_i\eta_j \geq \sigma|\eta|^2 \quad \text{whenever} \quad \alpha(\xi) = 1 \quad \text{and} \quad \nabla\alpha(\xi)\cdot\eta = 0, \tag{1.4}$$

then A satisfies (1.2) and (1.3). Note also that if $p \geq 2$ then (1.2) implies that

$$(A(\xi) - A(\eta)) \cdot (\xi - \eta) \geq \gamma|\xi - \eta|^p \tag{1.5}$$

for all $\xi, \eta \in \mathbb{R}^n$. By (1.1) – (1.3), we have

$$A(\xi) \cdot \xi = \alpha(\xi)^p, \tag{1.6}$$

$$\gamma|\xi|^p \leq A(\xi) \cdot \xi \leq \Gamma|\xi|^p \tag{1.7}$$

for all $\xi \in \mathbb{R}^n$. Let Ω be an open set in $\mathbb{R}^n$. By the p-Laplacian type operator we mean the operator $\mathcal{A} : u \mapsto \operatorname{div}A(-\nabla u)$ that assigns $\operatorname{div}A(-\nabla u) \in W^{-1,q}_{loc}(\Omega)$ to each $u \in W^{1,p}_{loc}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Thus if $f \in W^{-1,q}_{loc}(\Omega)$ is given, we mean by a solution of the problem

$$\operatorname{div}A(-\nabla u) = f \qquad \text{in} \quad \Omega$$

a function $u \in W^{1,p}_{loc}(\Omega)$ such that

$$-\int A(-\nabla u)\cdot\nabla\phi = < f, \phi >$$

for all $\phi \in C^1_o(\Omega)$.

EXAMPLES. We give a few examples of α satisfying (1.1) – (1.3).

1. $\alpha(\xi) = |\xi| = (\sum_{i=1}^n |\xi_i|^2)^{\frac{1}{2}}, A(\xi) = |\xi|^{p-2}\xi$.
2. $\alpha(\xi) = (\sum_{i=1}^n |\xi_i|^p)^{\frac{1}{p}}, A(\xi) = (|\xi_1|^{p-2}\xi_1, \cdots, |\xi_n|^{p-2}\xi_n)$.
3. $\alpha = \alpha_1 + \alpha_2$ with α_1 and α_2 satisfying (1.1) – (1.3).
4. The function $\alpha : \mathbb{R}^n \to \mathbb{R}$ defined implicitly by $\varphi(\xi/\alpha(\xi)) = 1$ where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^2(\mathbb{R}^n)$ satisfying that there exists $\sigma > 0$ such that

$$\sum_{i=1}^n \frac{\partial^2}{\partial \xi_i \partial \xi_j} \varphi(\xi)\eta_i\eta_j \geq \sigma|\eta|^2 \qquad \text{for all} \quad \xi, \eta \in \mathbb{R}^n$$

and that $\{\xi \in R^n | \varphi(\xi) < 1\}$ is a bounded neighborhood of the origin in $\mathbb{R}^n$.

When $p = 2$ and $\alpha(\xi) = |\xi|$, $\text{div}A(-\nabla u) = -\Delta u$. In this case, the convergence and existence results have been obtained. See for instance Struwe[9], Boccardo-Murat-Puel[3], Rabinowitz[8]. Also, note that the properties of solutions of p-Laplacian type operator have studied generally by Baek[2]. In this paper, we extend the results in Laplacian case to generalized versions in p-Laplacian type operator.

We would like to thank Professor Hyeong In Choi for many fruitful consultations.

2. Convergence Results For Nonlinear Elliptic Equation

In this section, we shall prove the existence of the solution of a nonlinear boundary value problem of the type

$$\begin{aligned} \text{div}A(-\nabla u) + u|\nabla u|^{p-2} &= h \qquad && \text{in} \quad \Omega \\ u &= 0 && \text{on} \quad \partial\Omega \end{aligned}$$

by an approximation method. At first we state a lemma for the basic property of α and prove a theorem stating that under certain compensated condition the gradients of approximate solutions converge as follows.

