CAUCHY PROBLEMS FOR A
PARTIAL DIFFERENTIAL EQUATION
IN WHITE NOISE ANALYSIS

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1. Introduction

The Gross Laplacian $\Delta_G$ was introduced by Gross for a function defined on an abstract Wiener space $(H, B)$ [1,7]. Suppose $\varphi$ is a twice $H$-differentiable function defined on $B$ such that $\varphi''(x)$ is a trace class operator of $H$ for every $x \in B$. Then the Gross Laplacian $\Delta_G \varphi$ of $\varphi$ is defined by

$$\Delta_G \varphi(x) = \text{trace}_H \varphi''(x).$$

If in addition $\varphi'(x) \in B^*$, then the number operator $N \varphi$ of $\varphi$ is defined by

$$N \varphi(x) = -\text{trace}_H \varphi''(x) + \langle x, \varphi'(x) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the $B - B^*$ pairing.

In [1], Gross studied the solution of the heat equation associated with the Gross Laplacian $\Delta_G$ on $(H, B)$:

$$\frac{\partial u}{\partial t}(x, t) = \frac{1}{2} \Delta_G u(x, t), \quad u(x, 0) = f(x).$$

In [15], Piech studied the solution of the heat equation associated with the number operator $N$ on $(H, B)$:

$$\frac{\partial u}{\partial t}(x, t) = -Nu(x, t), \quad u(x, 0) = f(x).$$

We note that in white noise space $(S'(\mathbb{R}), E, \mu)$, the white noise measure $\mu$ is supported in the space $S'_p$ (see section 2) for any $p > \frac{1}{2}$.

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Thus \((L^2(\mathbb{R}), S'_p(\mathbb{R}))\) is an abstract Wiener space. Therefore we can define \(\Delta_G \varphi\) and \(N\varphi\) for functions \(\varphi\) defined on \(S'_p(\mathbb{R})\).

In [9,10], Kuo has studied the heat equation associated with the Gross Laplacian \(\Delta_G\) in white noise analysis setting. In [5], Kang has studied the heat equation associated with the number operator \(N\) in white noise analysis setting. In this paper we will investigate the existence of a solution of the Cauchy problem associated with the operator \(\Delta_G + N\) in white noise analysis setting.

2. Preliminaries

Let \(S(\mathbb{R})\) be the Schwartz space of real valued rapidly decreasing smooth functions on \(\mathbb{R}\). The dual space \(S'(\mathbb{R})\) of \(S(\mathbb{R})\) is the space of tempered distributions on \(\mathbb{R}\). Then \(S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R})\) is a Gel'fand triple.

Let \(A = -\frac{d^2}{dx^2} + t^2 + 1\). Then \(A\) is the densely defined self-adjoint on \(L^2(\mathbb{R})\) and \(Ae_n = (2n + 2)e_n\), where \(\{e_n\}\) is an ONB of \(L^2(\mathbb{R})\) defined by

\[
e_n(u) = (-1)^n (\pi^{\frac{1}{2}} 2^n n!)^{-\frac{1}{2}} e^{\frac{-u^2}{2}} [\frac{d^n}{du^n} e^{-u^2}].
\]

For each \(p \geq 0\), define

\[|f|_p = |A^p f|_0, \quad f \in L^2(\mathbb{R}),\]

where \(|\cdot|_0\) is the \(L^2(\mathbb{R})\)-norm. Let \(S_p(\mathbb{R}) = \{f \in L^2(\mathbb{R}) | |f|_p < \infty\}\). Then \(S_p(\mathbb{R})\) is a real separable Hilbert space with the norm \(|\cdot|_p\) and the dual space \(S'_p(\mathbb{R})\) of \(S_p(\mathbb{R})\) is given by \(S_{-p}(\mathbb{R})\). Furthermore, we have \(S_q(\mathbb{R}) \subset S_p(\mathbb{R})\) for \(p < q\) and

\[
S(\mathbb{R}) = \bigcap_{p \geq 0} S_p(\mathbb{R}), \quad S'(\mathbb{R}) = \bigcup_{p \geq 0} S_{-p}(\mathbb{R}).
\]

