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ON COMBINATORICS OF KONHAUSER POLYNOMIALS

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1. Introduction

Let L be a linear functional on the vector space of polynomials in x. Let $w(x)$ be a polynomial in x of degree d, for some positive integer d. We consider two sets of polynomials, $\{R_n(x)\}_{n\geq 0}$, $\{S_n(x)\}_{n\geq 0}$, such that $R_n(x)$ is a polynomial in x of degree n and $S_n(x)$ is a polynomial in $w(x)$ of degree n. (So $S_n(x)$ is a polynomial in x of degree dn.) These two sets of polynomials are said to be biorthogonal with respect to a linear functional L if

$$L(R_m(x)S_n(x)) = 0 \text{ if and only if } m \neq n.$$

DEFINITION 1.1. Polynomials $\{R_n(x)\}_{n\geq 0}$, $\{S_n(x)\}_{n\geq 0}$ satisfying the above condition are called biorthogonal polynomials.

If $w(x) = x$ then biorthogonal polynomials become ordinary orthogonal polynomials [4]. If the linear functional L is given by the integral with respect to a weight function $\mu(x)$ over an interval $[\alpha, \beta]$, the above orthogonality condition can be written as

$$\int_\alpha^\beta R_m(x)S_n(x)\mu(x)\,dx = 0 \text{ if and only if } m \neq n.$$

When $w(x) = x^d$ and the linear functional L is given by the weight function $x^a e^{-x}$ on $(0, \infty)$, which is the weight function of Laguerre

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polynomials, the biorthogonal polynomials determined by L are called Konhauser polynomials, denoted $\{Y_n^{(a)}(x)\}_{n\geq 0}$ and $\{Z_n^{(a)}(x)\}_{n\geq 0}$, where

$$Y_n^{(a)}(x,d) = (-1)^n \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (a+s;d)_n$$

$$Z_n^{(a)}(x,d) = (-1)^n \sum_{t=0}^{n} (-1)^t \binom{n}{t} (a+dt)_{dn-dt} x^{dt},$$

where $(a;k)_n = \prod_{i=0}^{n-1}(a+ik)$. Konhauser polynomials are introduced in [2], [8], [9].

Combinatorial properties of orthogonal polynomials are studied extensively by many people. Laguerre, Hermite, Charlier, and their q-versions are given combinatorial interpretations [5], [6], [10], [12], [13], [14].

A combinatorial model for general biorthogonal polynomials based on the recurrence relations is given in [7]. The model, however, does not contain 'good' structures, because it depends on the recurrence relations alone.

The purpose of this paper is to give a combinatorial interpretation of Konhauser polynomials, proving their orthogonality by using a weight-preserving sign-reversing involution. Note that an involution ψ defined on a weighted set S with weight function ω is called weight-preserving sign-reversing if $\omega(\psi(\alpha)) = -\omega(\alpha)$ for all $\alpha \in S$ s.t. $\omega(\alpha) \neq \alpha$. This paper uses the following well-known lemma several times.

LEMMA 1.2. *If ψ is a weight-preserving sign-reversing involution on S and F is the set $\{\alpha \in S \mid \psi(\alpha) = \alpha\}$, then*

$$\sum_{\alpha\in S} \omega(\alpha) = \sum_{\alpha \in F} \omega(\alpha).$$

Proof. We partition S into orbits of ψ, i.e. into distinct $\{\alpha, \psi(\alpha)\}$'s. If $\psi(\alpha) \neq \alpha$, then $\omega(\{\alpha, \psi(\alpha)\}) = \omega(\alpha) + \omega(\psi(\alpha)) = 0$. Hence the sum in the left leaves only the orbits consisting of one element, i.e. $\{\alpha\}$'s with $\psi(\alpha) = \alpha$.

In this paper, $(a;k)_n$, $(a)_n$, $[a]_n$ denote $\prod_{i=0}^{n-1}(a+ik)$, $\prod_{i=0}^{n-1}(a+i)$, $\prod_{i=0}^{n-1}(a-i)$ respectively. Note that $[a]_n=(-1)^n(-a)_n$. We also use the notation $[n]$ for the set $\{1,2,\ldots,n\}$.

2. Konhauser polynomials

The weight function for Konhauser polynomials is $\mu(x)=x^a e^{-x}$ and the n-th moment $m_n=(a)_n$. It is known [3] that $Y_n^{(a)}(x,d)$ has a generating function

$$\sum_{n=0}^{\infty} Y_n^{(a)}(x,d)w^n=(1-w)^{-(1+a)/d}e^{x-x(1-w)^{-1/d}}$$

The biorthogonality relation is

$$\langle Y_m^{(a)}(x,d), Z_n^{(a)}(x,d)\rangle=\begin{cases} d^m m!\,(a)_{dm} & \text{if } m=n\\ 0 & \text{otherwise}\end{cases}$$

where $\langle f,g\rangle$ denotes the integral $\int_0^\infty f(x)g(x)x^a e^{-x}\,dx$.

