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WEAKLY WELL–DECOMPOSABLE OPERATORS AND AUTOMATIC CONTINUITY

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1. Introduction

Let X and Y be Banach spaces and consider a linear operator $\theta : X \to Y$. The basic automatic continuity problem is to derive the continuity of θ from some prescribed algebraic conditions. For example, if $\theta : X \to Y$ is a linear operator intertwining with $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, one may look for algebraic conditions on T and S which force θ to be continuous.

The study of continuity of a linear operator θ intertwining with T and S was initiated by Johnson and Sinclair [JS]. In [JS] necessary conditions on T and S for the continuity of θ were obtained for the operator S with countable spectrum.

In 1973 Vrbová presented an automatic continuity result concerning an intertwining operator with operators having suitable spectral decomposition properties [Vr72].

In 1986 Laursen and Neumann introduced super-decomposable operators in [LN] in order to consider necessary conditions for automatic continuity of intertwining operators; this class of operators contains most of interesting examples of decomposable operators. Since [LN], the study of automatic continuity of intertwining linear operators has been closely related to the classification of decomposable operators.

In this paper, we introduce the class of weakly well-decomposable operators to study the continuity of intertwining operators. This class is a subset of the class of operators having the weak 2-spectral decomposition property (weak 2-SDP), and contains all the super-decomposable

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operators. For this class, it is possible to give a very useful algebraic representation of the analytic spectral subspaces. From this representation we present some applications to automatic continuity theory. We give necessary and sufficient conditions on a decomposable operator T defined on a Banach space X and a weakly well-decomposable operator S defined on a Banach space Y in order that every linear operator $\theta : X \to Y$ which intertwines with T and S be automatically continuous. This generalizes the works of [AEN], [LN] and [LW].

2. Preliminaries

Throughout this paper we shall use the standard notions and some basic results on the theory of decomposable operators and automatic continuity theory as presented in [CF], [EL], [EW], [Va] and [Si76]. Let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on a Banach space X over the complex plane $\mathbb{C}$. Given an operator $T \in \mathcal{L}(X)$, let $\mathrm{Lat}(T)$ denote the collection of all closed T-invariant linear subspaces of X, and for an $Y \in \mathrm{Lat}(T)$ $T|Y$ denote the restriction of T on Y.

DEFINITION 2.1. Let $T : X \to X$ be a linear operator on a Banach space X. Let F be a subset of the complex plane $\mathbb{C}$. Consider the class of all linear subspaces Y of X which satisfy $(T-\lambda)Y = Y$ for all $\lambda \notin F$ and let $E_T(F)$ denote the span of all such subspaces Y of X. $E_T(F)$ is called an *algebraic spectral subspace* of T.

It is clear that $(T-\lambda)E_T(F) = E_T(F)$ for all $\lambda \notin F$ as well so that it is the largest linear subspace with this property. It is known that the algebraic spectral subspace preserves intersections of subsets F of $\mathbb{C}$ [La88]. It is also clear from the definition that

$$E_T(F) \subseteq \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T-\lambda)^n X.$$

A linear subspace Z of X is called a *T-divisible subspace* if

$$(T-\lambda)Z = Z \quad \text{for all } \lambda \in \mathbb{C}.$$

Hence $E_T(\emptyset)$ is precisely the largest T-divisible subspace. For an operator $T \in \mathcal{L}(X)$, it is easy to show that only closed T-divisible subspace is $\{0\}$. There is an operator which has non-trivial divisible subspaces. Indeed, the Volterra operator has a non-trivial divisible subspace [LN]. On the other hand, many important operators do not have non-trivial divisible subspaces. For example, hyponormal operators on Hilbert spaces do not have non-trivial divisible subspaces [Pu].

LEMMA 2.2. *Let $T \in \mathcal{L}(X)$ and let M be the maximal T-divisible subspace of X. Then M is characterized by M being maximal subspace with respect to*

$$(T-\lambda)M = M \quad \text{for all } \lambda \in \sigma(T)$$

where $\sigma(T)$ denotes the spectrum of T.

Proof. Let M be the maximal subspace with the property $(T-\lambda)M = M$ for all $\lambda \in \sigma(T)$. It is enough to show that $(T-\mu)M = M$ for all $\mu \in \rho(T)$, where $\rho(T)$ denotes the resolvent set of T. For each $\mu \in \rho(T)$

$$\begin{aligned}(T-\lambda)(T-\mu)^{-1}M &= (T-\mu)^{-1}(T-\lambda)M \\ &= (T-\mu)^{-1}M\end{aligned}$$

for all $\lambda \in \sigma(T)$, it follows that $(T-\mu)^{-1}M \subseteq M$ by the maximality of M. Hence we have

$$\begin{aligned}M &= (T-\mu)(T-\mu)^{-1}M \subseteq (T-\mu)M \\ &\subseteq (T-\lambda+(\lambda-\mu))M \\ &\subseteq M. \quad \square\end{aligned}$$