LEMMA 2.1. *Let $L^p(\mu)^n$ be the Banach space of all $\mathbb{R}^n$-valued μ-measurable funtions X with finite L^p - norm $||X||_{L^p(\mu)^n} = (\int |X|^p d\mu)^{\frac{1}{p}}$. If X_j is a sequence in $L^p(\mu)^n$ with the weak limit X such that $\int \alpha(X_j)^p d\mu \to \int \alpha(X)^p d\mu$, then $X_j \to X$ strongly in $L^p(\mu)^n$.*

Proof. The proof of the lemma is in [2]. But for the safe of completeness, it is presented here. If $X = 0$, then $\gamma \int ||X_j||^p d\mu \le \int \alpha(X_j)^p d\mu \to 0$. Assume $X \neq 0$. Put $Y_j = (X + X_j)/2$ and $Z_j = (X - X_j)/2$. By weak lower semicontinuity, we have

$$\liminf_{j\to\infty} \int \alpha(X + X_j/2)^p d\mu \ge \int \alpha(X)^p d\mu.$$

$$\begin{aligned}\alpha(X)^p + \alpha(X_j)^p - 2\alpha(Y_j)^p &\\ &= p\int_0^1 (A(Y_j + tZ_j) - A(Y_j - tZ_j)) \cdot Z_j dt\\ &\ge C_1|Z_j|^p \qquad \text{if} \quad p \ge 2\\ &\ge C_2(|X| + |X_j|)^{p-2}|Z_j|^2 \qquad \text{if} \quad 1 < p < 2\end{aligned}$$

If $1 < p < 2$, by Hölder inequality

$$\int (|X| + |X_j|)^{p-2}|Z_j|^2 d\mu \ge (\int (|X| + |X_j|)^p d\mu)^{p-2/p}(\int |Z_j|^p d\mu)^{2/p}$$

Since $\int \alpha(X)^p + \alpha(X_j)^p - 2\alpha(X + X_j/2)^p d\mu$ goes to zero, we obtain $\int |Z_j|^p d\mu \to 0$ as desired.

THEOREM 2.2. *Suppose $\{u_m\} \in H_o^{1,p}(\Omega)$ is a sequence of solutions to elliptic equation*

$$\begin{aligned} divA(-\nabla u_m) &= f_m \qquad in \quad \Omega\\ u_m &= 0 \qquad on \quad \partial\Omega\end{aligned}$$

in a smooth bounded domain Ω in $\mathbb{R}^n$. Let q be such that

$$\begin{cases} q > \dfrac{p^*}{p^* - 1} & if \quad 1 < p < n\\ q > 1 & if \quad p = n\\ q = 1 & if \quad p > n\end{cases}$$

where $p^* = \frac{np}{n-p}$. *Suppose* $u_m \rightharpoonup u$ *weakly in* $H_0^{1,p}(\Omega)$ *while* $\{f_m\}$ *is bounded in* $L^q(\Omega)$. *Then there is a subsequence such that* $\nabla u_m \to \nabla u$ *in* $L^p(\Omega)$ *and* $\nabla u_m \to \nabla u$ *pointwise almost everywhere.*

Proof. By weak lower semicontinuity

$$\liminf_{m\to\infty} \int \alpha(-\nabla u_m)^p dx \geq \int \alpha(-\nabla u)^p dx.$$

We want to show that $\limsup_{m\to\infty} \int \alpha(-\nabla u_m)^p dx \leq \int \alpha(-\nabla u)^p dx$. Note that

$$\begin{aligned}
&\alpha(-\nabla u)^p - \alpha(-\nabla u_m)^p - pA(-\nabla u_m)\cdot(-\nabla u + \nabla u_m)\\
&= p\int_0^1 (A((-\nabla u_m) + t(-\nabla u + \nabla u_m)) - A(-\nabla u_m))\cdot(-\nabla u + \nabla u_m)dt\\
&\geq \gamma p \int_0^1 (|-\nabla u_m + t(-\nabla u + \nabla u_m)| + |-\nabla u_m|)^{p-2}|-\nabla u + \nabla u_m|^2 \geq 0
\end{aligned}$$

By the uniform boundedness of (f_m) and the Rellich-Kondrakov theorem

$$-\int A(-\nabla u_m)\cdot(-\nabla u + \nabla u_m) = \int f_m(-u + u_m) \to 0 \qquad \text{as} \quad m \to \infty.$$

Therefore $\int \alpha(-\nabla u_m)^p dx \to \int \alpha(-\nabla u)^p$. By Lemma 2.1, $\nabla u_m \to \nabla u$ in $L^p(\Omega)$ and $\nabla u_m \to \nabla u$ pointwise almost everywhere.

We obtain the following theorem stating that a weak limit of approximate solutions is a solution of the given equation in case the operator is monotone and continuous in $\mathbb{R}^n$.

