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# CONTROLLABILITY OF NONLINEAR DELAY PARABOLIC EQUATIONS UNDER BOUNDARY CONTROL

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## 1. Introduction

Let $A(\xi, \partial)$ be a second order uniformly elliptic operator

$$A(\xi,\partial)u = -\sum_{j,k=1}^{n} \frac{\partial}{\partial \xi_i}(a_{jk}(\xi)\frac{\partial u}{\partial \xi_k}) + \sum_{j=1}^{n} b_j(\xi)\frac{\partial u}{\partial \xi_j} + c(\xi)u$$

with real, smooth coefficients $a_{jk}$, $b_j$, $c$ defined on $\xi \in \Omega$, $\Omega$ a bounded domain in $R^n$ with a sufficiently smooth boundary $\Gamma$.

In this paper, we consider the following parabolic equation:

$$\begin{aligned} &\frac{\partial x(t,\xi)}{\partial t} = A(\xi,\partial)x(t,\xi) + F(t,\xi,x_t(\cdot,\xi)), \quad (0,T]\times\Omega, \\ (1)\qquad &x(t,\xi) = \phi, [-r,0]\times\Omega \\ &Bx|_{\Gamma} = u, \quad (0,T]\times\Gamma \end{aligned}$$

Here,

$$x : R\times\bar{\Omega} \to R^n,$$

$$x_t(\theta,\xi) = x(t+\theta,\xi), \quad \forall\theta \in [-r,0],$$

$$F : R\times\bar{\Omega}\times C(R^-;R^n) \to R^n$$

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is nonlinear function and $B$ is an associated boundary operator of the usual form

$$B : x \to \alpha x + \beta x_{\mu} \text{ at } \Gamma$$

where $x_{\mu}$ is the exterior conormal derivative

$$x_{\mu} = \frac{\partial x}{\partial \mu} = \sum_{j,k} a_{jk} \frac{\partial u}{\partial \xi_k} n_k,$$

where $n =$ unit exterior normal to $\Gamma$, with "sufficiently smooth" (real) coefficients, normalized so that $\alpha^2 + \beta^2 \equiv 1$. We distinguish between - and admit - the two cases (Dirichlet and Neumann)

$$D : \alpha = 1, \; \beta = 0$$
$$N : \alpha = 0, \; \beta = 1.$$

In an attempt to develop results for nonlinear control systems several established techniques from nonlinear analysis have been employed. An early example of this was the use of Nussbaum's fixed point theorem([7]) to obtain controllability results for infinite-dimensional abstract nonlinear systems([1]). Recently, in [4], they was the use of Nussbaum's fixed point theorem to obtain controllability result for delay volterra control system.

In this paper, we will attempt to solve controllability for nonlinear delay parabolic equations with boundary control -a semigroup approach- using method by Nussbaum's fixed point theorem.

In section 2, establishing notation and formulating problem.

In section 3, we are to show the mild solution of (1) can be streered from the origin to any final state.

## 2. Notation and Formulation

Let $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ be the operator $Af = A(\xi, \partial)f$ for $f \in D(A)$ where

$$D(A) = \{x \in L^2(\Omega) : Ax \in L^2(\Omega), \; Bx|_{\Gamma} = 0\}.$$

It is well known that $A$ generates a analytic semigroup $S(t)$ on $L^2(\Omega)$. Without loss of generality, we can assume that the spectrum of $A$ is on the right of the complex plane and $0 \in \rho(A)$, so that the fractional powers of $A$ are well defined ([9]).

Let $X = L^2(\Omega)$, $W = L^2(\Gamma)$. With $\alpha$ a real non-negative index, let $\{X^\alpha\}$ and $\{W^\alpha\}$ be continua of Hilbert spaces such that

$$\left.\begin{array}{l} X^{\alpha_1} \subset X^{\alpha_2} \subset \cdots \subset X^0 \equiv X \\ W^{\alpha_1} \subset W^{\alpha_2} \subset \cdots \subset W^0 \equiv W \end{array}\right\}, \quad \alpha_1 > \alpha_2 \geq 0$$

where the injections are continuous.

We next extend the definition of $\{X^\alpha\}$ and $\{W^\alpha\}$ by setting

$$\begin{aligned} X^{-2\alpha} &= [D(A^*)^\alpha]' \\ W^{-2\alpha} &= [W^{2\alpha}]' \end{aligned}$$

for all $\alpha \geq 0$, so that the inclusions $X \subset \cdots X^{-\alpha_2} \subset X^{-\alpha_1}$ and $W \subset \cdots W^{-\alpha_2} \subset W^{-\alpha_1}$ also hold.

