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THE EXISTENCE OF PRODUCT BROWNIAN PROCESSES

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1. Introduction

Many authors have studied multiple stochastic integrals in pursuit of the existence of product processes in terms of multiple integrals. But there has not been much research into the structure of the product processes themselves. In this direction, a study which gives emphasis on sample path continuity and boundedness properties was initiated in Pyke[9]. For details of problem set-ups and necessary notations, see [9]. Recently the weak limits of U-processes are shown to be chaos processes, which is a product of the same Brownian measures, see [2] and [7]. A process $Z = \{Z(f) : f \in \mathcal{F}\}$ is called a *Brownian process*, where $(\mathbf{S}, \mathcal{S}, \mu)$ is a measure space and $\mathcal{F} \subset L^2(\mathbf{S}, \mu)$, if Z is a Gaussian process with $EZ(f) = 0$ and

$$Cov(Z(f), Z(g)) = \int fg d\mu, \qquad f, g \in \mathcal{F}.$$

Let $(Z_1, \mathcal{F}_1)$ and $(Z_2, \mathcal{F}_2)$ represent two Brownian processes with $\mathcal{F}_i \subset L^2(\mathbf{S}_i, \mu_i)$, $i = 1, 2$. Clearly we can define, by linearity, $Z := Z_1 \times Z_2$ on the field generated by $\mathcal{F}_1 \times \mathcal{F}_2$ as, for $f_i \in \mathcal{F}_i, i = 1, 2$, $Z(f_1 \otimes f_2) = Z_1(f_1)Z_2(f_2)$, where $\mathcal{F}_1 \times \mathcal{F}_2 = \{f_1 \otimes f_2 : f_i \in \mathcal{F}_i\}$ and $f_1 \otimes f_2(\mathbf{x}, \mathbf{y}) = f_1(\mathbf{x})f_2(\mathbf{y})$ with $\mathbf{x} \in \mathbf{S}_1$ and $\mathbf{y} \in \mathbf{S}_2$. We will refer $\mathcal{F}_1 \times \mathcal{F}_2$ as the set of product rectangle functions following set product analogy.

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In this paper we continue the study of sample path property of product processes. In the study of multiple stochastic integrals as well as weak limits of U-processes, however, only products of the same process are considered. When one concerns oneself with the study of product processes, there is no reason to restrict oneself to the same factor only. In section 2, tail probability bounds for product of Brownian processes are derived. In section 3, as an application, the results of section 2 are used to show the existence of product processes. The regularity on the-sample paths to be considered here is uniform continuity with respect to the L^2-metric. Finally we give some comments for constructions of product processes in section 4.

2. Probability Bounds

Let $(Z_1, \mathcal{F}_1)$ and $(Z_2, \mathcal{F}_2)$ represent two Brownian processes with $\mathcal{F}_i \subset L^2(\mathbf{S}_i, \mu_i)$, $i = 1, 2$. Let $\{f_{1i} : i = 1, 2, \cdots, n\}$ and $\{f_{2j} : j = 1, 2, \cdots, m\}$ be subsets of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, such that, for $i \neq i'$ and $j \neq j'$, $\int f_{1i} f_{1i'} d\mu_1 = \int f_{2j} f_{2j'} d\mu_2 = 0$. Let $f \in L^2(\mu_1 \times \mu_2)$ be of the form

$$(2.1) \qquad f = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} f_{1i} \otimes f_{2j} = \sum_{i=1}^{n} f_{1i} \otimes \left(\sum_{j \in K_i} c_{ij} f_{2j} \right),$$

where $c_{ij} \in \mathbf{R}$ and $K_i := \{j : c_{ij} \neq 0\}$. Then L^2- norm of f, $\|f\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} |c_{ij}|^2 \|f_{1i}\|^2 \|f_{2j}\|^2$. Define $X_i := Z_1(f_{1i})/\|f_{1i}\|$ and $X_j' := Z_2(f_{2j})/\|f_{2j}\|$ (when $\|f_{1i}\| = 0$ or $\|f_{2j}\| = 0$ it is understood X_i or X_j' to be 0 by convention). Then

$$\begin{aligned} Z(f) &= \sum_{i=1}^{n} Z_1(f_{1i}) \left(\sum_{j \in K_i} c_{ij} Z_2(f_{2j}) \right) \\ &= \sum_{i=1}^{n} \|f_{1i}\| X_i \left(\sum_{j \in K_i} c_{ij} \|f_{2j}\| X_j' \right). \end{aligned}$$

We will first condition on the second factor.