Since \(S(\mathbb{R})\) is a nuclear space, by the Bochner-Minlos theorem there exists a unique probability measure \(\mu\) on \(\sigma\)-algebra \(\mathcal{B}\) of Borel subsets of \(S'(\mathbb{R})\) such that

\[
\int_{S'(\mathbb{R})} e^{i(x, \xi)} \, d\mu(x) = e^{-\frac{1}{2} |\xi|_2^2}, \quad \xi \in S(\mathbb{R}),
\]
where $(\cdot, \cdot)$ is the $S'(\mathbb{R})-S(\mathbb{R})$ pairing. The triple $(S'(\mathbb{R}), B, \mu)$ is called the white noise space.

Let $(L^2)$ be the Hilbert space of $\mu$-square integrable functions on the white noise space $(S'(\mathbb{R}), B, \mu)$ with norm $||\cdot||_0$. By the Wiener-Itô decomposition theorem [3,14], every $\phi \in (L^2)$ admits a chaos decomposition:

$$\phi = \sum_{n=0}^{\infty} I_n(f_n),$$

where $I_n(f_n)$ denotes a multiple Itô integral of order $n$ with the kernel $f_n \in \widehat{L^2}(\mathbb{R}^n)$ (the symmetric $L^2$-space).

It is well-known (see [17]) that $I_n(f)(x) = \langle \cdot x^\otimes n : f \rangle$, where $\cdot x^\otimes n$ is the Wick ordering. Hence for each $\phi$ in $(L^2)$, we have

$$\phi(x) = \sum_{n=0}^{\infty} \langle \cdot x^\otimes n : f_n \rangle \quad f_n \in \widehat{L^2}(\mathbb{R}^n).$$

Moreover, we have

$$||\phi||_0^2 = \sum_{n=0}^{\infty} n!|f_n|_0^2.$$

The second quantization $\Gamma(A)$ of $A$ is densely defined on $(L^2)$ as follows: for $\phi \in (L^2)$ with $\phi(x) = \sum_{n=0}^{\infty} \langle \cdot x^\otimes n : f_n \rangle$,

$$\Gamma(A)\phi(x) = \sum_{n=0}^{\infty} \langle \cdot x^\otimes n : A^\otimes n f_n \rangle.$$

For $p \geq 0$, define

$$||\phi||_p = ||\Gamma(A)^p \phi||_0$$

and let $(S)_p = \{ \phi \in (L^2); ||\phi||_p < \infty \}$. Then $(S)_p$ is a Hilbert space with the norm $||\cdot||_p$. For $p < 0$, we define $(S)_p$ as the completion of $(L^2)$ with respect to $||\cdot||_p$. Then the dual space $(S)^*_p$ of $(S)_p$ is given by $(S)_{-p}$, and we have

$$(S)_q \subset (S)_p \subset (L^2) \subset (S)_{-p} \subset (S)_{-q},$$

where $q > p \geq 0$. The space $(S)$ of test functions is the projective limit of $\{(S)_p; p \geq 0\}$. The space $(S)^*$ of generalized functions (or Hida
distributions) is the dual space of \((S)\). Thus we have a Gel'fand triple \((S) \subset (L^2) \subset (S)^*\) and will use the symbol \(\langle \cdot, \cdot \rangle\) for the \((S)^*-\langle S\rangle\) pairing.