Finally, for the purpose of uniformity of notation, we fined it convenient to introduce the symbol

$$D(A^{-\alpha}) = [D(A^{*\alpha})]'$$

for all $\alpha \geq 0$.

We now formulate our basic assumptions concerning $D(A^\alpha)$ and the operator $G$.

Assume that $G \in L(W \to X)$ has regularity $\alpha_0$, $\alpha_0 \geq 0$, with respect to $(W^\alpha, X^\alpha)$; i.e. that

$$G : W^{2\alpha} \to X^{2\alpha+2\alpha_0}$$

is a linear bounded map for $\alpha \geq 0$.

$$(2) \qquad D(A^\alpha) = X^{2\alpha}, \quad \text{for} \quad 0 \leq \alpha < \alpha_0,$$

the identification being set theoretically and topologically; i.e.,

$$\|x\|_{X^{2\alpha}} \quad \text{equivalent to} \quad \|A^\alpha x\|_X.$$

That above assumption is fulfilled when where of course

$$X^s(\Omega), \quad s < \frac{1}{2} \quad \text{(Dirichlet)}, \quad s < \frac{3}{2} \quad \text{(Neumann)}$$

with topology given by

$$\|x\|_{D(A^\alpha)} = \|A^\alpha x\|_X, \quad \alpha < \frac{1}{4} \quad \text{(Dirichlet)}, \quad \alpha < \frac{3}{4}, \alpha \neq \frac{1}{4} \text{(Neumann)}$$

$G$ is the Green's map defined by

$$Gv = v \qquad \text{iff} \qquad A(\xi, \partial)v = 0 \quad \text{in} \quad \Omega$$

$$Bv|_\Gamma = u \qquad \text{on} \qquad \Gamma.$$

The Dirichlet map

(3)
$$G_D : W^{2\alpha} \to X^{2\alpha+2\cdot\frac{1}{4}}$$

the Neumann map

(4)
$$G_N : W^{2\alpha} \to X^{2\alpha+2\cdot\frac{3}{4}}$$

is the bounded linear operator. By (2), (3) and (4), we then have

$$\text{range of } G_D = G_D W \subset D(A^{s+\frac{1}{4}-\rho}), \quad \rho > 0,$$

$$\text{range of } G_N = G_N W \subset D(A^{s+\frac{3}{4}-\rho}), \quad \rho > 0.$$

With this preliminary background, by semigroup formulation with $s = 0$, it is given by

(5)
$$\begin{aligned} x_t(\phi)(0) =& x(t) \\ =& S(t)\phi(0) + \int_0^t [S(t-s)F(s, x_s(\phi)) \\ &+ A^{\frac{3}{4}+\rho} S(t-s) A^{\frac{1}{4}-\rho} G_D u(s)] ds \\ &; \text{ Dirichlet case} \end{aligned}$$

(6)
$$\begin{aligned} x_t(\phi)(0) =& x(t) \\ =& S(t)\phi(0) + \int_0^t [S(t-s)F(s, x_s(\phi)) \\ &+ A^{\frac{1}{4}+\rho} S(t-s) A^{\frac{3}{4}-\rho} G_N u(s)] ds \\ &; \text{ Neumann case} \end{aligned}$$

It follows directly ( or by the convolution theorem ) that

$$\int_0^t A^{\frac{3}{4}+\rho}S(t-s)A^{\frac{1}{4}-\rho} : L^\infty(0,T;W) \to C([0,T];X^{\frac{1}{2}-\epsilon}),\ \epsilon>0,$$

; Dirichlet case

$$\int_0^t A^{\frac{1}{4}+\rho}S(t-s)A^{\frac{3}{4}-\rho} : L^\infty(0,T;W) \to C([0,T];X^{\frac{3}{2}-\epsilon}),\ \epsilon>0,$$

; Neumann case

is linear and bounded. Put

$$\sigma = \begin{cases} \frac{1}{4}-\rho & : \text{ Dirichlet case} \\ \frac{3}{4}-\rho & : \text{ Neumann case,} \end{cases}$$

$$\alpha = \begin{cases} \frac{1}{4}-\frac{\epsilon}{2} & : \text{ Dirichlet case} \\ \frac{3}{4}-\frac{\epsilon}{2} & : \text{ Neumann case,} \end{cases}$$

$$D = \begin{cases} G_D & : \text{ Dirichlet case} \\ G_N & : \text{ Neumann case,} \end{cases}$$