We begin with some properties of Konhauser polynomials. Let

$$L_n^{(a)}=(-1)^n\sum_{i=0}^{n}(-1)^i\binom{n}{i}(a+i)_{n-i}x^i.$$

$L_n^{(a)}$ is the Laguerre polynomial, which is equal to $Y_n^{(a)}(x,1)$ or $Z_n^{(a)}(x,1)$ [4].

PROPOSITION 2.1. $Y_n^{(a)}(x,d)=\sum_{i=0}^{n}c_iL_i^{(a)}$ *where* c_i *is the following expression independent of* a*:*

$$c_i=(-1)^{n+i}\sum_{t=0}^{i}\frac{(-1)^t(-t;d)_n}{t!\,(i-t)!}$$

Proof. Since $i!(a)_ic_i=\langle Y_n^{(a)}(x,d),L_i^{(a)}\rangle$, we get

$$\begin{aligned}c_i=&\frac{(-1)^{n+i}}{i!\,(a)_i}\sum_{t=0}^{i}(-1)^t\binom{i}{t}(a+t)_{i-t}\sum_{r=0}^{n}\frac{(a)_{t+r}}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}(a+s;d)_n\\ =&\frac{(-1)^{n+i}}{i!}\sum_{t=0}^{i}(-1)^t\binom{i}{t}\sum_{r=0}^{n}\frac{(a+t)_r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}(a+s;d)_n\end{aligned}$$

We can evaluate the inside double sum by using the difference identity [p. 103][1],

$$F(x)=\sum_{k\geq 0}\frac{[x]_k}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}F(i).$$

We choose $F(x)=(a+x;d)_n$. and evaluate at $x=-a-t$. Since $(a+t)_n=(-1)^n[-a-t]_n$, we get the expression for c_i.

PROPOSITION 2.2. *$Z_n^{(a)}(x,d)n!d=\sum_{i=n}^{dn} b_iL_i^{(a)}$ where b_i is the following expression:*

$$b_i=\frac{(-1)^{n+i}(a)_{dn}}{i!\,(a)_i}\sum_{t=0}^{n}\frac{(-n)_t(-dt)_i}{t!}$$

Proof. This is similar to the previous proof and is omitted.

To interpret Konhauser polynomials combinatorially, we first interpret moments $(a)_n$ as in [Prop. 1.3.4, p. 19], 11], and generalize it to interpret $(a+s;d)_n$. Any permutation of $[n]$ can be represented uniquely as a product of disjoint cycles. For a permutation σ of $[n]$, define ρ and τ as follows:

$$\rho(\sigma)=\text{the number of cycles in }\sigma$$
$$\tau(\sigma)=n-(\text{the number of cycles in }\sigma).$$

LEMMA 2.3. *Let S_n be the permutations of $[n]=\{1,2,\ldots,n\}$. Assume that permutations are represented by disjoint cycles. Then*

$$(a)_n=\sum_{\sigma\in S_n}a^{\rho(\sigma)}.$$

Proof. We use induction on n. When $n=1$, it is obvious. Assume that it holds for $n-1$. We divide the set S_n into two groups: one group contains permutations σ where n forms a cycle of length 1, i.e. $\sigma(n)=n$, and the other contains permutations where n is contained in a cycle with at least two numbers, i.e. $\sigma(n)\neq n$. In the latter case, there are $n-1$ choices for $\sigma(n)$. Since $(a)_n=(a)_{n-1}(a+n-1)=(a)_{n-1}\cdot a+(a)_{n-1}\cdot(n-1)$, it holds for n.

LEMMA 2.4.
$$(a;d)_n = \sum_{\sigma \in S_n} a^{\rho(\sigma)} d^{\tau(\sigma)}.$$

Proof. We use induction on n as in the previous proof. When we add n to a permutation of $[n-1]$, if n forms a cycle by itself, then a factor a is multiplied and if n is added to a cycle, then a factor d is multiplied. So it agrees with the product $(a;d)_n = (a;d)_{n-1}(a + d(n-1))$.