THEOREM 2.3. *Let* $\{u_m\}$ *and* $\{f_m\}$ *be as in Theorem 2.2 and if* $f_m \rightharpoonup f$ *weakly in* $L^q(\Omega)$. *Take a subsequence of* $\{u_m\}$, *still called* $\{u_m\}$ *as in Theorem 2.2, then* u *is a weak solution of*

$$\begin{aligned}
divA(-\nabla u) &= f \qquad && in \quad \Omega\\
u &= 0 \qquad && on \quad \partial\Omega.
\end{aligned}$$

Proof. Since A is monotone,

$$0 \leq \int (A(-\nabla v) - A(-\nabla u_m))\cdot(-\nabla v + \nabla u_m)dx$$

for all $v \in H_o^{1,p}(\Omega)$. Furthermore, the identity

$$-\int A(-\nabla u_m) \cdot (-\nabla v + \nabla u_m) dx = \int f_m(-v + u_m) dx$$

holds. Now pass to the limit to get

$$0 \leq \int A(-\nabla v) \cdot (-\nabla v + \nabla u) + f(-v + u) dx.$$

Fix $\lambda > 0$, $w \in H_o^{1,p}(\Omega)$, and set $v = u + \lambda w$. Upon cancelling λ, we have

$$0 \geq -\int A(-\nabla u - \lambda \nabla w) \cdot \nabla w - f w dx.$$

Then send λ to zero to deduce

$$0 \leq -\int A(-\nabla u) \cdot \nabla w - f w dx.$$

Replacing w by $-w$, we obtain

$$0 = -\int A(-\nabla u) \cdot \nabla w - f w dx$$

for each $w \in H_o^{1,p}(\Omega)$.

To get the existence and uniqueness of solutions, we shall use the following theorem, due to Struwe[9], giving sufficient conditions for a functional to be bounded from below and to attain its infimum.

THEOREM 2.4. *Suppose V is a reflexive Banach space, and let M be its weakly closed subset. Suppose $E : M \to \mathbb{R} \cup \{+\infty\}$ is coercive on M with respect to V, and (sequencially) weakly lower semicontinuous on M with respect to V. Then E is bounded from below on M and attains its infimum in M.*

Proof. Refer to Theorem 1.2 in [9].

THEOREM 2.5. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $f \in H^{-1,q}(\Omega)$ be given. Then there exists a weak solution $u \in H_0^{1,p}(\Omega)$ of the boundary value problem*

$$\begin{cases} divA(-\nabla u) = f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega. \end{cases} \tag{2.1}$$

Proof. Set the corresponding functional

$$E(u) = \frac{1}{p}\int_\Omega \alpha(-\nabla u)^p dx - \int_\Omega f u dx$$

on the Banach space $H_o^{1,p}(\Omega)$; that is, problem (2.1) is of variational form. Note that $H_o^{1,p}(\Omega)$ is reflexive. Moreover, E is coercive. In fact,

$$\begin{aligned} E(u) &\geq \frac{1}{p}\gamma||u||^p_{H_o^{1,p}} - ||f||_{H^{-1,q}}||u||_{H_o^{1,p}} \geq \frac{\gamma}{p}(||u||^p_{H_o^{1,p}} - c||u||_{H_o^{1,p}}) \\ &\geq C_1||u||^p_{H_o^{1,p}} - C_2. \end{aligned}$$

Finally, E is weakly lower semicontinuous: It suffices to show that

$$\int f u_m dx \to \int f u dx.$$

for $u_m \rightharpoonup u$ weakly in $H_o^{1,p}(\Omega)$. This follows from the very definition of weak convergence, since $f \in H^{-1,q}(\Omega)$. Hence Theorem 2.4 implies that there is a minimizer $u \in H_o^{1,p}$.

REMARK. In the same way, a result like Theorem 2.5 is obtained for $f = f(x, u, \nabla u)$ with $|f(x, u, \nabla u)| \leq C$.

REMARK. Our operator is strictly monotone in the sense that

$$\begin{aligned} &\int (A(-\nabla u) - A(-\nabla v)) \cdot (-\nabla u + \nabla v) dx \\ &\qquad\qquad \geq \gamma \int (|-\nabla u| + |\nabla v|)^{p-2}|-\nabla u + \nabla v|^2 dx. \end{aligned}$$

Now it is bigger than $\gamma \int |-\nabla u+\nabla v|^p dx$ when $p \geq 2$. If $1<p<2$, we have

$$\int (|-\nabla u|+|-\nabla v|)^{p-2}|-\nabla u+\nabla v|^2 dx \geq \left(\int (|-\nabla u|+|-\nabla v|)^p dx\right)^{\frac{p-2}{p}} \left(\int |-\nabla u+\nabla v|^p dx\right)^{\frac{2}{p}}$$

by Hölder inequality. So, in particular, the solution u is unique.