Let \(G\) be a continuous linear operator from \((S)\) into itself defined by

\[
G\phi(y) = \langle \langle e^{-i\langle \cdot, y \rangle}, \phi \rangle \rangle, \quad y \in E^*
\]

where \(e^{-i\langle \cdot, y \rangle} \in (S)^*\). Then it is known [3,12] that the adjoint \(G^*\) of \(G\) is the Fourier transform \(\mathcal{F}\). For any \(\phi \in (S)\) with \(\phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : f_n \rangle, f_n \in \widehat{S(\mathbb{R}^n)}, G\phi\) has the following chaos decomposition

\[
(2.1) \quad G\phi(y) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : \sum_{m=0}^{\infty} \frac{(n + 2m)!}{n!m!2^m} (-i)^{n+2m} \tau^{\otimes m} \otimes f_{n+2m} \rangle.
\]

And it can be shown that \(G^3 = G^{-1}\), and for any \(\phi \in (S)\) with \(\phi(y) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : f_n \rangle, f_n \in \widehat{S(\mathbb{R}^n)}\), \(G^{-1}\phi\) is given by

\[
(2.2) \quad G^{-1}\phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : \sum_{m=0}^{\infty} \frac{(n + 2m)!}{n!m!2^m} (i)^{n+2m} \tau^{\otimes m} \otimes f_{n+2m} \rangle,
\]

where \(\tau^{\otimes m} \otimes f_{n+2m}(u) = \int_{\mathbb{R}^m} f(t_1, t_2, t_2, \ldots, t_m, t_m, u) dt_1 \cdots dt_m, u \in \mathbb{R}^n\). It also can be shown by using the duality that for any \(\Phi \in (S)^*\) with \(\Phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : F_n \rangle, F_n \in S'(\mathbb{R}^n), \mathcal{F}\Phi\) and \(\mathcal{F}^{-1}\Phi\) are given by

\[
(2.3) \quad \mathcal{F}\Phi(y) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-i)^n}{m!2^m} F_{n+2m} \otimes \tau^{\otimes m} \rangle,
\]

and

\[
(2.4) \quad \mathcal{F}^{-1}\Phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{i^n}{m!2^m} F_{n+2m} \otimes \tau^{\otimes m} \rangle.
\]

It is known [3] that for any \(\phi \in (S)\) with \(\phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : f_n \rangle, f_n \in \widehat{S(\mathbb{R}^n)}, \Delta_G\phi\) and \(N\phi\) are given by

\[
\Delta_G\phi(x) = \sum_{n=0}^{\infty} (n + 2)(n + 1) \langle x^{\otimes n} : \tau^{\otimes 2} f_{n+2} \rangle.
\]
and

\[ N\phi(x) = \sum_{n=0}^{\infty} n\langle x^{\otimes n} : f_n \rangle. \]

3. Cauchy problem associated with the operator \( \Delta_G + N \)

In this section we use the \( \mathcal{G} \)-and Fourier transforms to investigate the existence of a solution of the following Cauchy problems:

\[
(3.1) \quad \frac{\partial}{\partial t} u(x, t) = -(\Delta_G + N)u(x, t), \quad u(x, 0) = \phi(x),
\]

where \( \phi \in (S), \) and

\[
(3.2) \quad \frac{\partial}{\partial t} u(x, t) = -(\Delta_G^* + N)u(x, t), \quad u(x, 0) = \Phi(x),
\]

where \( \Phi \in (S)^* \) and \( \Delta_G^* \) is the adjoint of \( \Delta_G. \)

**Theorem 3.1.** For any \( \phi \in (S), \) we have

(i) \( \mathcal{G}(\Delta_G\phi) = -\Delta_G(\mathcal{G}\phi) \)

(ii) \( \mathcal{G}(N\phi) = (\Delta_G + N)\mathcal{G}\phi. \)

**Proof.** (i) Let \( \phi \in (S) \) be given by \( \phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n} : f_n \rangle. \) Then we have

\[
\Delta_G\phi(x) = \sum_{n=0}^{\infty} (n+1)(n+2)\langle x^{\otimes n} : \tau^{\otimes 2} f_{n+2} \rangle.
\]

Hence by (2.1), we obtain that

\[
\mathcal{G}(\Delta_G\phi)(y) = \sum_{n=0}^{\infty} \langle y^{\otimes n} : \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!2^m} (-i)^{n+2m} \tau^{\otimes m} \hat{\otimes} 2m \rangle 
\times ((n+2m+1)(n+2m+2)\tau^{\otimes 2} f_{n+2m+2}))
\]

\[
= -\sum_{n=0}^{\infty} (n+1)(n+2)\langle y^{\otimes n} : \sum_{m=0}^{\infty} \frac{(n+2m+2)!}{(n+2)!m!2^m} (-i)^{n+2m+2} \tau^{\otimes m+1} \hat{\otimes} 2(m+1) f_{n+2m+2} \rangle
\]

\[
= -\Delta_G(\mathcal{G}\phi)(x).
\]
Thus we have $G(\Delta_G \phi) = -\Delta_G (G\phi)$.