We consider an element of S_n as a labeled directed graph, consisting of directed cycles. For example, a permutation $(1\,2)(3\,4\,5)(6\,7\,8\,9)$ in disjoint cycle notation can be represented by the following directed graph:

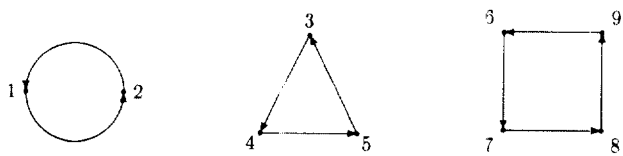

Figure 1: A directed cycle: $(1\,2)(3\,4\,5)(6\,7\,8\,9)$

To deal with $(a+s;d)_n$ for a nonnegative integer s, we consider more general objects for which labeled directed cycles are a special case. Let n, s be nonnegative integers. Consider $\sigma \in S_n$ as a set of directed cycles. We imagine that we have s labeled boxes. Put some of the cycles of σ in some boxes. Define $U_{n,s}$ to be the set of these objects. An element of $U_{14,2}$ is shown below:

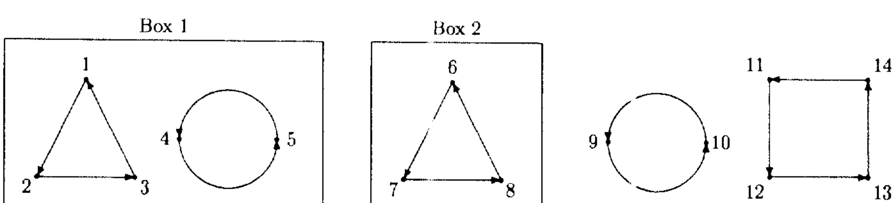

Figure 2: An element of $U_{14,2}$

As we can see, if we ignore boxes then any element of $U_{n,s}$ is an element of S_n, so that we can define the weight on elements of $U_{n,s}$ to be the weight of elements after ignoring the boxes. A precise definition of weight is given below. We show as before that:

LEMMA 2.5.

$$(a+s;d)_n = \sum_{\sigma \in U_{n,s}} a^{\rho(\sigma')} d^{\tau(\sigma'')},$$

where σ' denotes the object after deleting boxes and their contents, and σ'' denotes the set of cycles in the object. (The element in Figure 2 has weight a^2d^9.)

Proof. Similar to the previous case.

We now interpret $Y_n^{(a)}(x,d)$ as a generating function of some objects. For convenience' sake, we first interpret $(-1)^n n! Y_n^{(a)}(x,d)$ instead of $Y_n^{(a)}(x,d)$.

DEFINITION 2.6. For a permutation π of a subset of $[n]$, let $\hat{\pi}$ denote the underlying set of π, i.e. subset consisting of elements in π. Let A_n be the set of all ordered pairs (π, σ), where π is a permutation of a subset of $[n]$ and σ is an element in $U_{n,s}$ for some s, $0 \le s \le n - |\hat{\pi}|$, where the boxes in σ are labeled by some integers in $[n] \setminus \hat{\pi}$. The weight of (π, σ) is given by $(-1)^s x^{n-|\hat{\pi}|} a^{\rho(\sigma')} d^{\tau(\sigma'')}$, where σ' denotes the object after deleting boxes and their contents, and σ'' denotes the set of cycles in the object.

LEMMA 2.7. *The weight generating function of A_n is $(-1)^n n! Y_n^{(a)}(x,d)$, i.e.*

$$Y_n^{(a)}(x,d) = \frac{(-1)^n}{n!} \sum_{r=0}^{n} [n]_{n-r} x^r \sum_{s=0}^{r} (-1)^s \binom{r}{s} (a+s;d)_n$$

Proof. The factor $[n]_{n-r}$ counts permutations π of an r element subset of $[n]$ and $\binom{r}{s}$ counts the number of different sets of labels chosen from $[n] \setminus \hat{\pi}$. So we can show that the sum is the weight generating function of A_n. We omit the details.

The weighted set A_n admits a weight-preserving sign-reversing involution. The involution is described as follows:

> Among labels of empty boxes and the integers which are in $[n] \setminus \hat{\pi}$ but not chosen for labels, one chooses the smallest integer, say k. If k is the label of an empty box, then delete that box and make it 'unchosen', otherwise create an empty box with label k.

This operation fixes an element or changes only the number of boxes in σ by 1, so that it is weight-preserving sign-reversing. From the definition of this operation, it is clearly an involution. The fixed points of this involution are $(\pi, \sigma) \in A_n$ such that σ has no empty box and the number of boxes in σ is equal to $n - |\hat{\pi}|$. So we get the following:

> $(-1)^n n! Y_n^{(a)}(x,d)$ is equal to the weight generating function of the set consisting of (π, σ)'s, where π is a permutation of a subset of $[n]$ and $\sigma \in U_{n,n-|\hat{\pi}|}$ has no empty box and the boxes in σ are labeled by $[n] \setminus \hat{\pi}$.