We close this section by proving the existence of a solution of the following equation as a way of illustrating the use of results we obtained:

THEOREM 2.6. *Let Ω be a smooth and bounded domain in $\mathbb{R}^n$. Suppose $p \geq 2$ and $h \in L^\infty(\Omega)$. Then the following equation*

$$\begin{cases} divA(-\nabla u)+u|\nabla u|^{p-2}=h & in \quad \Omega \\ u=0 & on \quad \partial\Omega \end{cases} \tag{2.2}$$

has a solution in $H_o^{1,p}(\Omega)$.

Proof. Set the nonlinear term $g(u,\nabla u)=u|\nabla u|^{p-2}$ and approximate g by functions

$$g_\epsilon(u,\nabla u)=\frac{g(u,\nabla u)}{1+\epsilon|g(u,\nabla u)|}, \qquad \epsilon>0$$

satisfying $|g_\epsilon| \leq \frac{1}{\epsilon}$ and $g_\epsilon(u,\nabla u)u \geq 0$.

Now, since g_ϵ is uniformly bounded, the map $H_o^{1,p} \ni u \mapsto g_\epsilon(u,\nabla u) \in H^{-1,q}$ is compact and bounded for any $\epsilon>0$. Denote $F_\epsilon(u)=\mathcal{A}(u)+g_\epsilon(u,\nabla u)=\mathrm{div}A(-\nabla u)+g_\epsilon(u,\nabla u)$. The remark after Theorem 2.5 indicates that there is a solution $u_\epsilon \in H_o^{1,p}(\Omega)$ of the equation $F_\epsilon u_\epsilon=h$.

In addition, we have

$$\gamma||u_\epsilon||^p_{H_o^{1,p}} \leq \int \alpha(-\nabla u_\epsilon)^p dx \leq <u_\epsilon, F_\epsilon u_\epsilon> = <u_\epsilon, h> \leq ||u_\epsilon||_{H_o^{1,p}}||h||_{H^{-1,q}},$$

so $\{u_\epsilon\}$ is uniformly bounded in $H_o^{1,p}(\Omega)$. We also deduce the uniform L^q-bound of $g_\epsilon(u_\epsilon, \nabla u_\epsilon)$by letting $q = \frac{\delta p^*}{p^*-1}$ where $\delta = \frac{p(p^*-1)}{p+p^*(p-2)} > 1$. In fact,

$$\begin{aligned} ||g_\epsilon(u_\epsilon, \nabla u_\epsilon)||_{L^q}^q &\leq \int |u_\epsilon|\nabla u_\epsilon|^{p-2}|^q dx \\ &\leq (\int |u_\epsilon|^{p^*} dx)^{1-r} (\int |\nabla u_\epsilon|^p)^r \leq C \end{aligned}$$

where $r = \frac{\delta p^*(p-2)}{p(p^*-1)}$. We may assume that the sequence $\{u_m = u_{\epsilon_m}\}$ weakly converges in $H_o^{1,p}(\Omega)$ to a limit $u \in H_o^{1,p}(\Omega)$. By Theorem 2.2, moreover, we may assume u_m converges strongly in $H_o^{1,p}(\Omega)$ and u_m and ∇u_m converge pointwise almost everywhere. Finally, Theorem 2.3 implies that u weakly solves (2.2) as desired.

REMARK. In case of $f(x, u, \nabla u) = -|\nabla u|^{p-1} + h$ with $p \geq 1$ and $h \in L^\infty$, we can prove the existence of solution in the same way as in Theorem 2.6.

3. Existence Results For Quasilinear Elliptic Problem

In this section we deal with the existence of solutions of the quasilinear elliptic equation

$$\begin{aligned} \mathrm{div} A(-\nabla u) &= g(x, u) \qquad &\text{in} \quad \Omega \\ u &= 0 \qquad &\text{on} \quad \partial\Omega \end{aligned}$$

assuming Conditions (g1)-(g3).