Rewrite (5) and (6) then
(7)

$$x_t(\phi)(0) = S(t)\phi(0) + \int_0^t [S(t-s)F(s,x_s(\phi)) + A^{1-\sigma}S(t-s)A^\sigma Du(s)]ds.$$

Put $X_\alpha = D(A^\alpha) = X^{2\alpha}$ with norm

$$\|x\|_\alpha = \|x\|_{D(A^\alpha)} = \|A^\alpha x\|, \qquad x \in X_\alpha.$$

Let $W_0 = C_0(R^+;W)$, the space of all bounded and continuous $W$-valued functions with the usual supremum norm, then the boundary control or the perturbation function $u$ is given in $W_{\rm ad} \subset W_0$.

We shall denote by $C_\alpha$ the Banach space of continuous functions $C([-r,0];X_\alpha)$ with the norm

$$\|\phi\|_{C_\alpha} = \sup_{-r\le\theta\le 0} \|A^\alpha\phi(\theta)\|.$$

Let $\phi \in C([-r,0];X_\alpha)$.

If $x_t(\phi)$ is an element in $C([-r,0];X_\alpha)$ then which has point wise definition

$$x_t(\phi)(\theta) = x(\phi)(t+\theta), \quad \text{for} \quad \theta \in [-r,0].$$

In [8], they are to solve the existence problem of milde solution of (1) under following assumptions:

(F) the nonlinear function

$$F(\cdot,\cdot) : R \times C([-r,0] : X_\alpha) \to X$$

is continuous, $F(t,0) = 0$ for all $t > 0$, there exists an $L > 0$ and $\omega > 0$ so that

$$\|F(t,\phi) - F(t,\psi)\| \leq e^{-\omega t} L \|\phi - \psi\|_{C_\alpha}$$

and the following inequality holds for some constant $M_\alpha > 0$

$$\|A^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\delta t}.$$

## 3. Controllability

In this section, we consider the controllability of the system (7).

We desire to transfer the nonlinear system (7) from $x(0) = \phi$ to $x_T(\phi) = v$. Here $u \in L^2(0,T;W)$, a Banach space of possible control actions.

Assume that the following inequality holds for some constant $M_\alpha > 0$,

$$\|A^\alpha S(t)\| \leq M_\alpha t^{-\alpha}.$$

The result dependes on the exact controllability of the linear system

$$x_t(\phi)(0) = \int_0^t A^{1-\sigma} S(t-s) A^\sigma D u(s) ds. \tag{8}$$

We assume that it can be streered to the subspace $V$ then Range $G \supset V$ where

$$Gu = \int_0^T A^{1-\sigma} S(t-s) A^{\sigma} D u(s) ds.$$

Actually we can assume, without losing generality, that Range $G = V$ and that we can construct an invertible operator $\tilde{G}$ defined on $L^2(0,T;W)/KerG$([1]). Then, the control can be introduce

$$u(s) = \tilde{G}^{-1}[v - \int_0^T S(T-s)F(s,x_s(\phi))ds](s).$$

This control is substituted into (7) to provide the operator

$$\Phi x_t(\phi)(0) = \int_0^t S(t-s)F(s,x_s(\phi))ds$$
$$+ \int_0^t A^{1-\sigma} S(t-s) A^{\sigma} D \tilde{G}^{-1}[v - \int_0^T S(T-s)F(s,x_s(\phi))ds](s)ds.$$

Notice that $\Phi x_T(\phi)(0) = v$, which means that the control $u$ streers the nonlinear system from the origine to $v$ in time $T$ provided we can obtain a fixed point of the nonlinear operator $\Phi$.

We assume the following hypothesis:

(F1) the nonlinear function

$$F(\cdot,\cdot) : [0,T] \times C([-r,0] : X_\alpha) \rightarrow X$$

is continuous and satisfies a Lipschitz type condition

$$\|F(t,\phi) - F(t,\psi)\| \leq r(t)\|\phi - \psi\|_{C_\alpha}$$

where $r(\|\phi\|,\|\psi\|) = r(t)$ is continuous on $[0,T], r(t) \rightarrow 0$ as $t \rightarrow 0$ and $F(t,0) \equiv 0$, $0 \leq t \leq T$.

(H1) The linear system (8) is exactly controllable to the subspace $V$.