Since the fixed set consists of all the elements (π, σ) where σ has exactly $n - |\hat{\pi}|$ boxes none of which is empty, we can interpret $(-1)^n Y_n^{(a)}$ (x, d) directly, trimming the factor $n!$ from the above set. If we just ignore π and labels in σ, then we don't have the factor $n!$ and the boxes in σ will be distinguished only by their contents. The precise definition follows:

DEFINITION 2.8. Let Y_n be the set of partitioned permutations. A permutation σ is a partitioned permutation if its cycles are divided into several blocks, at most one of which is empty. We distinguish a block, called the principal block, from other blocks, called secondary blocks. Secondary blocks are assumed to be indistinguishable, i.e. they are distinguished by their contents only. Assume that only the principal block may be empty but none of the secondary blocks can be empty.

The weight of an element in Y_n is $(-x)^k a^l d^m$, where k denotes the number of secondary blocks and l denotes the number of cycles in the principal block and m denotes n minus the number of cycles in all blocks (Figure 3).

THEOREM 2.9. *$(-1)^n Y_n^{(a)}(x, d)$ is equal to the weight generating function of the set Y_n.*

Proof. Let α be an element in Y_n. Suppose α has r secondary blocks. Then we can make $n!$ objects in A_n from α by choosing π, a permutation of $n-r$ elements, and labeling r secondary blocks. These objects are counted by $(-1)^n n! Y_n^{(a)}(x,d)$. Hence $(-1)^n Y_n^{(a)}(x,d)$ is the generating function of Y_n.

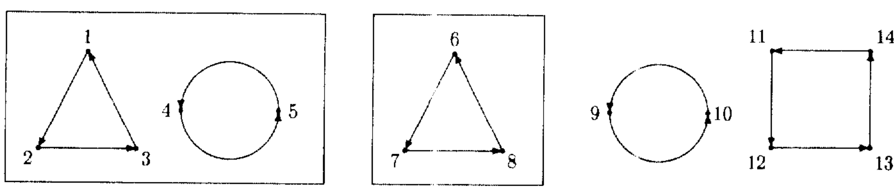

Figure 3: An element of Y_{14} with two secondary blocks with weight $(-1)^2 x^2 a^2 d^9$.

We now express $(-1)^n Z_n^{(a)}(x,d)$ as a weight generating function of a certain weighted set. There are nd colored boxes, d boxes for each color i, $i=1,2,\ldots,n$. We choose t colors and make permutations π with $d(n-t)$ boxes of unchosen colors. Distribute some of the cycles in π among dt boxes of chosen colors. Let Z_n be the set of all these objects. Put a weight ω on $\alpha \in Z_n$ by $\omega(\alpha) = (-1)^t a^l x^{dt}$, where t is the number of the chosen colors and l is the number of cycles outside boxes. Then the weight generating function of Z_n is equal to $(-1)^n Z_n^{(a)}(x,d)$. We draw an element of Z_n below (Figure 4). Let $n=5$ and $d=2$. We use labels $1,2;3,4;5,6;7,8;9,10$ to represent 10 colored boxes, two boxes for each color. Assume that only the second color, Color 2, is chosen, i.e. the boxes 3 and 4 are chosen. We distribute some of the cycles made with $1,2,5,6,7,8,9,10$ into boxes of the chosen colors.

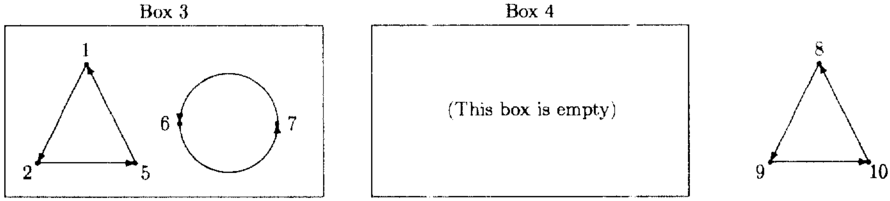

Figure 4: An example of Z_n of weight $-ax^2$ ($n=5$, $d=2$, $t=1$). Color 2 is chosen. Boxes 3 and 4 are two boxes of Color 2.

Using these combinatorial interpretations for $Y_n^{(a)}(x,d)$ and $Z_n^{(a)}(x,d)$ and a suitable weight-preserving sign-reversing involution, one can establish the orthogonality relation of these two polynomials combinatorially.