Let $T \in \mathcal{L}(X)$ and $x \in X$. Then $f(\lambda) = (T-\lambda)^{-1}x$ is an analytic function on $\rho(T)$. This function may have an analytic extension to an open set properly containing $\rho(T)$. If any two such extensions must agree on their common domain, then T is said to have the *single-valued extension property*. In this case $f(\lambda)$ must have the maximal analytic extension which we denote by $x(\lambda)$. We now define the *local resolvent* $\rho_T(x)$ of x as follows:

$\rho_T(x) := \{\lambda \in \mathrm{Dom}(x(\lambda)) : (T-\lambda)x(\lambda) \equiv x$ on a neighborhood of $\lambda\}$

and the *local spectrum* $\sigma_T(x)$ of x is defined to be $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. The set $\sigma_T(x)$ is a compact subset of $\mathbb{C}$ and $\sigma_T(x) \subseteq \sigma(T)$. It is easy to show that $\sigma_T(x) = \emptyset$ if and only if $x = 0$. It is also easy to see that for $F \subseteq \mathbb{C}$

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}.$$

is a T-invariant (in fact, hyper-invariant) linear subspace of X. This space is said to be an *analytic spectral subspace.* In general, this space $X_T(F)$ need not be closed even if F is closed. For a given $T \in \mathcal{L}(X)$ having the single-valued extension property, using a standard argument of the theory of local spectral theory it is easy to show that the inclusion $X_T(F) \subseteq E_T(F)$ for all $F \subseteq \mathbb{C}$.

The single-valued extension property passes to the restrictions of the given operator. The proof of the following lemma is in [EL].

LEMMA 2.3. *Let $T \in \mathcal{L}(X)$ have the single-valued extension property and let $Y \in Lat(T)$. Then $T|Y$ has the single-valued extension property and*

$$\sigma_T(y) \subseteq \sigma_{T|Y}(y) \quad \textit{for every } y \in Y.$$

The proof of the following proposition is in [CF].

PROPOSITION 2.4. *Consider $T \in \mathcal{L}(X)$ with the single-valued extension property. If $X_T(F)$ is closed, then*

$$\sigma(T|X_T(F)) \subseteq \sigma(T) \cap F.$$

An operator $T \in \mathcal{L}(X)$ is called *decomposable* if, for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there exist $Y, Z \in \mathrm{Lat}(T)$ such that

$$\sigma(T|Y) \subseteq U, \quad \sigma(T|Z) \subseteq V \text{ and } Y + Z = X. \tag{1}$$

In (1), if it is only required that the sum $Y + Z$ be dense in X, then we say that the operator T has the *weak 2-spectral decomposition property (weak 2-SDP).*

Let $\mathcal{F}(\mathbb{C})$ denote the family of all closed subsets of $\mathbb{C}$ and let $\mathcal{S}(X)$ denote the family of all closed linear subspaces of X.

DEFINITION 2.5. (1) A map $\mathcal{E}(\cdot): \mathcal{F}(\mathbb{C}) \to \mathcal{S}(X)$ is called *stable* if it satisfies the following two conditions:
(i) $\mathcal{E}(\emptyset) = \{0\}$, $\mathcal{E}(\mathbb{C}) = X$.
(ii) $\mathcal{E}(\bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} \mathcal{E}(F_n)$ for any sequence $\{F_n\}$ in $\mathcal{F}(\mathbb{C})$.

(2) A map $\mathcal{E}(\cdot): \mathcal{F}(\mathbb{C}) \to \mathcal{S}(X)$ is called a *spectral capacity* if $\mathcal{E}(\cdot)$ is stable and satisfies the following condition:
(iii) $X = \sum_j \mathcal{E}(\overline{G}_j)$ for every finite open cover $\{G_j\}$ of $\mathbb{C}$.

We say that $\mathcal{E}(\cdot)$ is *order preserving* if it preserves the inclusion order. Clearly a stable map is order preserving. It is well known that T is decomposable if and only if there exists a spectral capacity $\mathcal{E}(\cdot)$ such that $\mathcal{E}(F) \in \operatorname{Lat}(T)$ and $\sigma(T|\mathcal{E}(F)) \subseteq F$ for each closed set $F \subseteq \mathbb{C}$. In this case the spectral capacity of a closed subset F of $\mathbb{C}$ is uniquely determined and it is the analytic spectral subspace $X_T(F)$.

We shall also consider an important subclass of the class of all decomposable operators on X. An operator $T \in \mathcal{L}(X)$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi: C^{\infty}(\mathbb{C}) \to \mathcal{L}(X)$ satisfying $\Phi(1) = I$ and $\Phi(z) = T$ where I is the identity operator on X and z is the identity function on $\mathbb{C}$. Every linear operator on a finite dimensional space as well as every spectral operator of finite type is a generalized scalar operator.