(ii) For any exponential vector $\phi_\xi(x) = \sum_{n=0}^\infty \langle : x^\otimes n : , \frac{\xi^n}{n!} \rangle$, $\xi \in E$, we have $N \phi_\xi(x) = \sum_{n=0}^\infty n \langle : x^\otimes n : , \frac{\xi^n}{n!} \rangle$. Hence we have

$$G(N \phi_\xi)(x) = \frac{\xi^{n+2m}}{(n+2m)!} \sum_{m=0}^\infty \frac{(n+2m)!}{n! m! 2^m} (-i)^{n+2m} \otimes m \hat{\otimes} 2m \{ \langle x^\otimes n : , \frac{\xi^{n+2m}}{(n+2m)!} \rangle \}$$

$$= \frac{\xi^{n+2m}}{n! m! 2^m} \sum_{m=0}^\infty \langle x^\otimes n : , \frac{\xi^{n+2m}}{n! m! 2^m} \rangle$$

$$+ \frac{\xi^{n+2m}}{n! m! 2^m} \sum_{m=0}^\infty \langle x^\otimes n : , \frac{\xi^{n+2m}}{n! m! 2^m} \rangle$$

$$= \exp\{ -\frac{1}{2} |\xi|^2 \} \sum_{n=0}^\infty \langle x^\otimes n : , \frac{(-i\xi)^{n}}{n!} \rangle$$

By noting that

$$\Delta_G (G\phi_\xi)(x) = -|\xi|^2 \exp\{ -\frac{1}{2} |\xi|^2 \} \sum_{n=0}^\infty \langle x^\otimes n : , \frac{(-i\xi)^{n}}{n!} \rangle$$

and

$$N(G\phi_\xi)(x) = \exp\{ -\frac{1}{2} |\xi|^2 \} \sum_{n=0}^\infty \langle x^\otimes n : , \frac{(-i\xi)^{n}}{n!} \rangle$$

we have $G(N \phi_\xi) = (\Delta_G + N)G \phi_\xi$. But since $\{ \phi_\xi, \xi \in S(\mathbb{R}) \}$ spans a dense subspace of $(S)$, it follows from the continuity of $\Delta_G$, $N$ and the $G$-transform that for any $\phi \in (S)$, we have $G(N \phi) = (\Delta_G + N)G \phi$.

**Theorem 3.2.** For any Hida-distribution $\Phi$, we have

(i) $F(\Delta_G^* \Phi) = -\Delta_G^* F \Phi$

(ii) $F(N \Phi) = N F \Phi + \Delta_G^* F \Phi$.

**Proof.** By the duality of Theorem 3.1, for any $\phi \in (S)$ we have

(i) $\langle \langle \Delta_G^* F \Phi, \phi \rangle \rangle = \langle \langle \Phi, G(\Delta_G \phi) \rangle \rangle = \langle \langle \Phi, -\Delta_G G \phi \rangle \rangle = \langle \langle -\Delta_G^* \Phi, G \phi \rangle \rangle$

$$= \langle \langle -F(\Delta_G^* \Phi), \phi \rangle \rangle.$$


(ii) \( \langle \mathcal{F}(N\Phi), \phi \rangle = \langle (N\Phi, G\phi) \rangle = \langle \Phi, NG\phi \rangle = \langle \Phi, G(N\phi) - \Delta G\phi \rangle \\
= \langle \Phi, G(N\phi) + G(\Delta G\phi) \rangle = \langle \mathcal{F}\Phi, N\phi + \Delta G\phi \rangle \\
= \langle (N\mathcal{F}\Phi + \Delta^* G\mathcal{F}\Phi, \phi) \rangle. \)

This completes the proof.