For this, define $\sigma(\{Z_2(f_{2j})\})$ to be the σ−field generated by $Z_2(f_{2j})$ with $1 \le j \le m$. By normality

$$P(Z(f) > \eta) = E\{P(Z(f) > \eta | \sigma(\{Z_2(f_{2j})\}))\} = E\{P(\xi > \eta/V|V)\},$$

where ξ is an $N(0,1)$ random variable and V is nonnegative and independent of ξ with distribution given by

$$V^2 = E\{(Z(f))^2 | \sigma(\{Z_2(f_{2j})\})\},$$

the indicated conditional variance of $Z(f)$. In order to have a probability bound of $Z(f)$, we need some information of V^2. For this we will use Hanson-Wright inequality which is the following. Let $\mathbf{A} = (a_{ij}) = (|c_{ij}| \|f_{1i}\| \|f_{2j}\|)_{i,j \in \mathbf{N}}$ and let $\|\mathbf{A}\|$ be the norm of $\mathbf{A}$ considered as an operator norm l_2, the index on the sequences in l_2 taking on the values $1, 2, \cdots$. Let $\Lambda^2(\mathbf{A}) = \sum_{i,j} a_{ij}^2$.

Lemma 2.1. *Let $\{X_i\}_{n=1}^{\infty}$ be a sequence of random variables such that for all i and all $x \ge 0$*

$$P(|X_i| \ge x) \le M \int_x^{\infty} e^{-\gamma t^2} dt,$$

where M and γ are positive constants. Let

$$S_N = \sum_{i=n}^{N} \sum_{j=1}^{N} a_{ij}(X_i X_j - E X_i X_j)$$

be a quadratic form of $\{X_i\}_{n=1}^{\infty}$. Then there exist constants C_1 and C_2 depending on M and γ(but not on the coefficients a_{ij}) such that for every $\varepsilon > 0$,

$$P(S_N \ge \varepsilon) \le \exp(-\min\{C_1\varepsilon/\|\mathbf{A}\|, C_2\varepsilon^2/\Lambda^2\}).$$

Lemma 2.2. *For any $x > 0$ we have*

$$P(V^2 > x) \le \exp\left(-C_1 \frac{x - \|f\|^2}{\|f\|^2}\right) + \exp\left(-C_2 \frac{(x - \|f\|^2)^2}{\|f\|^4}\right).$$

Proof. First notice that

$$V^2 = \sum_{i=1}^{n} \|f_{1i}\|^2 \left(\sum_{j \in K_i} c_{ij} Z_2(f_{2j}) \right)^2 ,$$

which is a quadratic form of $\{Z_2(f_{2j})\}$. Hence it is possible to apply Hanson-Wright inequality to obtain a bound on the tails of V^2. For this we need to identify the matrix of V^2 corresponding to f. Let $\mathbf{K}_i$ denote an $m \times 1$ matrix and $\mathbf{K}$ an $m \times m$ matrix in the following way:

$$\mathbf{K}_i := (|c_{i1}| \|f_{1i}\| \, \|f_{21}\|, |c_{i2}| \|f_{1i}\| \, \|f_{22}\|, \cdots , |c_{im}| \|f_{1i}\| \, \|f_{2m}\|)^t, \quad 1 \le i \le n$$

and

$$\mathbf{K} = \left(\mathbf{K}_1^t, \mathbf{K}_2^t, \cdots , \mathbf{K}_n^t\right),$$

where $\mathbf{K}^t$ denotes the transposed matrix of $\mathbf{K}$. Let $\mathbf{A} = \mathbf{K}^t\mathbf{K}$. Then we have: $V^2 = Y\mathbf{A}^tY$, where $Y = (X_1', X_2', \dots , X_m')$. Also a simple calculation shows that $\Lambda^2(\mathbf{K}) = \|f\|^2$ and $\Lambda^2(\mathbf{A}) \le \|f\|^4$. Next note that since $\|\mathbf{A}\|_2$ is the maximum of the absolute value of the eigenvalues of $\mathbf{A}$ and since the largest eigenvalue of a square matrix B is bounded by the square root of the trace of B^tB,

$$\|\mathbf{A}\|_2 \le (\mathrm{Trace}(\mathbf{A}^t\mathbf{A}))^{1/2} = \Lambda^2(\mathbf{A})^{1/2} \le \|f\|^2.$$