(H2) $S(t)x \in X \cap V$ for all $x \in X$, $t \geq 0$

$$\|S(t)x\|_V \leq g(t)\|x\|, \qquad \|g\|_{L^2(0,T:X)} = c_1 < \infty.$$

(H3) $\gamma$ is chosen so that the following conditions hold

$$\sup_{\|\phi\| \leq \gamma} (c_2 + c_3 c_1(t)) r(\|\phi(t)\|, 0) \leq k < 1.$$

(H4) The semigroup $S(t)$ on $X$ is compact operator for each $t > 0$.

THEOREM 1. ([7]) *Suppose that $S$ is closed, bounded convex subset of a Banach space $X$. Suppose that $\Phi_1, \Phi_2$ are continuous mappings from $S$ into $X$ such that*

(i)

$$(\Phi_1 + \Phi_2)S \subset S,$$

(ii)

$$\|\Phi_1 x - \Phi_1 x'\|_X \le k\|x - x'\| \qquad \text{for all} \qquad x, x' \in S$$

*where $k$ is constant and $0 \le k \le 1$.*

(iii) $\overline{\Phi_2(S)}$ *is compact.*

*Then the operator $\Phi_1 + \Phi_2$ has a fixed point in $S$.*

LEMMA 1. *Let constant $\alpha$ satisfy $1 - \alpha > 0$ then, for $0 \le t \le T$,*

$$\| \int_0^t A^\alpha S(t-s)x(s)ds \| \le c_2 \|x\|_{L^2(0,T:X)}.$$

By similar estimation of Lemma 1, we get following Lemma 2.

LEMMA 2. *Let constant $\alpha, \sigma$ satisfy $1 - 2(\alpha + 1 - \sigma) > 0$ then, for $0 \le t \le T$,*

$$\| \int_0^{\cdot} A^{\alpha+1-\sigma} S(\cdot - s) A^\sigma Du(s)ds \| \le c_3(\cdot) \|u\|_{L^2(0,T:V)}$$

*where $c_3(\cdot)$ is increasing, $c_3(0) = 0$.*

THEOREM 2. *Hypothesis (H1)-(H4) and (F1) are satisfied. Then the state of the system (7) can be streeded from the $\phi$ to any final state $v$, satisfying*

$$\|v\|_V \le \frac{(1-k)\gamma}{c_3}$$

*in the time interval $[0, T]$.*

*Proof.* We now defined

$$\Phi_1 x_t(\phi)(0) = \int_0^t S(t-s)F(s, x_s(\phi))ds$$

and

$$\Phi_2 x_t(\phi)(0)$$
$$= \int_0^t A^{1-\sigma}S(t-s)A^{\sigma}D\tilde{G}^{-1}[v - \int_0^T S(T-s)F(s, x_s(\phi))ds](s)ds.$$

We can now employ Theorem 1 with

$$S = \{x_t(\phi)(\cdot) \in C([-r,T] : X_\alpha); \|x_t(\phi)\|_{C_\alpha} \leq \gamma\}.$$

Then the set $S$ is closed, bounded and convex. From the definition

$$\Phi x_t(\phi)(0) = \Phi_1 x_t(\phi)(0) + \Phi_2 x_t(\phi)(0).$$

Thus for any $x_t(\phi)(\cdot) \in S$, using Lemma 1, 2

$$\begin{aligned}
&\|\Phi x_t(\phi)\|_\alpha = \|A^\alpha \Phi x_t(\phi)(\theta)\| \\
&= \|A^\alpha \Phi x_{t+\theta}(\phi)(0)\| \\
&= \| \int_0^{t+\theta} A^\alpha S(t+\theta-s)F(s, x_s(\phi))ds + \int_0^{t+\theta} A^{\alpha+1-\sigma}S(t+\theta-s) \\
&\qquad \times A^\sigma D\tilde{G}^{-1}[v - \int_0^T S(T-s)F(s, x_s(\phi))ds](s)ds\| \\
&\leq c_2 r(t)\|x_s(\phi)\|_{C_\alpha} + c_3\|v\|_{L^2(0,T:X)} + c_3 c_1 r(t)\|x_s(\phi)\|_{C_\alpha} \\
&= (c_2 + c_1 c_3) r(t)\|x_s(\phi)\|_{C_\alpha} + c_3\|v\|_{L^2(0,T:X)} \\
&\leq k\gamma + (1-k)\gamma = \gamma, \qquad -h \leq \theta \leq 0.
\end{aligned}$$

Hence

$$\sup_{-h\leq\theta\leq 0} \|\Phi x_t(\phi)\|_\alpha = \|\Phi x_t(\phi)\|_{C_\alpha} \leq \gamma.$$

Hence $\Phi_1 x_t(\phi)(0) + \Phi_2 x_t(\phi)(0) \in S$ for all $x_t(\phi) \in S$, which means that part (i) of Theorem 1 is satisfied.