REMARK 2.6. For a generalized scalar operator $T \in \mathcal{L}(X)$ and a closed set F of $\mathbb{C}$, Vrbová proved the existence of a natural number p such that

$$X_T(F) = \bigcap_{\lambda \notin F} (T-\lambda)^p X$$

[Vr73]. From this equality, we have

$$E_T(F) \subseteq \bigcap_{\lambda \notin F, n \in \mathbb{N}} (T-\lambda)^n X \subseteq \bigcap_{\lambda \notin F} (T-\lambda)^p X = X_T(F).$$

Hence $X_T(F) = E_T(F)$ for a closed subset F of $\mathbb{C}$. In particular, generalized scalar operators do not have non-trivial divisible subspaces.

Let θ be a linear operator from a Banach space X into a Banach space Y. The space

$$\mathfrak{S}(\theta) := \{y \in Y : \text{ there is a sequence } x_n \to 0 \text{ in } X \text{ and } \theta x_n \to y\}$$

is called the *separating space* of θ. It is easy to see that $\mathfrak{S}(\theta)$ is a closed linear subspace of Y. By the closed graph theorem, θ is continuous if and only if $\mathfrak{S}(\theta) = \{0\}$. The following lemma is found in [Si76].

LEMMA 2.7. *Let X and Y be Banach spaces. If R is a continuous linear operator from Y to a Banach space Z, and if $\theta : X \to Y$ is a linear operator, then $(R\mathfrak{S}(\theta))^{-} = \mathfrak{S}(R\theta)$. In particular, $R\theta$ is continuous if and only if $R\mathfrak{S}(\theta) = \{0\}$.*

The next lemma states that a certain descending sequence of separating space which obtained from θ via a countable family of continuous linear operators is eventually constant. This lemma is proved in [JS], [La75] and [Si76].

STABILITY LEMMA. *Let $\theta : X_0 \to Y$ be a linear operator between the Banach spaces X_0 and Y with separating space $\mathfrak{S}(\theta)$, and let $\langle X_i : i = 1, 2, \dots \rangle$ be a sequence of Banach spaces. If each $T_i : X_i \to X_{i-1}$ is continuous linear operator for $i = 1, 2, \dots$, then there is an $n_0 \in \mathbb{N}$ for which*

$$\mathfrak{S}(\theta T_1 T_2 \dots T_n) = \mathfrak{S}(\theta T_1 T_2 \dots T_{n_0}) \quad \textit{for all } n \geq n_0.$$

The following lemma, known as *localization of the singularities*, is adopted from [La92].

LEMMA 2.8. *Let X and Y be Banach spaces. Suppose that $\mathcal{E}_X : \mathcal{F}(\mathbb{C}) \to \mathcal{S}(X)$ is an order preserving map such that $X = \mathcal{E}_X(\overline{U}) + \mathcal{E}_X(\overline{V})$ whenever $\{U, V\}$ is an open cover of $\mathbb{C}$. And suppose that $\mathcal{E}_Y : \mathcal{F}(\mathbb{C}) \to \mathcal{S}(Y)$ is a stable map. If $\theta : X \to Y$ is a linear operator for which*

$$\mathfrak{S}(\theta | \mathcal{E}_X(F)) \subseteq \mathcal{E}_Y(F) \quad \textit{for every } F \in \mathcal{F}(\mathbb{C}),$$

then there is a finite set $\Lambda \subseteq \mathbb{C}$ for which $\mathfrak{S}(\theta) \subseteq \mathcal{E}_Y(\Lambda)$.

This lemma tells us that under appropriate assumptions on a linear operator which have a large lattice of closed invariant subspaces the separating space will be contained eventually in a small closed invariant subspace.

We need the next theorem, known as Mittag-Leffler Theorem of Bourbaki, which is found in [Bo].

MITTAG-LEFFLER THEOREM. *Let $\langle X_n : n = 0, 1, 2, \dots \rangle$ be a sequence of complete metric spaces, and for $n = 1, 2, \dots$, let $f_n : X_n \to X_{n-1}$ be a continuous map with $f_n(X_n)$ dense in X_{n-1}. Let $g_n = f_1 \circ \cdots \circ f_n$. Then $\bigcap_{n=1}^{\infty} g_n(X_n)$ is dense in X_0.*

3. Weakly well-decomposable operators and automatic continuity

In this section we introduce a class of operators which have a new type of spectral decomposition that we call weakly well-decomposable operators. And we consider the automatic continuity of the operators of this type.

For a given $T \in \mathcal{L}(X)$, we define the commutator $C(T)$ acting on $\mathcal{L}(X)$ by

$$C(T)A := TA - AT \quad \text{for } A \in \mathcal{L}(X).$$

For a natural number n, define $C(T)^n$ to be the n-th composition of the operator $C(T)$. That is

$$C(T)^n A := C(T)^{n-1}(TA - AT) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k T^{n-k} A T^k.$$

We define $\mathcal{I}(T)$ as follows:

$$\mathcal{I}(T) := \{A \in \mathcal{L}(X) : C(T)^n A = 0 \text{ for some } n \in \mathbb{N}\}.$$

DEFINITION 3.1. A bounded linear operator T on a Banach space X is said to be *weakly well-decomposable* if for every open covering $\{U, V\}$ of $\mathbb{C}$ there exist $Y, Z \in \mathrm{Lat}(T)$ such that

$$\sigma(T|Y) \subseteq U, \quad \sigma(T|Z) \subseteq V$$

and there exist sequences $\langle P_j \rangle$, $\langle Q_j \rangle$ in $\mathcal{I}(T)$ such that

$$P_j(X) \subseteq Y, \; Q_j(X) \subseteq Z \text{ and } P_j + Q_j \to I\,(\mathbf{WOT}).$$

Here, $P_j + Q_j \to I\,(\mathbf{WOT})$ denotes that the sequence $\langle P_j + Q_j \rangle$ converges to I in the weak operator topology in $\mathcal{L}(X)$.