THEOREM 2.10. *$\{Y_n^{(a)}(x,d)\}_{n\geq 0}$ and $\{Z_n^{(a)}(x,d)\}_{n\geq 0}$ are orthogonal with respect to weight function $\mu(x)=x^a e^{-x}$*

Proof. Let $S_{n,m}$ be the set of all triples (A,B,C) where $A\in Y_n$, $B\in Z_m$, and C is an appropriate disjoint cycle described below. If A has u secondary blocks and B has t chosen colors, then $C\in S_{u+dt}$. C is given the weight defined in Lemma 2.3. Since secondary blocks in A are distinguished by their contents and dt boxes with chosen colors in B are labeled, C is regarded as a permutation of $u+dt$ distinct boxes. Define the weight of (A,B,C) to be the product of the weight of each component. It is clear that the weight of the set $S_{n,m}$ is the integral $\langle Y_n^{(a)}(x,d), Z_n^{(a)}\rangle(x,d)$ [12].

We define an involution ψ on $S_{n,m}$ such that

(1) if $\psi((A,B,C))\neq(A,B,C)$ then $\omega(\psi((A,B,C)))=-\omega((A,B,C))$, i.e. ψ is a weight-preserving sign-reversing involution,
(2) if $n=m$, then the fixed set has weight $d^n n!(a)_{dn}$,
(3) if $n\neq m$, then the fixed set is empty.

Note that the sign of the weight of (A,B,C) is $(-1)^{u+t}$. So the first condition says that if (A,B,C) is not fixed by ψ then the parity of the quantity $u+t$ is changed. We will achieve this objective in seven steps.

Step 1: We partition the set $S_{n,m}$ into $\cup_{u,t}T_{u,t}$ where $T_{u,t}$ denotes the set of all elements $(A,B,C)\in S_{n,m}$ such that A has u secondary blocks and B has t chosen colors. The weight of $T_{u,t}$ has a factor $(a+dt)_{dm-dt}(a)_{u+dt}$, which can be rearranged to give the factor $(a)_{dm}(a+dt)_u$. We can easily describe this rearrangement combinatorially. We omit this description. After factoring out, we are left with the factor $(a+dt)_u$.

We now regard this as the weight of the set of all objects obtained by distributing cycles in a permutation of $[u]$, representing u secondary blocks in A, into one principal box and t secondary boxes, representing t chosen colors, where the weight of each cycle in the principal box is a and the weight of each cycle in a secondary box is d.

Step 2: The set $T_{u,t}$ can be regarded as the set of all objects (π, E, F) where π is a permutation of dm boxes of all colors and E is an element in Y_n with u secondary blocks and F is an element in the set described above whose weight is $(a+dt)_u$, where $[u]$ represents the u secondary blocks in E and t is the number of chosen colors. We define an involution ψ_1 on $\cup_{u,t} T_{u,t}$ (Figure 5). Let (π, E, F) be an element in $T_{u,t}$. If all m colors are chosen and none of them is empty, i.e. each color is used at least once, then set $\psi_1(\pi, E, F) = (\pi, E, F)$. Otherwise find the least integer k among the labels of $m-t$ unchosen colors in F and the labels of chosen colors which are not used in F. If integer k comes from an unused chosen color, then make color k chosen, otherwise make color k unchosen. Let $\psi_1(\pi, E, F)$ be the result of this change. This is certainly a weight preserving involution and changes the parity of t for elements not in the fixed set of ψ_1.

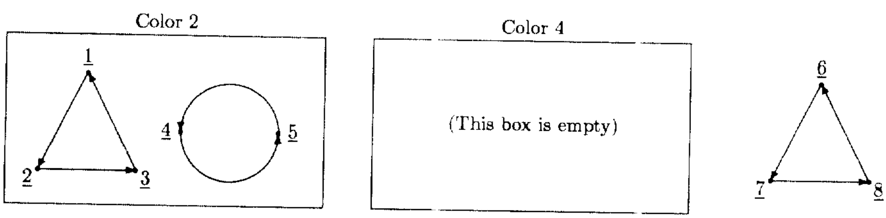

Figure 5: The F component in an element (π, E, F) in $T_{u,t}$. Integers $\underline{1}$ through $\underline{8}$ represent 8 secondary blocks in E. ψ_1 will make Color 1 chosen in this case. ($m=5$, $t=2$, $u=8$)