Let V be a Banach space. Recall that an operator $\mathcal{F} : V \to V^*$ is said to be pseudo-monotone if

(1) $\mathcal{F}$ is bounded

(2) $u_j \rightharpoonup u$ in V and $\limsup_{j\to\infty}(\mathcal{F}(u_j), u_j - u) \leq 0$ imply

$$\liminf_{j\to\infty}(\mathcal{F}(u_j), u_j - v) \geq (\mathcal{F}(u), u - v) \qquad \forall v \in V. \tag{3.1}$$

The following lemma, whose proof is given below, is taken from [7].

LEMMA 3.1. *A pseudo-monotone operaror $\mathcal{F}$ has the following property:*
If $u_j \rightharpoonup u$ in V, $\mathcal{F}(u_j) \rightharpoonup \chi$ in V^* and $\limsup_{j\to\infty}(\mathcal{F}(u_j), u_j) \le (\chi, u)$, then $\chi = \mathcal{F}(u)$.

THEOREM 3.2. *The p-Laplacian type operator $\mathcal{A} : H_0^{1,p} \to H^{-1,q}$ given by $\mathcal{A}(u) = divA(-\nabla u)$ is pseudo-monotone. Thus $\mathcal{A}$ has the property as in Lemma 3.1.*

Proof. First, note that

$$\begin{aligned} ||\mathcal{A}(v)||_{H^{-1,q}} &= \sup_{||\varphi||_{H_o^{1,p}}=1} \int A(-\nabla v)\nabla\varphi dx \\ &\le \sup \int \Gamma|-\nabla v|^{p-1}|\nabla\varphi|dx \\ &\le \sup \Gamma\left(\int |-\nabla v|^p dx\right)^{\frac{p-1}{p}}\left(\int |\nabla\varphi|^p dx\right)^{\frac{1}{p}} = \Gamma||v||_{H_o^{1,p}}^{p-1} \end{aligned}$$

imply the boundedness of $\mathcal{A}$. Next, if u_j satisfy the hypotheses of (2) above, then

$$(\mathcal{A}(u_j), u_j - u) \to 0 \tag{3.2}$$

In fact, since $\mathcal{A}$ is monotone and $u_j - u \rightharpoonup 0$ in $H_o^{1,p}(\Omega)$,

$$(\mathcal{A}(u_j), u_j - u) \ge (\mathcal{A}(u), u_j - u) \to 0$$

Suppose $w = (1-\epsilon)u + \epsilon v$, $\quad \epsilon \in (0,1)$; we have

$$(\mathcal{A}(u_j) - \mathcal{A}(w), u_j - w) \ge 0$$

Therefore

$$\epsilon(\mathcal{A}(u_j), u - v) \ge -(\mathcal{A}(u_j), u_j - u) + (\mathcal{A}(w), u_j - u) - \epsilon(\mathcal{A}(w), v - u).$$

By (3.2),

$$\epsilon \liminf_{j\to\infty}(\mathcal{A}(u_j), u - v) \ge -\epsilon(\mathcal{A}(w), v - u).$$

dividing by ϵ and using (3.2) again, we have

$$\liminf_{j\to\infty}(\mathcal{A}(u_j), u_j - v) \geq (\mathcal{A}(w), u - v).$$

Passing $\epsilon \to 0$ in this equation, we deduce (3.1) as desired.

Proof of Lemma 3.1. We shall still use the same notations as in Theorem 3.2. Suppose $u_j \rightharpoonup u$ in $H_o^{1,p}(\Omega)$, $\mathcal{A}(u_j) \rightharpoonup \chi$ in $H^{-1,q}(\Omega)$ and
$\limsup_{j\to\infty}(\mathcal{A}(u_j), u_j) \leq (\chi, u)$. Then,

$$\limsup_{j\to\infty}(\mathcal{A}(u_j), u_j - u) \leq 0$$

and by (3.1),

$$(\mathcal{A}(u), u - v) \leq \liminf_{j\to\infty}(\mathcal{A}(u_j), u_j - v) \leq (\chi, u - v) \qquad \forall v \in H_o^{1,p}(\Omega).$$

Therefore $\chi = \mathcal{A}(u)$.

To obtain the result we want, we shall use the famous mountain pass lemma, see Ambrosetti and Rabinowitz [1].

THEOREM 3.3. *Suppose $E \in C^1(V)$ satisfies (P.-S.). Suppose*
(1) $E(0) = 0$;
(2) $\exists \rho > 0, \alpha > 0 : ||u|| = \rho \Rightarrow E(u) \geq \alpha$;
(3) $\exists u_1 \in V : ||u_1|| \geq \rho$ and $E(u_1) \leq \alpha$.
Define

$$\mathcal{P} = \{p \in C^o([0,1]; V); p(0) = 0, p(1) = u_1\}.$$

Then

$$\beta = \inf_{p\in\mathcal{P}} \sup_{u\in p} E(u)$$

is a critical value.