The notion of super-decomposability of a linear operator is introduced in [LN]. Every super-decomposable operator is weakly well-decomposable and every weakly well-decomposable operator has the weak 2-SDP, by the definition. Spectral operators in the sense of Dunford, normal operators on Hilbert spaces, and operators with totally disconnected spectrums are super-decomposable. Hence these operators are all weakly well-decomposable.

THEOREM 3.2. *If $T \in \mathcal{L}(X)$ is weakly well-decomposable, then T has the single-valued extension property.*

Proof. Let T be weakly well-decomposable, D an open subset of $\mathbb{C}$, and $f : D \to X$ an analytic X-valued function satisfying

$$(T - \lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in D.$$

We may also suppose that D is connected. Let G_1 and G_2 be open discs in D with $\overline{G}_1 \cap \overline{G}_2 = \emptyset$. Choose another open set H_1 such that $\{G_1, H_1\}$ covers $\mathbb{C}$ and $G_1 \setminus \overline{H}_1 \neq \emptyset$. Since T is weakly well-decomposable, there exist X_1 and $Y_1 \in \mathrm{Lat}(T)$ with

$$\sigma(T|X_1) \subseteq G_1, \quad \sigma(T|Y_1) \subseteq H_1$$

and there are sequences $\langle P_j \rangle$ and $\langle Q_j \rangle$ in $\mathcal{I}(T)$ such that for each $x \in X$, $u \in X^*$

$$(2) \quad \lim_{j\to\infty} |u(x - P_j x - Q_j x)| = 0, \quad P_j(X) \subseteq X_1 \text{ and } Q_j(X) \subseteq Y_1.$$

For each $j \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that $C(T)^n Q_j = 0$. It follows that $C(T - \lambda)^n Q_j = 0$ for all $\lambda \in D$. Hence $C(T - \lambda)^n Q_j f(\lambda) = 0$ for all $\lambda \in D$. That is

$$(3) \quad \sum_{k=0}^{n} \binom{n}{k} (-1)^k (T - \lambda)^{n-k} Q_j (T - \lambda)^k f(\lambda) = 0 \quad \text{for all } \lambda \in D.$$

In (3), since $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in D$, it follows that

$$(T - \lambda)^n Q_j f(\lambda) = 0 \quad \text{for all } \lambda \in D.$$

Since $Q_j f(\lambda) \in Y_1$ for all $\lambda \in D$ and $\sigma(T|Y_1) \subseteq H_1$, we have

$$Q_j f(\lambda) = 0 \text{ for all } j \in \mathbb{N} \text{ and } \lambda \in G_1 \setminus \overline{H}_1.$$

In (2), set $x = f(\lambda)$ for a $\lambda \in G_1 \setminus \overline{H}_1$. Then we have

$$\lim_{j\to\infty} |u(f(\lambda)) - u(P_j f(\lambda))| = 0. \tag{4}$$

Since X_1 is a weakly closed subspace, from (4) and $P_j f(\lambda) \in X_1$ we have

$$f(\lambda) \in X_1 \quad \text{for all } \lambda \in G_1 \setminus \overline{H}_1.$$

Because D is connected, by the Hahn-Banach theorem and the identity theorem of analytic function, we have

$$f(\lambda) \in X_1 \quad \text{for all } \lambda \in D. \tag{5}$$

By a similar argument we can find $X_2 \in \mathrm{Lat}(T)$ with $\sigma(T|X_2) \subseteq G_2$ and

$$f(\lambda) \in X_2 \quad \text{for all } \lambda \in D. \tag{6}$$

Then by (5) and (6)

$$f(\lambda) \in X_1 \cap X_2 \quad \text{for all } \lambda \in D.$$

Since G_1 is disc and $\sigma(T|X_1 \cap X_2)$ can be partitioned by the bounded components of $\rho(T|X_1)$, $\sigma(T|X_1 \cap X_2) \subseteq G_1$. Also we obtain $\sigma(T|X_1 \cap X_2) \subseteq G_2$. Since $G_1 \cap G_2 = \emptyset$, we have

$$X_1 \cap X_2 = \{0\}.$$

Hence $f = 0$ on D, this completes the proof. $\square$

THEOREM 3.3. *If $T \in \mathcal{L}(X)$ is weakly well-decomposable, then $X_T(F)$ is closed for every closed subset F of $\mathbb{C}$.*