Step 3: We now examine the fixed set of ψ_1. An element $(\pi, E, F) \in T_{u,t}$ is in the fixed set of ψ_1 iff $t=m$ and all m colors are chosen in F and each of the secondary boxes representing m chosen colors is nonempty. Since the first component π is arbitrary, we let $F_u^{(1)}$ denote the set of all (E, F) such that (π, E, F) is a fixed point of ψ_1 in $T_{u,m}$ for any permutation π of dm boxes. The weight of $S_{n,m}$ is the product of $(a)_{dm}$ and the weight of $F_u^{(1)}$. Recall that E has one principal block and u secondary blocks and F has one principal box and m secondary boxes none of which is empty. We identify the u secondary blocks of E with integers in $[u]$ used in F. Then the element $(E, F) \in F_u^{(1)}$ will be represented as (P, Q, R) where P is the principal block in E which consists of cycles with elements in $[n]$ and Q is the principal box of F

which consists of cycles formed with u secondary blocks in E and R is the set of m secondary boxes of F filled with cycles formed with u secondary blocks in E.

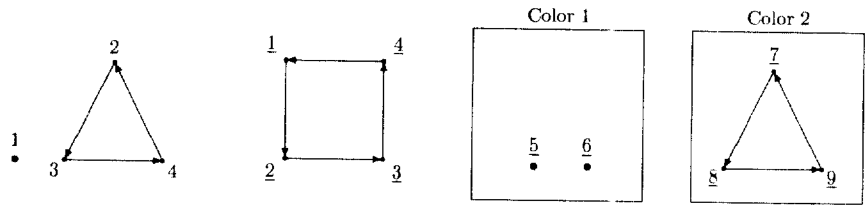

Figure 6: An element (P, Q, R) in $F_u^{(1)}$.
Integers $\underline{1}$ through $\underline{9}$ represent 9 secondary blocks in Q.
Secondary blocks are filled with cycles with elements $\{5, \ldots, 15\}$.
$(m = 2,\ n = 15,\ u = 9)$

Note that we can describe $F_u^{(1)}$ directly (Figure 6). We begin with a permutation of $[n]$. Distribute the cycles of the permutation among one principal block and u secondary blocks. The principal block may be empty but none of the secondary blocks is empty. The secondary blocks are not labeled. They are distinguished by the contents in the blocks. Form a permutation with u secondary blocks. Distribute the cycles of the permutation among the principal box and m secondary boxes. The principal box may be empty but none of the m secondary boxes is empty. Let $F_u^{(1)}$ be the object made in this way. Let P, Q, R denote the principal block, the principal box and the m secondary boxes, respectively. The weight of $(P, Q, R) \in F_u^{(1)}$ is $(-1)^{u+m} a^k d^l$ where k is the sum of the number of cycles in P and the number of cycles in Q, cycles of blocks, and l is the sum of $n - \alpha$ where α is the number of cycles in (P, Q, R) when we ignore boxes and blocks and the number of cycles in R, cycles of u secondary blocks.

Step 4: We now describe an involution ψ_2 on $\cup_u F_u^{(1)}$. Let (P, Q, R) be an element in $F_u^{(1)}$. We order the cycles with $[n]$ by the maximal integer contained in it. We will call a 1-cycle in Q 'lonely' if its only second block consists of exactly one cycle formed with $[n]$. If P is not empty or Q has any 'lonely' 1-cycles, then choose the smallest cycle C among the cycles in P and the cycles contained in secondary blocks in 'lonely' 1-cycles of Q. If C comes from P, then form a new block with

C alone and put it inside Q as a 'lonely' 1-cycle. If C comes from a secondary block in a 'lonely' 1-cycle in Q, then delete the 'lonely' 1-cycle from Q and add C to P as a new cycle. If P is empty and Q has no 'lonely' 1-cycles, then we do nothing. Let the result of this operation be $\psi_2(P, Q, R)$. Let $F_u^{(2)}$ be the set of all element $(P, Q, R) \in F_u^{(1)}$ which are fixed by ψ_2 (Figure 7). It is clear that ψ_2 is an weight preserving involution and the parity of u changes for elements not in the fixed set $\cup_u F_u^{(2)}$.

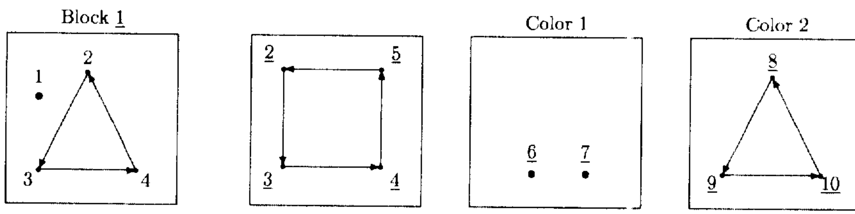

Figure 7: An element $(\emptyset, Q, R)$ in $F_u^{(2)}$.
Integers $\underline{1}$ through $\underline{10}$ represent 10 secondary blocks in Q.
Secondary blocks $\underline{2}$ through $\underline{10}$ are filled with cycles
with elements $\{5, \ldots, 15\}$. ($m = 2$, $n = 15$, $u = 10$)