REMARK. The conclusion of Theorem 3.3 remains valid at level β under the weaker assumption, which we call (P.-S.)$_\beta$ condition, that (P.-S.)-sequences $\{u_m\}$ for E such that $E(u_m) \to \beta$ are relatively compact.

THEOREM 3.4. *Let Ω be a smooth, bounded domain in $\mathbb{R}^n, n > p$ and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with primitive $G(x,v) = \int_0^u g(x,v)dv$. Suppose the following conditions hold:*
(1) $\limsup_{u\to 0} \frac{g(x,u)}{|u|^{p-1}} = 0$ *uniformly in* $x \in \Omega$;
(2) $\exists s < p^* = \frac{np}{n-p}, C : |g(x,u)| \leq C(1+|u|^{s-1})$, *for almost every* $x \in \Omega$, $u \in R$;
(3) $\exists t > p, R_o : 0 < tG(x,u) \leq g(x,u)u$ *for almost every* $x \in \Omega$, *if* $|u| \geq R_o$.
Then the problem

$$\begin{cases} divA(-\nabla u) = g(x,u) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{3.3}$$

admits non-trivial solutions $u^+ \geq 0 \geq u^-$.

REMARK. A similar result in a Laplacian case was proved by Struwe [9; p.102].

Proof. The problem (3.3) corresponds to the Euler-Lagrange equation of the functional

$$E(u) = \frac{1}{p}\int_\Omega \alpha(-\nabla u)^p dx - \int_\Omega G(x,u)dx$$

on the space $H_o^{1,p}(\Omega)$. Note that

$$\begin{aligned}
&||\mathrm{div}A(-\nabla u) - \mathrm{div}A(-\nabla v)||_{H^{-1,q}} \\
&= \sup_{||\varphi||_{H_o^{1,p}}=1} |\int_\Omega (-A(-\nabla u) + A(-\nabla v)) \cdot \nabla\varphi dx| \\
&\leq \sup \int_\Omega |A(-\nabla u) + A(-\nabla v)||\nabla\varphi| dx \\
&\leq \sup (\int_\Omega |A(-\nabla u) + A(-\nabla v)|^{\frac{p}{p-1}} dx)^{\frac{p-1}{p}} (\int_\Omega |\nabla\varphi|^p dx)^{\frac{1}{p}} \\
&\leq \Gamma[\int_\Omega [(|-\nabla u| + |-\nabla v|)^{p-2}|-\nabla u + \nabla v|]^{\frac{p}{p-1}} dx]^{\frac{p-1}{p}}.
\end{aligned}$$

Now it is less than $\Gamma(\int_\Omega |-\nabla u+\nabla v|^p dx)^{\frac{p-1}{p}}$ when $1<p\leq 2$. If $p\geq 2$, we have

$$\int_\Omega ((|-\nabla u|+|-\nabla v|)^{p-2}|-\nabla u+\nabla v|)^{\frac{p}{p-1}}dx$$
$$\leq (\int_\Omega (|-\nabla u|+|-\nabla v|)^p dx)^{\frac{p-2}{p-1}}(\int_\Omega |-\nabla u+\nabla v|^p dx)^{\frac{1}{p-1}}$$

by Hölder inequality. Therefore if $u\to v$ in $H_o^{1,p}(\Omega)$, then $\mathrm{div}A(-\nabla u)\to \mathrm{div}A(-\nabla v)$ in $H^{-1,q}(\Omega)$. This fact and assumption (2) imply that E is of class C^1

To see that E satisfies $(\text{P.-S.})_\beta$, we claim that

$$||u_m||_{H_o^{1,p}}\leq C$$

for a sequence $\{u_m\}$ in $H_o^{1,p}$ such that $E(u_m)\to\beta$ and $DE(u_m)\to 0$ in $H^{-1,q}$. We obtain

$$C+o(1)||u_m||_{H_o^{1,p}}\geq tE(u_m)-<u_m,DE(u_m)>$$
$$=t(\frac{1}{p}\int\alpha(-\nabla u_m)^p dx-\int G(x,u_m)dx)-\int\alpha(-\nabla u_m)^p dx$$
$$+\int g(x,u_m)u_m dx$$
$$=\frac{t-p}{p}\int\alpha(-\nabla u_m)^p dx+\int(g(x,u_m)u_m-tG(x,u_m))dx$$
$$\geq\frac{t-p}{p}\gamma||u_m||^p_{H_o^{1,p}}+\mathcal{L}^n(\Omega)\ \inf_{x\in\Omega,v\in R}(g(x,v)v-tG(x,v))$$

where $o(1)\to 0$ as $m\to\infty$.