Summing-up, finally we have

$$\begin{aligned} P(V^2 > x) &= P\left(V^2 - \|f\|^2 > x - \|f\|^2\right) \\ &\le \exp\left(-C_1 \frac{x - \|f\|^2}{\|\mathbf{A}\|_2}\right) + \exp\left(-C_2 \frac{(x - \|f\|^2)^2}{\Lambda^2(\mathbf{A})}\right) \\ &\le \exp\left(-C_1 \frac{x - \|f\|^2}{\|f\|^2}\right) + \exp\left(-C_2 \frac{(x - \|f\|^2)^2}{\|f\|^4}\right). \end{aligned}$$

THEOREM 2.3. *Let $f \in L^2(\mu_1 \times \mu_2)$ be of the form in (2.1). Then, for $\eta > 0$,*

$$P(Z(f) > \eta) \leq K_1 \exp(-K_2 \eta / \|f\|),$$

where K_1 and K_2 are constants not depending on f.

Proof. By lemma 2.2 and the standard bound for the tail of a normal random variable after conditioning, we have

$$\begin{aligned} P(Z(f) > \eta) &\leq E\{(V/\eta)\exp(-\eta^2/2V^2) \wedge 1\} \\ &\leq E((V/\eta)\exp(-\eta^2/2V^2) \wedge 1 \cdot 1_{[V^2 > x]}) \\ &\qquad + E((V/\eta)\exp(-\eta^2/2V^2) \wedge 1 \cdot 1_{[V^2 \leq x]}) \\ &\leq \exp\left(-C_1 \frac{x - \|f\|^2}{\|f\|^2}\right) + \exp\left(-C_2 \frac{(x - \|f\|^2)^2}{\|f\|^4}\right) \\ &\qquad + \sqrt{x}/\eta \exp(-\eta^2/2x). \end{aligned}$$

Since $\eta > 2\|f\|$, we can assume that $x > 2\|f\|^2$. Then

$$P(Z(f) > \eta) \leq 2\exp(-c_1 x / 2\|f\|^2) + \sqrt{x}/\eta \exp(-\eta^2/2x).$$

Making the two exponents equal, we have $x = \eta\|f\|$ and we finish the proof.

Next we will consider a product of the same Brownian process. Let $(Z_1, \mathcal{F}_1)$ denote a Brownian process indexed by $\mathcal{F}_1$ and $\mathcal{F}_1 \subset L^2(\mu)$. Let $f \in L^2(\mu \times \mu)$ be such that $f = \sum_{i=1}^m \sum_{j=1}^m c_{ij} f_i \otimes f_j$, where c_{ij} are real constants, and $\{f_i : i = 1, 2, \cdots, m\}$ a subset of $(Z_1, \mathcal{F}_1)$ such that $\int f_i f_j d\mu = 0$, for $i \neq j$. For f,

$$Z_1 \times Z_1(f) = \sum_{i=1}^m \|f_i\| X_i \left(\sum_{j \in K_i} c_{ij} \|f_j\| X_j \right). \tag{2.2}$$

where $X_i := Z_1(f_i)/\|f_i\|$, for $i = 1, 2, \cdots, m$, and $\{X_i : 1 \leq i \leq m\}$ are independent $N(0,1)$ random variables. Denote $\Delta = \sum_{i=1}^m c_{ii}\|f_i\|^2$. Then we have the following theorem.