Proof. Let F be a closed subset of $\mathbb{C}$. For each $\lambda \notin F$ we define

$$G_\lambda = \{\mu \in \mathbb{C} : |\mu - \lambda| < \frac{1}{2}\mathrm{dist}(\lambda, F)\},$$

$$H_\lambda = \{\mu \in \mathbb{C} : |\mu - \lambda| > \frac{1}{3}\mathrm{dist}(\lambda, F)\}.$$

Clearly $\{G_\lambda, H_\lambda\}$ covers $\mathbb{C}$, hence by the weak well-decomposability of T there exist $Y_\lambda, Z_\lambda \in \mathrm{Lat}(T)$ and $\langle P_j \rangle$, $\langle Q_j \rangle$ in $\mathcal{I}(T)$ such that

$$\sigma(T|Y_\lambda) \subseteq G_\lambda, \quad \sigma(T|Z_\lambda) \subseteq H_\lambda$$

and for each $x \in X$, $u \in X^*$

$$(7) \qquad \lim_{j\to\infty} |u(x - P_j x - Q_j x)| = 0, \quad P_j(X) \subseteq Y_\lambda \text{ and } Q_j(X) \subseteq Z_\lambda.$$

For a given $x \in X_T(F)$ and for each $j \in \mathbb{N}$ we want to show that $P_j x = 0$. Since $P_j \in \mathcal{I}(T)$, there is some $n \in \mathbb{N}$ such that $C(T)^n P_j = 0$. Let $x(\lambda)$ be the X-valued analytic function on $\rho_T(x)$ satisfying

$$(8) \qquad (T - \lambda)x(\lambda) \equiv x.$$

Define $y(\lambda) : \rho_T(x) \to X$ by

$$(9) \qquad y(\lambda) = \sum_{k=0}^{n-1} (-1)^k C(T)^k P_j \frac{x^{(k)}(\lambda)}{k!}.$$

Clearly $y(\lambda)$ is X-valued and analytic on $\rho_T(x)$. We shall show that

$$(T - \lambda)y(\lambda) \equiv P_j x \quad \text{on } \rho_T(x).$$

For this, if we differentiate the identity (8) k-times, then we have

$$(10) \qquad (T - \lambda)x^{(k)}(\lambda) = kx^{(k-1)}(\lambda) \quad \text{for all } \lambda \in \rho_T(x).$$

Using the relation (8), (9) and (10), we obtain

$$
\begin{aligned}
(T-\lambda)y(\lambda) &= \sum_{k=0}^{n-1}(-1)^k(T-\lambda)C(T)^kP_j\frac{x^{(k)}(\lambda)}{k!}\\
&= \sum_{k=0}^{n-1}(-1)^kTC(T)^kP_j\frac{x^{(k)}(\lambda)}{k!} - \sum_{k=0}^{n-1}(-1)^kC(T)^kP_j\frac{\lambda x^{(k)}(\lambda)}{k!}\\
&= \sum_{k=0}^{n-1}(-1)^kC(T)^{k+1}P_j\frac{x^{(k)}(\lambda)}{k!} + \sum_{k=0}^{n-1}(-1)^kC(T)^kP_jT\frac{x^{(k)}(\lambda)}{k!}\\
&\qquad - \sum_{k=0}^{n-1}(-1)^kC(T)^kP_j\frac{\lambda x^{(k)}(\lambda)}{k!}\\
&= \sum_{k=0}^{n-1}(-1)^kC(T)^{k+1}P_j\frac{x^{(k)}(\lambda)}{k!} + \sum_{k=0}^{n-1}(-1)^kC(T)^kP_j(T-\lambda)\frac{x^{(k)}(\lambda)}{k!}\\
&= \sum_{k=0}^{n-1}(-1)^kC(T)^{k+1}P_j\frac{x^{(k)}(\lambda)}{k!} + P_j(T-\lambda)x(\lambda)\\
&\qquad + \sum_{k=1}^{n-1}(-1)^kC(T)^kP_j(T-\lambda)\frac{x^{(k)}(\lambda)}{k!}\\
&= \sum_{k=0}^{n-1}(-1)^kC(T)^{k+1}P_j\frac{x^{(k)}(\lambda)}{k!} + P_j(x)\\
&\qquad + \sum_{k=1}^{n-1}(-1)^kC(T)^kP_j\frac{x^{(k-1)}(\lambda)}{(k-1)!}\\
&= P_j(x).
\end{aligned}
$$

So we have

$$\sigma_T(P_jx) \subseteq \sigma_T(x).$$

But by Lemma 2.3 we obtain $\sigma_T(P_jx) \subseteq G_\lambda$. Hence

$$\sigma_T(P_jx) \subseteq \sigma_T(x) \cap G_\lambda \subseteq F \cap G_\lambda = \emptyset.$$

Therefore, we have

$$P_jx = 0 \quad \text{for all } x \in X_T(F) \text{ and } j \in \mathbb{N}. \tag{11}$$