Thus we may assume that $u_m\rightharpoonup u$ weakly in $H_o^{1,p}(\Omega)$. Since the map $u\mapsto g(\cdot,u): H_o^{1,p}(\Omega)\xrightarrow{\text{cpt}}L^s(\Omega)\xrightarrow{g(\cdot,u)}L^{\frac{s}{s-1}}(\Omega)\xrightarrow{\text{cpt}}H^{-1,q}(\Omega)$ is compact, we also may assume that

$$\begin{aligned}
u_m &\rightharpoonup u && \text{weakly in}\quad H_o^{1,p}(\Omega)\\
u_m &\to u && \text{in}\quad L^s(\Omega)\\
u_m &\to u && \text{a.e.}\quad x\in\Omega\\
g(\cdot,u_m) &\to g(\cdot,u) && \text{in}\quad H^{-1,q}(\Omega)\\
\mathrm{div}A(-\nabla u_m) &\rightharpoonup \chi && \text{weakly in}\quad H^{-1,q}(\Omega).
\end{aligned}$$

Since $\mathrm{div}A(-\nabla u_m) - g(x,u_m) = \zeta_m$ where $\zeta_m \to 0$ in $H^{-1,q}(\Omega)$, then for any $\varphi \in H_o^{1,p}(\Omega)$

$$< \mathrm{div}A(-\nabla u_m), \varphi > - < g(x,u_m), \varphi > = < \zeta_m, \varphi > .$$

Passing to the limit $m \to \infty$, we have $\chi = g(x,u)$. Also,

$$\begin{aligned} &\limsup_{m\to\infty} < \mathrm{div}A(-\nabla u_m), u_m > \\ &\quad \le \limsup_{m\to\infty} \int_\Omega g(x,u_m)u_m dx + o(1)||u_m||_{H_o^{1,p}} \\ &\quad = \int_\Omega g(x,u)u dx = < \chi, u > \end{aligned}$$

Thus Theorem 3.2 implies $\chi = \mathrm{div}A(-\nabla u)$. Moreover, since

$$\begin{aligned} &||\mathrm{div}A(-\nabla u_m) - \mathrm{div}A(-\nabla u)||_{H^{-1,q}} \\ &\le ||\mathrm{div}A(-\nabla u_m) - g(x,u_m)||_{H^{-1,q}} + ||g(x,u_m) - \chi||_{H^{-1,q}}, \end{aligned}$$

$\mathrm{div}A(-\nabla u_m) \to \mathrm{div}A(-\nabla u)$ in $H^{-1,q}(\Omega)$. So, E satisfies $(\mathrm{P.}\text{-}\mathrm{S.})_\beta$.

From assumption (1), for any $\epsilon > 0$ there is $\delta > 0$ such that $|u| < \delta$ implies $\frac{g(x,u)}{|u|^{p-1}} < \epsilon$. Then

$$G(x,u) = \int_0^u g(x,v)dv \le \frac{\epsilon}{p}|u|^p$$

if $|u| < \delta$. Also by (2) we obtain

$$G(x,u) \le C(\epsilon)|u|^s$$

for some constant $C(\epsilon)$, if $|u| \ge \delta$. Thus

$$G(x,u) \le \epsilon|u|^p + C(\epsilon)|u|^s$$

for all $u \in R$ and almost every $x \in \Omega$. It follows that

$$\begin{aligned} E(u) &\ge \frac{1}{p}\int_\Omega \alpha(-\nabla u)^p dx - \epsilon\int_\Omega |u|^p dx - C(\epsilon)\int_\Omega |u|^s dx \\ &\ge \frac{1}{p}\gamma||u||^p_{H_o^{1,p}} - \frac{\Gamma\epsilon}{\lambda_1}||u||^p_{H_o^{1,p}} - C(\epsilon)||u||^s_{H_o^{1,p}} \\ &= \left(\frac{\gamma}{p} - \frac{\Gamma\epsilon}{\lambda_1} - C(\epsilon)||u||^{s-p}_{H_o^{1,p}}\right)||u||^p_{H_o^{1,p}} \ge \alpha > 0 \end{aligned}$$

if $||u||_{H_o^{1,p}} = \rho$ is sufficiently small. Here, we have used the fact that

$$\lambda_1 \leq \frac{\int \alpha(-\nabla u)^p dx}{\int |u|^p dx} \leq \frac{\Gamma ||u||^p_{H_o^{1,p}}}{||u||^p_{L^p}}$$

and the fact that $H_o^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$.