THEOREM 2.4. *Let $f \in L^2(\mu \times \mu)$ and $\mathbf{A} = (c_{ij}\|f_i\| \cdot \|f_j\|)$ be the corresponding matrix of f. Then, for all $\eta > 4\Delta + \|f\|$ and for some constants K_1 and K_2 independent of $\mathbf{A}$,*

$$P(Z_1 \times Z_1(f) > \eta) \leq K_1 \exp\left(-K_2 \frac{(\eta - 4\Delta)}{2\|f\|}\right).$$

Proof. First, assume that $\mathbf{A}$ is symmetric. Lemma 2.1 gives the following.

$$P(Z_1 \times Z_1(f) > \eta) \leq \exp(-K_1(\eta - \Delta)^2/\|f\|^2) + \exp(-K_2(\eta - \Delta)/\|f\|),$$

for all $\eta > \Delta$ and for some constants K_1 and K_2. Next consider the case when $\mathbf{A}$ is upper (or lower) triangular matrix. Let f' be the function corresponding to the transpose $\mathbf{A}^t$ of $\mathbf{A}$ and let $\mathbf{A} + \mathbf{A}^t$ be the corresponding matrix to the function $f + f'$. Then $\mathbf{A} + \mathbf{A}^t$ is symmetric and we have the following equality.

$$Z_1 \times Z_1(f + f') = Z_1 \times Z_1(f) + Z_1 \times Z_1(f') \quad \text{in law}$$

$$Z_1 \times Z_1(f) = Z_1 \times Z_1(f') \quad \text{in law.}$$

Thus, we have, from $\|f + f'\|^2 \leq 4\|f\|^2$,

$$\begin{aligned} P(Z_1 \times Z_1(f) > \eta) \leq\ & \exp(-K_1(\eta - 2\Delta)^2/\|f\|^2) \\ & + \exp(-K_2(\eta - 2\Delta)/\|f\|). \end{aligned}$$

for all $\eta > 2\Delta$ and for some constants K_1 and K_2. Lastly consider a general case. Write $\mathbf{A} = \mathbf{A}^l + \mathbf{A}^u$, where $\mathbf{A}^l$ and$\mathbf{A}^u$ are the lower and the upper triangular matrix of $\mathbf{A}$ in the obvious sense. Let f^l and f^u be the functions corresponding to A^l and A^u respectively, then $f = f^l + f^u$ and

$$Z_1 \times Z_1(f) = Z_1 \times Z_1(f^l) + Z_1 \times Z_1(f^u) \quad \text{in law.}$$

By applying the second case to the two terms of the right hand side, we have the theorem.

3. Existence of Product Brownian Processes

Let $(Z_1, \mathcal{F}_1)$ and $(Z_2, \mathcal{F}_2)$ represent two independent Brownian processes or $(Z_1, \mathcal{F}_1)$ and $(Z_2, \mathcal{F}_2)$ be the same Brownian process with $\mathcal{F}_i \subset L^2(\mathbf{S}_i, \mu_i)$, $i = 1, 2$. Set $Z = Z_1 \times Z_2$. To prove the existence theorem, we need to restrict our index family in terms of metric entropy. Let $(\mathbf{S}, \rho)$ denote a separable pseudo metric space. For $\delta > 0$, let $\mathbf{S}(\delta)$ denote a finite δ-net for $\mathbf{S}$ with respect to the metric ρ.

DEFINITION 3.1. For $\delta > 0$, let

$$\nu(\delta, \mathbf{S}, \rho) = \min\{\text{card}\mathbf{S}(\delta) : \mathbf{S}(\delta) \text{is a } \delta\text{-net for } \mathbf{S} \text{ with respect to the metric } \rho\}.$$

Then the metric entropy of $\mathbf{S}$ with respect to ρ is defined by

$$H(\delta, \mathbf{S}, \rho) = \log \nu(\delta, \mathbf{S}, \rho).$$

It is well known that $\int_0^1 H(u, \mathbf{S}, \rho)^{\frac{1}{2}} du < \infty$ endures the a.s. sample path continuity of Brownian process $(Z, \mathbf{S})$. In this section, the metric space is $(\mathcal{F}, d_{L^2})$, where $\mathcal{F}_1 \times \mathcal{F}_2 \subset \mathcal{F} \subset L^2(\mu_1 \times \mu_2)$ and $d_{L^2}(f, g) = \|f - g\|$.

Assume that $\mathcal{F}$ satisfies the following:

ASSUMPTION I.

(1) $\int_0^1 H(u, \mathcal{F}, d_{L^2}) du < \infty$.
(2) $H(u) \geq \log(\log u)^2$.