Step 5: The fixed set $\cup_u F_u^{(2)}$ consists of elements $(P, Q, R) \in \cup_u F_u^{(1)}$ such that P is empty and each 1-cycle of secondary blocks in Q consists of a secondary block which contains more than one cycles formed with $[n]$, i.e. each cycle in Q contains at least two cycles formed with $[n]$. This fixed set is still too large for our purpose. We now define an involution ψ_3 on $\cup_u F_u^{(2)}$. Let (P, Q, R) be an element of $\cup_u F_u^{(2)}$. If there exists a cycle C in Q or R such that either the secondary block containing C has more than one cycles, or C forms a secondary block alone which belongs to a cycle of length greater than one, then choose the largest cycle $\hat{C}$ among such cycles; otherwise define $\psi_3(P, Q, R) = (P, Q, R)$. Let $\hat{U}$ be the secondary block containing $\hat{C}$. If $\hat{U}$ contains only $\hat{C}$, then contract the cycle containing $\hat{U}$ by adding $\hat{C}$ to the next block after $\hat{U}$ in the cycle and deleting the secondary block $\hat{U}$ from the cycle; otherwise delete the cycle $\hat{C}$ from $\hat{U}$ and form a secondary block consisting of $\hat{C}$ alone and insert it in the cycle before the block $\hat{U}$ (Figure 8). Let the result of this operation be $\psi_3(P, Q, R)$. The map

ψ_3 is a weight preserving involution and the parity of u changes for the elements not in the fixed set. Let $F_u^{(3)}$ be the set of all element $(P, Q, R) \in F_u^{(2)}$ which are fixed by ψ_3.

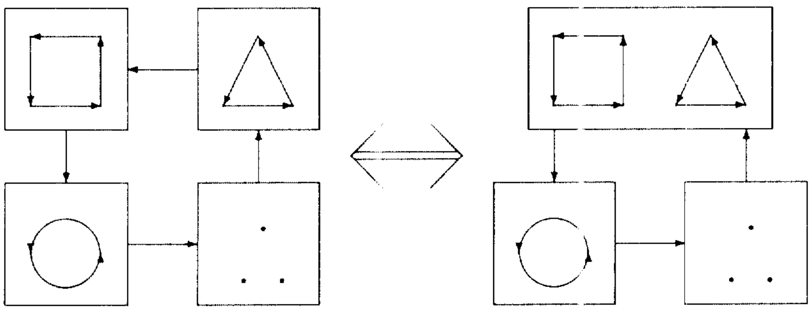

Figure 8: A correspondence for ψ_3. The 3-cycle is assumed to be the largest cycle that can be moved.

Step 6: The fixed set $\cup_u F_u^{(3)}$ consists of (P, Q, R) such that P is empty and each of the cycles in Q and R formed with the secondary blocks is a 1-cycle and each secondary block consists of exactly one cycle. Note that then $(P, Q, R) \in F_u^{(3)}$ implies that Q is empty as well, because if (P, Q, R) is in $\cup_u F_u^{(2)}$ then each cycle of secondary blocks in Q should contain at least two cycles of $[n]$. So $\cup_u F_u^{(3)}$ consists of (P, Q, R) such that P and Q are empty and each of the cycles in R formed with the secondary blocks is a 1-cycle and each secondary block consists of exactly one cycle.

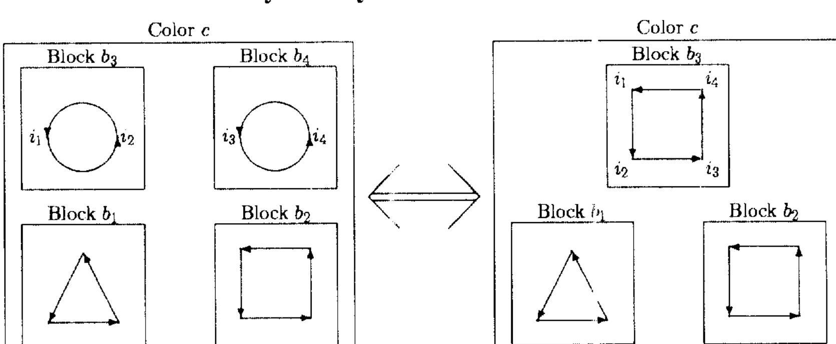

Figure 9: A correspondence for ψ_4. i_1 is the largest and i_3 is the smallest in Color c.