Thus for each $x \in X_T(F)$ and $u \in X^*$, (7) and (11) imply

$$(12) \qquad \lim_{j\to\infty} |u(x) - u(Q_j x)| = 0.$$

Since $Q_j x \in Z_\lambda$ and Z_λ is a weakly closed subspace, (12) implies that $x \in Z_\lambda$. Hence $X_T(F) \subseteq Z_\lambda$. Since λ is an arbitrary element of the complement of F, we obtain

$$(13) \qquad X_T(F) \subseteq \bigcap\{Z_\lambda : \lambda \notin F\}.$$

We will show that the reverse inclusion of (13) holds.
Let $x \in \bigcap\{Z_\lambda : \lambda \notin F\}$. Since $\sigma(T|Z_\lambda) \subseteq H_\lambda$,

$$\sigma_T(x) \subseteq \sigma_{T|Z_\lambda}(x) \subseteq \sigma(T|Z_\lambda) \subseteq H_\lambda \quad \text{for all } \lambda \notin F.$$

Then

$$\sigma_T(x) \subseteq \bigcap\{H_\lambda : \lambda \notin F\} = F.$$

So $x \in X_T(F)$.

Therefore, $X_T(F) = \bigcap\{Z_\lambda : \lambda \notin F\}$, thus, $X_T(F)$ is closed. $\square$

THEOREM 3.4. *If $T \in \mathcal{L}(X)$ is weakly well-decomposable and has no non-trivial divisible subspaces, then*

$$X_T(F) = E_T(F) \quad \textit{for each closed subset } F \textit{ of } \mathbb{C}.$$

Proof. Let F be a given closed subset of $\mathbb{C}$. Since T has the single-valued extension property, the inclusion $X_T(F) \subseteq E_T(F)$ is clear. Thus we need only to prove $E_T(F) \subseteq X_T(F)$. Since $X_T(\cdot)$ preserves countable intersection, it suffices to prove that $E_T(F)$ is contained in $X_T(\overline{V})$, where V denotes an arbitrary open neighborhood of F. We choose an open subset U of $\mathbb{C}$ such that $F \subseteq U \subseteq \overline{U} \subseteq V$. Then $\{V, \mathbb{C}\setminus\overline{U}\}$ is an open cover of $\mathbb{C}$. Since T is weakly well-decomposable, there are sequences $\langle P_j\rangle$ and $\langle Q_j\rangle$ in $\mathcal{I}(T)$ such that

$$P_j + Q_j \to I\,(\mathbf{WOT})$$

and there are $Y, Z \in \mathrm{Lat}(T)$ such that

$$(14) \qquad \sigma(T|Y) \subseteq V \subseteq \overline{V}, \quad \sigma(T|Z) \subseteq \mathbb{C}\setminus\overline{U} \subseteq \mathbb{C}\setminus U$$

and

$$P_j(X) \subseteq Y, \ \ Q_j(X) \subseteq Z \ \text{ for all } j \in \mathbb{N}.$$

By Lemma 2.3 and (14) we have

$$P_j(X) \subseteq X_T(\overline{V}), \ \ Q_j(X) \subseteq X_T(\mathbb{C} \setminus U). \tag{15}$$

Let M be the maximal subspace of $X_T(\mathbb{C} \setminus U)$ such that

$$(T - \lambda)M = M \ \text{ for all } \lambda \notin F.$$

Then by Proposition 2.4, we have

$$\sigma(T | X_T(\mathbb{C} \setminus U)) \subseteq \mathbb{C} \setminus U \subseteq \mathbb{C} \setminus F.$$

Hence the maximality of the space M implies that M is actually T-divisible. Therefore, $M = \{0\}$ by our assumption on T.

Now we shall prove that $E_T(F) \subseteq X_T(\overline{V})$. For this, it is enough to show that $Q_j(E_T(F)) = \{0\}$ for all $j \in \mathbb{N}$. Indeed, for each $x \in E_T(F)$, from the fact $(P_j + Q_j)x \to x$ weakly in X and (15), we obtain $x \in X_T(\overline{V})$ provided that $Q_j(E_T(F)) = \{0\}$ for all $j \in \mathbb{N}$. We will prove that $Q_j(E_T(F)) = \{0\}$. Given Q_j there is an $n \in \mathbb{N}$ such that $C(T)^n Q_j = 0$. Consider the algebraic linear subspace

$$W := Q_j(E_T(F)) + TQ_j(E_T(F)) + \cdots + T^{n-1} Q_j(E_T(F))$$

of X. Since $Q_j(E_T(F)) \subseteq X_T(\mathbb{C} \setminus U)$ and since the latter space is invariant under T, we conclude that $W \subseteq X_T(\mathbb{C} \setminus U)$. Moreover, from $C(T)^n Q_j = 0$ we deduce that W is invariant under T. In particular, it follows that $(T - \lambda)W \subseteq W$ holds for all $\lambda \notin F$. In order to prove the opposite inclusion $W \subseteq (T - \lambda)W$, let $\lambda \notin F$ and $w \in W$ be arbitrarily given. Then we have

$$w = \sum_{k=0}^{n-1} T^k Q_j a_k \ \text{ for suitable } a_0, a_1, \ldots, a_{n-1} \in E_T(F).$$