**Theorem 3.3.** For any \( \phi \in (S) \), \( \sigma_{e^{-t}} \phi \in (S) \) satisfies the Cauchy problem (3.1), where \( \sigma_\lambda \phi(x) = \phi(\lambda x) \).

**Proof.** Let \( v(y, t) = \mathcal{G}u(y, t) \). Then by Theorem 3.1, \( v(y, t) \) satisfies the following equation:

\[
\frac{\partial}{\partial t} v(y, t) = -N v(y, t), \quad v(y, 0) = \mathcal{G}\phi(y).
\]

It is well-known [5] that \( q_t(\mathcal{G}\phi)(y) = \int_{S^\infty} \mathcal{G}\phi(e^{-t}y + \sqrt{1 - e^{-2t}}x)d\mu(x) \) satisfies the equation (3.3). Hence \( \mathcal{G}^{-1}(q_t(\mathcal{G}\phi))(x) \) satisfies the equation (3.1). Note that for any \( \xi \in E_C \),

\[
\mathcal{G}\phi_\xi = e^{-\frac{i}{2}(\xi, \xi)} \phi_{-i\xi}, \quad q_t(\mathcal{G}\phi_\xi) = e^{-\frac{i}{2}(\xi, \xi)} \phi_{-i\xi},
\]

and

\[
\sigma_\lambda \phi_\xi(x) = \phi_\xi(\lambda x) = e^{\frac{i}{2}(\lambda^2 - 1)(\xi, \xi)} \phi_{\lambda \xi}.
\]

Thus by (2.2), (3.4) and (3.5), we obtain that

\[
\mathcal{G}^{-1}(q_t(\mathcal{G}\phi_\xi)) = e^{\frac{i}{2}(e^{-2t} - 1)(\xi, \xi)} \phi_{t-i\xi}
\]

\[= \sigma_{e^{-t}} \phi_\xi, \quad \xi \in E_C. \]

But since \( \{\phi_\xi : \xi \in E_C\} \) spans a dense subspace of \( (S) \), it follows from the continuity of \( \sigma_\lambda \) [16] that for any \( \phi \in (S) \), we have \( \mathcal{G}^{-1}(q_t(\mathcal{G}\phi)) = \sigma_{e^{-t}} \phi \). Hence we complete the proof.

**Remark.** It is well-known that \( \Delta_G + N = \int_{\mathbb{R}} (\partial_t + \partial_t^*) \partial_t dt = \int_{\mathbb{R}} x(t) \partial_t dt \). Thus \( \Delta_G + N \) is an infinite dimensional analogue of a first order differential operator \( \sum_{j=1}^n x_j \frac{\partial}{\partial x_j} \) on \( \mathbb{R}^n \). Hence the solution \( u(t, x) = \phi(e^{-t}x) \) of the Cauchy problem (3.1) is indeed the solution of a first order differential equation with variable coefficients in white noise analysis setting.
EXAMPLE. Let $B$ be a bounded operator from $\mathcal{S}'(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. Consider the following Cauchy problem

\begin{equation}
\frac{\partial}{\partial t}u(x, t) = -(\Delta_G + N)u(x, t), \quad u(x, 0) = \langle x, Bx \rangle.
\end{equation}

Since $(L^2(\mathbb{R}), S_p'(\mathbb{R}))$ is an abstract Wiener space for any $p > \frac{1}{2}$, $B$ is a trace class operator of $L^2(\mathbb{R})$ and $\mu$ has the support contained in $S_1'(\mathbb{R}) = S_{-1}(\mathbb{R})$. Hence

$$G(\langle \cdot, B \cdot \rangle)(x) = \text{trace}_{L^2(\mathbb{R})} B - \langle x, Bx \rangle.$$ 