Finally we define one more involution ψ_4 on $\cup_u F_u^{(3)}$. Let $(P,Q,R) \in F_u^{(3)}$. If each of m secondary box consists of exactly one cycle which in turn consists of exactly one cycle of length one, i.e. each secondary box contains exactly one integer, then define $\psi_4(P,Q,R) = (P,Q,R)$; otherwise let α be the largest integer in $[n]$ such that the secondary box containing the integer α contains other integers also. Let Z be the secondary box containing α. Let β be the smallest integer contained in Z. If β is in the cycle containing α, say $(\alpha,\ldots,x,\beta,\ldots,y)$, then we split the cycle into two cycles $(\alpha,\ldots,x)$ and $(\beta,\ldots,y)$ and make two secondary blocks with each of them and make two 1-cycles with each of the two blocks; otherwise we combine two cycles (Figure 9). Let the result of this operation be $\psi_4(P,Q,R)$. The map ψ_4 is a weight preserving involution on $(P,Q,R) \in F_u^{(3)}$ and the parity of u changes for elements not in the fixed set. The fixed set $V_{n,m}$ for ψ_4 consists of all the elements (P,Q,R) such that P and Q are empty and each of m secondary boxes contains exactly one integer chosen from $[n]$. Since each integer in $[n]$ must appear in some secondary box in R, the fixed set is empty unless $n=m$. If $n=m$ then $V_{n,n}$ corresponds to the set of all permutation of $[n]$ where the weight is d^n for each element. So we get

$$\text{the weight of } V_{n,m} = \begin{cases} d^n n!, & \text{if } n=m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Step 7: By combining the steps, we can actually define an involution ψ which satisfies the properties in the beginning of the proof. The weight of the fixed set of ψ, equivalently the weight of $S_{n,m}$, is the product of $(a)_{dm}$ and the weight of $V_{n,m}$. Hence we get

$$\omega(S_{n,m}) = \begin{cases} d^n n!(a)_{dn}, & \text{if } n=m, \\ 0 & \text{if } n \neq m. \end{cases}$$

This shows that $Y_n^{(a)}(x,d)$'s and $Z_n^{(a)}(x,d)$'s are orthogonal.

3. Remarks

In this section we write the results of our experiments done in Maple. Combinatorial interpretations of Laguerre polynomials give the linearization coefficients of the product of $L_n^{(a)}(x)$'s [6]. The polynomials

$y_m^{(a)}(x,d)$'s and $Z_n^{(a)}(x,d)$'s have combinatorial interpretations and the linearization coefficients can be computed. We find, in experiments, that $Z_n^{(a)}(x,d)$'s have interesting linearization coefficients.

Assume that L denote the linear functional for Konhauser polynomials.

- (Conjecture) Let $Z_{n_1}^{(a)}(x,d)Z_{n_2}^{(a)}(x,d)\cdots Z_{n_k}^{(a)}(x,d) = \sum A_n Z_n^{(a)}(x,d)$. Then A_n is a polynomial in a of constant sign
- (Conjecture) Let $Y_{n_1}^{(a)}(x,d)Y_{n_2}^{(a)}(x,d)\cdots Y_{n_k}^{(a)}(x,d) = \sum B_n Y_n^{(a)}(x,d)$. Then B_n is a polynomial in a of constant sign for all $n \geq n_0$, for some n_0. What is n_0? For small n's, even for $k=2$, B_n's are not a polynomial in a of constant sign.
- (Guess) $L(Z_{n_1}^{(a)}(x,d_1)Z_{n_2}^{(a)}(x,d_2)\cdots Z_{n_k}^{(a)}(x,d_k))$ is a polynomial in a with nonnegative coefficients.
- (Guess) $L(Y_{n_1}^{(a)}(x,d_1)Y_{n_2}^{(a)}(x,d_2)\cdots Y_{n_k}^{(a)}(x,d_k))$ is a polynomial in a with nonnegative coefficients, for $d_i \leq 2$.
- (Guess) $L(Y_{n_1}^{(a)}(x,d_1)Y_{n_2}^{(a)}(x,d_2)\cdots Y_{n_k}^{(a)}(x,d_k))$ is a polynomial in a with nonnegative coefficients. Even though we assume that d is a positive integer, we can substitute $-d$ for d since d appears in only the coefficients of the polynomials.

Any proof, analytic or combinatorial, of the above will be interesting. Orthogonality of $Y_n^{(a)}(x,d)$'s and $Z_n^{(a)}(x,d)$'s follows from Propositions 2.1 and 2.2 and the orthogonality of Laguerre polynomials. But the proofs of the propositions are not combinatorial. Combinatorial proofs of them will be interesting.

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