To show that $\Phi_1$ and $\Phi_2$ are completely continuous. We consider

$$\begin{aligned}
&\|\Phi_1(x_t(\phi)+\eta) - \Phi_1 x_t(\phi)\|_\alpha \\
&= \|A^\alpha \Phi_1(x_{t+\theta}(\phi)+\eta)(0) - A^\alpha \Phi_1 x_{t+\theta}(\phi)(0)\| \\
&= \| \int_0^{t+\theta} A^\alpha S(t+\theta-s)[F(s, x_s(\phi)+\eta) - F(s, x_s(\phi))]ds\| \\
&\leq c_2 r(t)\|\eta\|_{C_\alpha} \to 0 \qquad \text{as} \qquad \eta \to 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\|\Phi_2(x_t(\phi)+\eta')-\Phi_2 x_t(\phi)\|_\alpha \\
&=\|A^\alpha\Phi_2(x_{t+\theta}(\phi)+\eta')(0)-A^\alpha\Phi_2(x_{t+\theta}(\phi))(0)\| \\
&\le c_3c_1r(t)\|\eta'\|_{C_\alpha}, \qquad -h\le\theta\le 0,\ \ 0\le t\le T.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\sup_{-h\le h\le 0}\|\Phi_2(x_t(\phi)+\eta')-\Phi_2(x_t(\phi))\|_\alpha \\
&=\|\Phi_2(x_t(\phi)+\eta')-\Phi_2(x_t(\phi))\|_{C_\alpha} \\
&\le c_3c_1r(t)\|\eta'\|_{C_\alpha}\to 0 \qquad \text{as} \qquad \eta'\to 0.
\end{aligned}$$

Thus $\Phi_1$ and $\Phi_2$ are continuous.

Using the Arzela-Ascoli Theorem we show that $\Phi_2$ maps $S$ into a precompact subset of $S$. We consider

$$\begin{aligned}
\Phi_2 x_t(\phi)(\theta) &= \int_0^{t+\theta} A^{1-\sigma}S(t+\theta-s) \\
&\quad\times A^\sigma D\tilde{G}^{-1}[v-\int_0^T S(T-s)F(s,x_s(\phi))ds](s)ds.
\end{aligned}$$

We now define

$$\begin{aligned}
\Phi_{2-\epsilon} x_t(\phi)(\theta) &= \int_0^{t+\theta-\epsilon} A^{1-\sigma}S(t+\theta-s) \\
&\quad\times A^\sigma D\tilde{G}^{-1}[v-\int_0^T S(T-s)F(s,x_s(\phi))ds](s)ds
\end{aligned}$$

for all $x_t(\phi)\in S$. Then

$$\begin{aligned}
\Phi_{2-\epsilon} x_t(\phi)(\theta) &= S(\epsilon)\int_0^{t+\theta-\epsilon} A^{1-\sigma}S(t+\theta-s) \\
&\quad\times A^\sigma D\tilde{G}^{-1}[v-\int_0^T S(T-s)F(s,x_s(\phi))ds](s)ds.
\end{aligned}$$

By hypothesis (H4), $S(\epsilon)$ is compact operator. Thus the set

$$K_{2-\epsilon}[x_t(\phi)(\theta)] = \{\Phi_{2-\epsilon}x_t(\phi)(\theta) : x_t(\phi) \in S\}.$$

is precompact. Also

$$\begin{aligned}
&\|\Phi_2 x_t(\phi) - \Phi_{2-\epsilon}x_t(\phi)\|_\alpha \\
&= \| \int_{t+\theta-\epsilon}^{t+\theta} A^{\alpha+1-\sigma}S(t+\theta-s)A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s,x_s(\phi))ds](s)ds\| \\
&\leq c_3(\epsilon)[\|v\|_{L^2(0,T:X)} + c_1 r(t)\|x_s(\phi)\|_{C_\alpha}] \to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sup_{-h\leq\theta\leq 0} \|\Phi_2 x_t(\phi) - \Phi_{2-\epsilon}x_t(\phi)\|_\alpha \\
&= \|\Phi_2 x_t(\phi) - \Phi_{2-\epsilon}x_t(\phi)\|_{C_\alpha} \\
&\leq c_3(\epsilon)[\|v\|_{L^2(0,T:X)} + c_1 r(t)\|x_s(\phi)\|_{C_\alpha}] \to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

Thus there are precompact sets arbitrarily close to the set

$$K_2[x_t(\phi)(\theta)] = \{\Phi_2 x_t(\phi)(\theta) : x_t(\phi) \in S\}$$

and therefore $K_2[x_t(\phi)(\theta)]$ is precompact.