Observe that $E(0) = 0$. Finally, (3) can be restated in the form

$$u|u|^t \frac{d}{du}(|u|^{-t}G(x,u)) \geq 0 \qquad \text{for} \quad |u| \geq R_o$$

Upon integration, we have

$$G(x,u) \geq \gamma_o(x)|u|^t$$

with $\gamma_o(x) = R_o^{-t}\min\{G(x,R_o), G(x,-R_o)\} > 0$, if $|u| \geq R_0$. Hence,

$$\begin{aligned} E(\lambda u) &= \frac{\lambda^p}{p}\int_\Omega \alpha(-\nabla u)^p dx - \int_\Omega G(x,\lambda u)dx \\ &\leq \frac{\Gamma}{p}\lambda^p ||u||^p_{H_o^{1,p}} - \lambda^t \int_{x\in\Omega, |u|\geq R_o} \gamma_o(x)|u|^t dx \\ &\quad + \mathcal{L}^n(\Omega)\ \inf_{x\in\Omega,|v|\leq R_o}|G(x,v)| \to -\infty \qquad \text{as} \quad \lambda \to \infty. \end{aligned}$$

We may let $u_1 = \lambda u$ for fixed $u \neq 0$ and sufficiently large $\lambda > 0$. Therefore we obtain, from Theorem 3.3 the existence of a nontrivial solution to (3.3)

In order to obtain a solution $u^+ \geq 0$, we may truncate g below $u = 0$, replacing g by

$$g^+(x,u) = \begin{cases} g(x,u) & \text{if} \quad u \geq 0 \\ 0 & \text{if} \quad u \leq 0 \end{cases}$$

with primitive $G^+(x,u) = \int_o^u g^+(x,v)dv$. Note that (1), (2) remain valid for g^+ while (3) will hold for $u > R_o$, almost everywhere in Ω. Moreover for $u \leq 0$ all terms in (3) vanish. Denote

$$E^+(u) = \frac{1}{p}\int_\Omega \alpha(-\nabla u)^p dx - \int_\Omega G^+(x,u)dx.$$

Our previous reasoning then yields a nontrivial critical point u^+ of E^+ which weakly solves the equation

$$\operatorname{div} A(-\nabla u^+) = g^+(x, u^+) \qquad \text{in} \quad \Omega.$$

Rewriting it as

$$\operatorname{div} A(-\nabla u^+) + N(g^+(x, u^+)) = P(g^+(a, u^+))$$

where $P(a) = \max(a, 0)$ and $N(a) = \max(-a, 0)$, we have

$$-\int A(-\nabla u^+) \cdot \nabla\varphi + \int N(g^+(x, u^+))\varphi \geq 0$$

for all $\varphi \in H_0^{1,p}(\Omega)$ with $\varphi \geq 0$. Substituting $\varphi = N(u^+)$, we deduce that

$$\int_{\{u^+<0\}} A(-\nabla u^+) \cdot (-\nabla u^+) - \int_{\{u^+<0\}} N(g^+(x, u^+))(-u^+) \leq 0$$

while the left hand side is not less than a positive constant multiple of $\int_{\{u^+<0\}} |-\nabla u^+|^p$. Therefore $N(u^+) = 0$, that is, $u^+ \geq 0$ a.e. in Ω. Hence we conclude that u^+ is a weak solution of the original equation (3.3). Similarly, we can show that $u^- \leq 0$ is also a weak solution of (3.3) as desired.

REMARK. We note that if $u \in H_0^{1,p}(\Omega)$ weakly solves (3.3) with g satisfying the hypotheses of Theorem 3.4, then u weakly solves the equation

$$\operatorname{div} A(-\nabla u) = a(x)(1 + |u|^{p-1})$$

with

$$a(x) = \frac{g(x, u(x))}{1 + |u|^{p-1}} \in L^{\frac{n}{p}}(\Omega)$$

We can then deduce that $u \in L^q$ for any $q < \infty$, therefore $u \in C^{1,\alpha}$ with some $\alpha > 0$; see [2], [6], [10].

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