ASSUMPTION II. *There exists a geometrically decreasing sequence* $\{\delta_n\}_{n \geq 0}$ *such that for all* n,

(1) *any member of the* δ_n*-net of* $\mathcal{F}$ *is a rectilinear function,*
(2) *the difference of any two members of the* δ_n*-net of* $\mathcal{F}$ *is rectilinear,*
(3) *the difference of any two members of the* δ_n*-net and* δ_{n+1}*-net of* $\mathcal{F}$ *is also rectilinear and*
(4) *any member* f *of the net* $\mathcal{F}(\delta_n)$ *have* $\Delta f = 0$.

REMARK. The condition (4) of assumption II will be used in the proof of product of the same Brownian process. If the case, we have the same type of probability bounds in theorem 2.3 and 2.4. Also this restriction appears in the study of multiple Wiener integrals, see [3] and [10].

THEOREM 3.1. *Let $\{(Y_n, \mathbf{S})\}$ be a sequence of random elements with $\mathbf{S}$ a complete separable metric space and suppose that there exist a sequence of positive numbers ε_n which satisfies the following:*

(1) $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.
(2) $\sum_{n=1}^{\infty} P(\|Y_n - Y_{n-1}\| \geq \varepsilon_n) < \infty$.

Then there is a random element Y such that $Y = \lim_{n\to\infty} Y_n$.

Proof. Let $A_n = [\|Y_n - Y_{n-1}\| \geq \varepsilon_n]$ and $A_\infty = \limsup_{n\to\infty} A_n$. From (2) and the Borel-Cantelli lemma $P(A_\infty) = 0$, so that $P(\bar{A}_\infty) = 1$. The relation $\omega \in \bar{A}_\infty$ means that ω belongs only to finitely many A_n; therefore there exists an $N_1 = N_1(\omega)$ such that $\|Y_n - Y_{n-1}\| < \varepsilon_n$ for all $n \geq N_1(\omega)$. Since

$$\|Y_{n+k} - Y_n\| \leq \|Y_{n+1} - Y_n\| + \|Y_{n+2} - Y_{n+1}\| + \cdots + \|Y_{n+k} - Y_{n+k-1}\|,$$

we have

$$\|Y_{n+k} - Y_n\| \leq \varepsilon_{n+1} + \varepsilon_{n+2} + \cdots + \varepsilon_{n+k}$$

when $n \geq N_1(\omega)$. Let $\varepsilon > 0$ be given. Since the series $\sum_{n=1}^{\infty} \varepsilon_n$ converges, there is an $N_2 = N_2(\varepsilon)$ such that $\sum_{k=n+1}^{\infty} \varepsilon_k < \varepsilon$ for $n \geq N_2$. Let $N_0 = N_0(\omega, \varepsilon) = \max[N_1(\omega), N_2(\varepsilon)]$, then it follows that $\|Y_{n+k} - Y_n\| \leq \varepsilon$ for $n \geq N_0$. This implies that the sequence $\{Y_n(\omega)\}$ converges for all $\omega \in \bar{A}_\infty$.

COROLLARY 3.2. *Let $\{X_n\}$ be a sequence of random variables and suppose that there esists a sequence of positive numbers ε_n such that*

(1) $\sum_{n=1}^{\infty} \varepsilon_n < \infty$
(2) $\sum_{n=1}^{\infty} P(|X_n| \geq \varepsilon_n) < \infty$.

Then the infinite series $\sum_{n=1}^{\infty} X_n$ is almost surely convergent.

THEOREM 3.3. *Assume that $\mathcal{F}$ satisfies Assumption I and II. Then for any $f \in \mathcal{F}$, $Z(f)$ is defined uniquely as a limit of a sequence $\{Z(f_n)\}$, where $\{f_n\}$ denote an approximating sequence of f.*