From simple calculation, the system of linear equations

$$\sum_{i=k}^{n-1} \binom{i}{k} (-\lambda)^{i-k} b_i = a_k \ \text{ for } k = 0, 1, \ldots, n-1$$

has a unique solution $b_0, b_1, \dots, b_{n-1}$ in $E_T(F)$. For this solution we obtain

$$\sum_{i=0}^{n-1}(T-\lambda)^i Q_j b_i = \sum_{i=0}^{n-1}\sum_{k=0}^{i}\binom{i}{k}(-\lambda)^{i-k}T^k Q_j b_i = \sum_{k=0}^{n-1} T^k Q_j a_k = w.$$

Hence $w = (T-\lambda)u + Q_j b_0$ for some $u \in W$. Now, by the defining property of $E_T(F)$, there exists some $c_0 \in E_T(F)$ such that $b_0 = (T-\lambda)^n c_0$. Since

$$0 = C(T)^n Q_j = C(T-\lambda)^n Q_j = \sum_{k=0}^{n}\binom{n}{k}(-1)^k (T-\lambda)^{n-k} Q_j (T-\lambda)^k,$$

we obtain the representation

$$Q_j b_0 = Q_j (T-\lambda)^n c_0 = \sum_{k=0}^{n-1}\binom{n}{k}(-1)^{n+k+1}(T-\lambda)^{n-k} Q_j (T-\lambda)^k c_0$$

and therefore $Q_j b_0 = (T-\lambda)v$ for some $v \in W$. We have shown that $w \in (T-\lambda)W$. Hence W satisfies $(T-\lambda)W = W$ for all $\lambda \notin F$ and consequently $W \subseteq M = \{0\}$. In particular, it follows that $Q_j(E_T(F)) = \{0\}$. Hence we conclude that

$$E_T(F) \subseteq X_T(F).$$

This complete the proof. $\square$

For the weakly well-decomposable operators, this theorem allows us to combine the analytic tools associated with the space $X_T(F)$ with the algebraic tools associated with the space $E_T(F)$.

Let T and S be bounded linear operators on Banach spaces X and Y, respectively. A linear operator $\theta : X \to Y$ is said to be an *intertwining linear operator* (or *intertwiner*) with T and S if $S\theta = \theta T$.

PROPOSITION 3.5. *Assume that $T \in \mathcal{L}(X)$ has the single-valued extension property and that a weakly well-decomposable operator $S \in$*

$\mathcal{L}(Y)$ has no non-trivial divisible subspaces. Then every linear transformation $\theta : X \to Y$ with the property $S\theta = \theta T$ necessarily satisfies the following:

$$\theta X_T(F) \subseteq Y_S(F) \quad \textit{for all closed subsets } F \textit{ of } \mathbb{C}.$$

Proof. Since $X_T(F) \subseteq E_T(F)$,

$$\theta X_T(F) \subseteq \theta E_T(F) = \theta(T - \lambda)E_T(F) = (S - \lambda)\theta E_T(F)$$

for every $\lambda \in \mathbb{C} \setminus F$. This shows that $\theta E_T(F) \subseteq E_S(F)$ and since $E_S(F) = Y_S(F)$, by Theorem 3.4, the proof is complete. □

There are two results implying the existence of discontinuous intertwining linear operators which are presented below as Proposition 3.6 and Theorem 3.7. To discuss these results, we need some definitions. Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$. A complex number $\lambda \in \mathbb{C}$ is said to be a *critical eigenvalue* of the pair (T, S) if $(T - \lambda)X$ is of infinite codimension in X and λ is an eigenvalue of S. An operator T is called *algebraic* if there is a non-zero polynomial p such that $p(T) = 0$. From Caley-Hamilton theorem every linear operator on a finite dimensional space is algebraic. It is easy to see that T is algebraic if and only if the spectrum of T consists of a finite number of eigenvalues [Au].

The following proposition is found in [JS] but we include the proof of the proposition for convenience.

PROPOSITION 3.6. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$. If (T, S) has a critical eigenvalue, then there is a discontinuous linear operator $\theta : X \to Y$ with $S\theta = \theta T$.*

Proof. Let μ be a critical eigenvalue of (T, S). Since $X/(T - \mu)X$ is of infinite dimension, we can find a discontinuous linear functional f on X such that $f((T - \mu)X) = \{0\}$. Let $y \neq 0$ be a μ-eigenvector of S in Y, and let $\theta : X \to Y$ be defined by $\theta(x) = f(x)y$ for all $x \in X$. Then θ is discontinuous and $\theta(T - \mu I) = (S - \mu I)\theta = 0$ and so $S\theta = \theta T$. □

The following theorem is proved in [Si74] but the proof is not simple and we do not present here.