Therefore, we have

$$v(x, t) = \text{trace}_{L^2(\mathbb{R})} B$$

$$- \int_{S_{-1}(\mathbb{R})} \langle e^{-t}x + \sqrt{1 - e^{-2t}} y, B(e^{-t}x + \sqrt{1 - e^{-2t}} y) \rangle d\mu(y)$$

$$= \text{trace}_{L^2(\mathbb{R})} B - (1 - e^{-2t}) \text{trace}_{L^2(\mathbb{R})} B - e^{-2t} \langle x, Bx \rangle$$

$$= e^{-2t} \text{trace}_{L^2(\mathbb{R})} B - e^{-2t} \langle x, Bx \rangle.$$ 

Hence $u(x, t) = G^{-1}(v(\cdot, t))(x)$ satisfies the equation (3.6) and is given by

$$u(x, t) = G^{-1}(v(\cdot, t))(x)$$

$$= e^{-2t} \text{trace}_{L^2(\mathbb{R})} B - e^{-2t} \int_{S_{-1}(\mathbb{R})} \langle y + ix, B(y + ix) \rangle d\mu(y)$$

$$= e^{-2t} \langle x, Bx \rangle.$$ 

THEOREM 3.4. For any $\Phi \in (\mathcal{S})^*$ with $\Phi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, : F_n \rangle$, the Hida distribution

$$u(x, t) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, \sum_{m=0}^{[\frac{n}{2}]} \frac{1}{m!2^m} \left( \sum_{l=0}^{n} \binom{m}{l} (-1)^l e^{-(n-2l)t} \right) F_{n-2m} \hat{\tau}^{\otimes l} \rangle$$

satisfies the equation (3.2).

Proof. By taking the Fourier transform in the equation (3.2), we have

\begin{equation}
\frac{\partial}{\partial t} v(y, t) = -N v(y, t), \quad v(y, 0) = \mathcal{F} \Phi(y).
\end{equation}
where \( v(y, t) = \mathcal{F}(u(\cdot, t))(y) \). Since by (2.3)

\[
\mathcal{F}\Phi(y) = \sum_{n=0}^{\infty} \langle \cdot, y^n \rangle \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-i)^n}{m!2^m} F_{n-2m} \wedge \tau^m,
\]

we can easily check that

\[
v(y, t) = \sum_{n=0}^{\infty} e^{-nt} \langle \cdot, y^n \rangle \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-i)^n}{m!2^m} F_{n-2m} \wedge \tau^m
\]

satisfies the equation (3.7). By taking the inverse Fourier transform, we have

\[
u(x, t) = \mathcal{F}^{-1} v(\cdot, t)(x) = \sum_{n=0}^{\infty} \langle \cdot, x^n \rangle \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{i^n}{m!2^m} G_{n-2m} \wedge \tau^m,
\]

where \( G_n = e^{-nt} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-i)^n}{m!2^m} F_{n-2m} \wedge \tau^m \). And we note that

\[
\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{i^n}{m!2^m} G_{n-2m} \wedge \tau^m
\]

\[
= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{i^n}{m!2^m} (e^{-(n-2m)t} \sum_{l=0}^{\lfloor \frac{n-2m}{2} \rfloor} (-i)^{n-2m} \frac{2^l}{l!2^l} F_{n-2m-2l} \wedge \tau^l \wedge \tau^m)
\]

\[
= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{l=0}^{\lfloor \frac{n-2m}{2} \rfloor} \frac{(-1)^m}{l!m!2^{m+l}} (e^{-(n-2m)t} F_{n-2m-2l} \wedge \tau^l \wedge \tau^m)
\]

\[
= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!2^k} \sum_{s=0}^{k} \binom{k}{s} (-1)^s e^{-(n-2s)t} F_{n-2k} \wedge \tau^s \wedge \tau^k.
\]

Hence

\[
u(x, t) = \sum_{n=0}^{\infty} \langle \cdot, x^n \rangle \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{m!2^m} \left( \sum_{l=0}^{m} \binom{m}{l} (-1)^l e^{-(n-2l)t} F_{n-2m} \wedge \tau^l \right)
\]

satisfies the equation (3.2).
References


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