Proof. Assume f is itself rectilinear, then, as we mentioned above, $Z(f)$ can be defined easily. Assume that f is any member of $\mathcal{F}$, let f_δ denote a member of the δ-net $\mathcal{F}(\delta)$ satisfying $\|f - f_\delta\| < \delta$. Note that by Assumption II, f_δ is rectilinear. Let η_n, $n \geq 0$ be a sequence of positive real numbers whose specific values will be determined later. Let $\{\delta_n\}_{n\geq 0}$ be the same as in Assumption I. Write f_n for f_{δ_n} and consider

$$\varepsilon_n := P\left[|Z(f_n) - Z(f_{n+1})| > \eta_n\right].$$

If we can show $\sum_{n=1}^{\infty} \eta_n < \infty$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$, then by theorem 3.1, $Z(f_n)$ converges almost surely to a limit. Since $|Z(f_n) - Z(f_{n+1})| = |Z(f_n - f_{n+1})|$ we have $P(|Z(f_n) - Z(f_{n+1})| > \eta_n) \leq \max_{\mathcal{F}} P(|Z(f)| > \eta_n)$, where the maximum on the right hand side is over all f in $\mathcal{F}$ with $\|f\| \leq 2\delta_n$ that are rectilinear functions. By theorem 2.3 and 2.4,

$$\max P(|Z(f)| > \eta_n) \leq K_1 \exp\left(-K_2\eta_n/2\delta_n\right)$$

where K_1 and K_2 are constants.

Now set $\delta_n = \delta_0\beta^n$ for some $\delta_0 \in (0,1)$ and $\beta \in (0,1)$. And let $\eta_n = K_3\beta^n \log n$ with $K_2K_3 = 4\delta_0$. Then $\sum_{n=1}^{\infty} \eta_n < \infty$ and since $\exp(-K_2\eta_n/2\delta_n) = 1/n^2$, we have $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Notice that we did not use Assumption II yet, but to prove the independence of the choice of approximating sequences we need that assumption. For this let $\{g_n\}$ be another such approximating sequence of f. Consider

$$\varepsilon_n := P\left[|Z(f_n) - Z(g_n)| > \eta_n\right]. \tag{3.1}$$

Again if we can show $\sum_{n=1}^{\infty} \eta_n < \infty$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$, then by corollary 3.2, we have $\sum_{n=1}^{\infty}(Z(f_n) - Z(g_n))$ is almost surely convergent, which says that $Z(f_n) - Z(g_n)$ converges to zero almost surely, that is, $\{Z(f_n)\}$ and $\{Z(g_n)\}$ have the same limits.

Now $|Z(f_n) - Z(g_n)| = |Z(f_n - g_n)|$, we have $P(|Z(f_n) - Z(g_n)| > \eta_n) \leq \max_{\mathcal{F}} P(|Z(f)| > \eta_n)$, where the maximum on the right hand side is over all f in $\mathcal{F}$ with $\|f\| \leq 2\delta_n$ that are rectilinear functions. By theorem 2.3 and 2.4,

$$\max P(|Z(f)| > \eta_n) \leq K_1 \exp\left(-K_2\eta_n/2\delta_n\right). \tag{3.2}$$

where K_1 and K_2 are constants. Combining (3.1) and (3.2) gives

$$\epsilon_n \le K_1 \exp\left(2H(\delta_n) - K_2\eta_n/2\delta_n\right).$$

Take $\eta_n = 6{K_2}^{-1}H(\delta_n)\delta_n$ and $\delta_n = \delta_0\beta^n$ for some $\delta_0 \in (0,1)$, then $\epsilon_n \le K_1 \exp(-H(\delta_n))$ and

$$\int_0^{\delta_0} H(u)du \ge \sum_{n=2}^{\infty}(\delta_n - \delta_{n+1})H(\delta_n) = (1-\beta)\sum_{n=2}^{\infty}\delta_n H(\delta_n) = C\sum_{n=1}^{\infty}\eta_n,$$

where $C = K_2\beta(1-\beta)/6$. Thus Assumption I implies the convergence of the series $\sum_{n=1}^{\infty}\eta_n$ and, since

$$\exp(-H(\delta_n)) \le \exp(-\log(\log\delta_n)^2) \le (n\log\beta + \log\delta_0)^{-2},$$

which implies $\sum_{n=1}^{\infty}\varepsilon_n < \infty$. This finishes the proof of the theorem 3.3.