THEOREM 3.7. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$. If T is not algebraic, and if Y has a non-trivial S-divisible subspace, then there is a discontinuous linear operator $\theta : X \to Y$ satisfying $S\theta = \theta T$.*

THEOREM 3.8. *Suppose that $T \in \mathcal{L}(X)$ is decomposable and that $S \in \mathcal{L}(Y)$ is weakly well-decomposable. Then the following assertions are equivalent:*

(a) *Every linear operator $\theta : X \to Y$ for which $\theta T = S\theta$ is necessarily continuous.*

(b) *The pair (T, S) has no critical eigenvalues, and either T is algebraic or S has no non-trivial divisible subspaces.*

Proof. (a) $\Rightarrow$ (b) By Proposition 3.6 and Theorem 3.7, it is clear.

(b) $\Rightarrow$ (a) Assume that the condition (b) is fulfilled, and consider an arbitrary linear operator $\theta : X \to Y$ satisfying $S\theta = \theta T$. To prove the continuity of θ, it suffices to construct a non-trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$. Indeed if we do so, all injective factors $S - \lambda$ of $p(S)$ may be removed from $p(S)$; what is left still annihilate $\mathfrak{S}(\theta)$. Thus we have obtained a polynomial p, all of whose roots are eigenvalues of S. Let λ be a root of p. Since (T, S) has no critical eigenvalues, $(T - \lambda)X$ is of finite codimension in X. This means that $p(T)X$ is of finite codimension in X. Hence the open mapping theorem implies that $p(T)X$ is closed and that $p(T)$ is an open mapping from X onto $p(T)X$. Since $p(S)\mathfrak{S}(\theta) = \{0\}$, by Lemma 2.7, $p(S)\theta = \theta p(T)$ is continuous on X, and hence θ is continuous.

Now, if T is algebraic, we choose a non-zero polynomial p satisfying $p(T) = 0$ and observe that

$$p(S)\mathfrak{S}(\theta) \subseteq p(S)\overline{\theta(X)} \subseteq \overline{\theta(p(T)(X))} = \{0\}.$$

Hence θ is continuous.

It remains to consider the case that S has no non-trivial divisible subspaces. From Proposition 3.5, we infer that $\theta X_T(F) \subseteq Y_S(F)$ for all closed subsets F of $\mathbb{C}$. Since $X_T(F)$ is the spectral capacity and $Y_S(F)$ is stable, by Lemma 2.8, there is a finite set Λ of $\mathbb{C}$ such that $\mathfrak{S}(\theta) \subseteq Y_S(\Lambda)$. An application of the Stability Lemma to the sequence $T - \lambda$, where $\lambda \in \Lambda$, yields a polynomial p for which

$$\mathfrak{S}(\theta p(T)) = \mathfrak{S}(\theta p(T)(T - \lambda)) \quad \text{for every } \lambda \in \Lambda.$$

Since θ intertwines T and S, this means that by Lemma 2.7

$$((S-\lambda)p(S)\mathfrak{S}(\theta))^- = (p(S)\mathfrak{S}(\theta))^- \quad \text{for every } \lambda \in \Lambda.$$

Applying Mittag-Leffler Theorem, there exists a dense subspace $W \subseteq (p(S)\mathfrak{S}(\theta))^-$ for which $(S-\lambda)W = W$ for every $\lambda \in \Lambda$. This means that $W \subseteq E_S(\mathbb{C} \setminus \Lambda)$ by the definition of algebraic spectral subspaces. Since $W \subseteq \mathfrak{S}(\theta) \subseteq E_S(\Lambda)$, we obtain that

$$W \subseteq E_S(\Lambda) \cap E_S(\mathbb{C} \setminus \Lambda) = E_S(\emptyset).$$

Hence $W = \{0\}$, by the assumption on S. Consequently, $p(S)\mathfrak{S}(\theta) = \{0\}$. Hence θ is continuous. So the proof is complete. □

Given $a \in \mathbb{R} \setminus \{0\}$ and a function $f : \mathbb{R} \to \mathbb{C}$, the *shift operator* T_a is defined as usual by $(T_a f)(t) := f(t-a)$.

COROLLARY 3.9. *Let $p, q \in [1, \infty)$ and consider a linear operator $\theta : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ such that $T_a\theta = \theta T_a$ for some $a \in \mathbb{R} \setminus \{0\}$. Then θ is automatically continuous.*

Proof. It is well known that the shift operator T_a has no eigenvalues. Define a map $\Phi : C^\infty(\mathbb{C}) \to \mathcal{L}(L^p(\mathbb{R}))$ by

$$\Phi(f) := \sum_{n=-\infty}^{\infty} \widehat{f}(n) {T_a}^n \quad \text{for all } f \in C^\infty(\mathbb{C}),$$

where $\widehat{f}(n)$ denotes the n-th Fourier coefficient of the restriction of f to the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Since $\|{T_a}^k\| = 1$ for all $k \in \mathbb{Z}$ and $\widehat{f}(n) = o(n^{-k})$ as $|n| \to \infty$ for any $k \in \mathbb{N}$, Φ is well defined and Φ is a continuous algebra homomorphism for which $\Phi(1) = I$ and $\Phi(z) = T_a$. Hence T_a is a generalized scalar operator. In particular, T_a is weakly well-decomposable. Moreover, by Remark 2.6, T_a has no non-trivial divisible subspaces. Hence the continuity of θ follows from Theorem 3.8. □

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