We next show that $\Phi_2$ maps the function in $S$ into an equicontinuous family of functions. For equicontinuity from the left we take $t > \epsilon > t' > 0$ then

$$\begin{aligned}
&\|\Phi_2 x_t(\phi) - \Phi_2 x_{t-t'}(\phi)\|_\alpha \\
&= \| \int_0^{t+\theta} A^{\alpha+1-\sigma}S(t+\theta-s)A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s,x_s(\phi))ds](s)ds \\
&\quad - \int_0^{t-t'+\theta} A^{\alpha+1-\sigma}S(t-t'+\theta-s)A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s,x_s(\phi))ds](s)ds\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \| \int_0^{t+\theta-\epsilon} A^{\alpha+1-\sigma} S(t+\theta-s) A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s, x_s(\phi))ds](s)ds \\
&\quad - \int_0^{t+\theta-\epsilon} A^{\alpha+1-\sigma} S(t-t'+\theta-s) A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s, x_s(\phi))ds](s)ds \| \\
&\quad + \| \int_{t+\theta-\epsilon}^{t+\theta} A^{\alpha+1-\sigma} S(t+\theta-s) A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s, x_s(\phi))ds](s)ds \| \\
&\quad + \| \int_{t+\theta-\epsilon}^{t+\theta-t'} A^{\alpha+1-\sigma} S(t+\theta-t'-s) A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s, x_s(\phi))ds](s)ds \| \\
&\leq \|S(t'+\epsilon) - S(\epsilon)\| \int_0^{t+\theta-\epsilon} \|A^{\alpha+1-\sigma} S(t-t'+\theta-s-\epsilon) A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s, x_s(\phi))ds](s)\|ds \\
&\quad + \int_{t+\theta-\epsilon}^{t+\theta} \|A^{\alpha+1-\sigma} S(t+\theta-s) A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s, x_s(\phi))ds](s)ds \\
&\quad + \int_{t+\theta-\epsilon}^{t+\theta-t'} \|A^{\alpha+1-\sigma} S(t+\theta-t'-s) A^\sigma D\tilde{G}^{-1} \\
&\qquad \times [v - \int_0^T S(T-s)F(s, x_s(\phi))ds](s)\|ds \\
&\leq \|S(t'+\epsilon) - S(\epsilon)\| c_3(t+\theta-\epsilon)\|u\| + c_3(\epsilon)\|u\| + c_3(\epsilon-t')\|u\| \ \to\ 0
\end{aligned}
$$

as $\epsilon \to 0$, by $c_3(t) \to 0$ as $t \to 0$ and $S(t)$ is continuous. Thus we have

$$\begin{aligned}&\sup_{-h\leq\theta\leq 0} \|\Phi_2 x_t(\phi) - \Phi_2 x_{t-t'}(\phi)\|_\alpha \\ &= \|\Phi_2 x_t(\phi) - \Phi_2 x_{t-t'}(\phi)\|_{C_\alpha} \to 0 \qquad \text{as} \qquad t' \to 0.\end{aligned}$$

The equicontinuity from the right is similar. Finally we must have a Lipschitz condition for the operator $\Phi_1$.

Consider $x_t(\phi)$, $\hat{x}_t(\phi) \in S$,

$$\begin{aligned}&\|\Phi_1 x_t(\phi) - \Phi_1 \hat{x}_t(\phi)\|_\alpha \\ &= \|\int_0^{t+\theta} A^\alpha S(t-s)[F(s, x_s(\phi)) - F(s, \hat{x}_s(\phi))]ds\| \\ &\leq c_2 r(t)\|x_s(\phi) - \hat{x}_s(\phi)\|_{C_\alpha}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\sup_{-h\leq\theta\leq 0} \|\Phi_1 x_t(\phi) - \Phi_1 \hat{x}_t(\phi)\|_\alpha \\ &= \|\Phi_1 x_t(\phi) - \Phi_1 \hat{x}_t(\phi)\|_{C_\alpha} \\ &\leq c_2 r(t)\|x_s(\phi) - \hat{x}_s(\phi)\|_{C_\alpha}.\end{aligned}$$

Therefore, by Theorem 1, the proof of Theorem 2 is complete.

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