THEOREM 3.4. *Assume that $\mathcal{F}$ satisfies Assumption I and II. Then a product Brownian measure Z can be defined on $\mathcal{F}$ whose trajectories are uniformly continuous with respect to the $\|\cdot\|$-metric.*

Proof. In view of theorem 3.3, it suffices to prove that Z is uniformly continuous over $\mathcal{F}$ almost surely. Let $\epsilon > 0$ and $\eta > 0$ be given. Claim that there exists $\delta = \delta(\epsilon,\eta) > 0$ such that

$$P\left(\sup\{|Z(f) - Z(g)| : f, g \in \mathcal{F}, \|f-g\| \le \delta\} > \eta\right) < \epsilon.$$

For any f, $g \in \mathcal{F}$ and $n \ge 1$, we have

$$(3.3) \quad |Z(f)-Z(g)| \le |Z(f)-Z(f_n)|+|Z(g)-Z(g_n)|+|Z(f_n)-Z(g_n)|.$$

Let n be such that $\sum_{k=n}^{\infty}\eta_k < \eta/3$ and $\sum_{k=n}^{\infty}\epsilon_k < \epsilon/3$ with $\{\eta_k\}$ and $\{\varepsilon_k\}$ as the same as in the proof of the last half of theorem 3.3, then we have

$$P\left(\sup_{\mathcal{F}}|Z(f) - Z(f_n)| > \eta/3\right) < \epsilon/3,$$

which bounds the first two terms on the right hand side of (3.3). For the third term, note that if $\|f-g\| < \delta_n$, then by the triangular inequality, $\|f_n - g_n\| < 3\delta_n$. Following lines of the proof of theorem 3.3 we have

$$
\begin{aligned}
&P\left[\max_{f,g\in\mathcal{F},\|f-g\|<3\delta_n} |Z(f_n) - Z(g_n)| > \eta/3\right] \\
&\leq K_1 \exp(2H(\delta_{n+1}) - \eta/3\delta_n). \qquad (3.4)
\end{aligned}
$$

By assumptiom II, $\delta H(\delta) \leq \int_0^\delta H(u)du \to 0$, as $\delta \to 0$. It follows therefore that the bound in (3.4) is less than $\epsilon/3$ when δ_n is sufficiently small. This is always possible just by increasing n if necessary, and this increase does not affect the two previous restrictions placed on n. Take $\delta = \delta(\epsilon, \eta)$ to be the resulting value of δ_n. This completes the proof of the theorem.

Let F and G denote the envelopes of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. And let

$$W_1 = Z_1 - Z_1(F)\mu_1 \qquad \text{and} \qquad W_2 = Z_2 - Z_2(G)\mu_2$$

be tied-down Brownian processes. Then the product tied-down Brownian process is defined by

$$
\begin{aligned}
W_1 \times W_2 = Z_1 \times Z_2 &- Z_1(F)(\mu_1 \times Z_2) \\
&- Z_2(G)(Z_1 \times \mu_2) + Z_1(F)Z_2(G)(\mu_1 \times \mu_2),
\end{aligned}
$$

where the first and the fourth terms are the product of (independent or the same) Brownian processes and the product measure of μ_1 and μ_2 respectively, and $Z_1 \times \mu_2$ (or $\mu_1 \times Z_2$) is a Gaussian process. Thus theorem 3.3 and 3.4 give the following corollary.

COROLLARY 3.5. *Assume that $\mathcal{F}$ satisfies Assumption I and II. Then a product of tied-down Brownian processes can be defined on $\mathcal{F}$, whose sample paths are uniformly continuous with respect to the $\|\cdot\|$-norm on $L^2(\mu_1 \times L_2)$.*

4. Other Possible Constructions

In this section we give some comments and surveys about possible (implicit) construction of product Brownian process as a limit of (some) sequence of product processes, appearing in [2], [7] and [9]. Let $\xi_1, \xi_2, \dots$ be independent observations taken from a distribution P on a set X, and $\mathcal{F}$ be a class of real-valued symmetric functions on $X \times X$. We define the U-process $\{S_n(f) : f \in \mathcal{F}\}$ by

$$S_n(f) = \sum_{1 \le i \ne j \le n} f(\xi_i, \xi_j) \quad \text{for} \quad f \in \mathcal{F}.$$

If $\mathcal{F}$ is degenerate, that is, if $\int f(x, \cdot) dP = 0$ for all $f \in \mathcal{F}$ and $x \in X$, then the covariance kernel $c(f,g)$ is identically zero. In this cace for each f, the random variables $n^{-1} S_n(f)$ converge in distribution to an infinite weighted sum of independent χ^2 random variables. In [2] and [7], they showed that under some conditions, $\{n^{-1} S_n\}$ converges in distribution to a limit Q, a version with sample paths in $C(\mathcal{F}, P \otimes P)$. In this result we are interested in the process Q, which defined as follow: Consider the Hermite polynomials $\{h_k, k \in \mathbf{N}\}$ on $\mathbf{R}$ defined by the series expansion

$$\exp(\lambda x - \frac{\lambda^2}{2}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{k!}} h_k(x), \quad \lambda, \quad x \in \mathbf{R}.$$

The Hermite polynomials form an orthonomal basis of $L_2(\mathbf{R}, \gamma_1)$, where γ_1 is the canonical Gaussian measure. If $\underline{k} \in \mathbf{N}^{\mathbf{N}}$, i.e. $\underline{k} = (k_1, k_2, \dots)$, $k_i \in \mathbf{N}$, with $|\underline{k}| = \sum_i k_i < \infty$, set, for

$$H_{\underline{k}}(x) = h_{k_1}(x_1) h_{k_2}(x_2) \cdots .$$

Then $\{H_{\underline{k}}, \underline{k} \in \mathbf{N}^{\mathbf{N}}\}$ forms an orthonomal basis of $L_2(\mathbf{R}^{\mathbf{N}}, \gamma)$ where $\gamma = \gamma_\infty$ is the canonical Gaussian product measure on $\mathbf{R}^{\mathbf{N}}$. A function f in $L_2(\gamma)$ can be written as $f = \sum_{\underline{k}} H_{\underline{k}} f_{\underline{k}}$ where $f_{\underline{k}} = \int f H_{\underline{k}} d\gamma$ and the sum runs over all $\underline{k}$'s in $\mathbf{N}^{\mathbf{N}}$. We can also write

$$f = \sum_{d=0}^{\infty} \left(\sum_{|\underline{k}|=d} H_{\underline{k}} f_{\underline{k}} \right) = \sum_{d=0}^{\infty} Q_d f.$$

$Q_d f$ is named the chaos of degree d of f. Since $h_0 = 1$, $Q_0 f$ is simply the mean of f; $h_1(x) = x$, so chaos of degree 1 are Gaussian series $\sum_i g_i \alpha_i$. Chaos of degree 2 are of the type

$$\sum_{i \neq j} g_i g_j \alpha_{ij} + \sum_i (g_i^2 - 1)\alpha_i, \tag{4.1}$$

where $(g_i)_{i \in \mathbf{N}}$ is a sequence of independent standard Gaussian random variables. The first term of the right hand side of (4.1) is the same modulo coefficients as (2.2).

Second possible construction can arise as a weak limit of a sequence of product partial sum posscess, as mentioned in [9]. We define the product(smoothed) partial sum process S_n corresponding to $\{X_{\mathbf{i}}\}$ and $\{Y_{\mathbf{j}}\}$ indexed by subsets of the d-dimentional unit cube $\mathbf{I}^d$ by, for $A \in \mathcal{A}$,

$$S_n(A) := \sum_{|\mathbf{i}| \le n} \sum_{|\mathbf{j}| \le n} X_{\mathbf{i}} Y_{\mathbf{j}} |nA \cap C_{\mathbf{ij}}|,$$

where $C_{\mathbf{ij}}$ is d-dimentional unit cube whose Lebesgue measure is 1 and the upper right corner has a coordinate $(\mathbf{i}, \mathbf{j})$ with $\mathbf{i} \in \mathbf{N}^{d_1}$ and $\mathbf{j} \in \mathbf{N}^{d_2}$. When $A = B \times C$, by nomalizing with $n^{-d/2}$, $n^{-d/2} S_n(A) = n^{-d/2} S_n(B \times C)$ converges to product of Gaussian random variables depending on B and C. From this observation, it is possible to see that a product Brownian process could be the weak limit of a sequence of product partial sum processes. See [5] for the proof of the tightness of $n^{-d/2} S_n$ under some conditions on the random variables.

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