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ROUGH ISOMETRY AND HARNACK INEQUALITY

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§1. Introduction

Certain analytic behavior of geometric objects defined on a Riemannian manifold depends on some very crude properties of the manifold. Some of those crude invariants are the volume growth rate, isoperimetric constants, and the likes. However, these crude invariants sometimes exercise surprising control over the analytic behavior.

Let us take, for example, the celebrated Harnack inequality for positive harmonic function. Moser showed that if a manifold diffeomorphic to $\mathbb{R}^n$ is equipped with a Riemannian metric which is uniformly equivalent to the flat metric, then the Harnack inequality for positive harmonic function is valid [5]. Moser's result can be proved by his famous iteration argument which requires only the volume growth rate, the Sobolev inequality, and the Neumann type Poincaré inequality, all of which are valid for $\mathbb{R}^n$ equipped with the metric quasi-isometric to the flat Euclidean metric.

This line of idea is proved quite useful in geometry. For example, Saloff-Coste exploited it to obtain many interesting results [6]. One major advantage of the quasi-isometry condition is that one does not need any curvature condition. However, in terms of topology of the underlying manifold, the quasi-isometry condition does not yield any more information.

This quasi-isometry condition can be relaxed quite a bit by introducing the following much cruder. hence more powerful, concept introduced by Kanai. A (not necessarily continuous) map $\varphi : X \to Y$

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between two metric spaces X and Y is called a *rough isometry*, if the following conditions hold:

(1) there exists a constant $\tau > 0$ such that

$$Y = B_\tau(\varphi(X)),$$

in which case φ is called τ-full;

(2) there exist constants $a \geq 1$ and $b \geq 0$ such that

$$\frac{1}{a}d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq ad(x_1, x_2) + b$$

for all $x_1, x_2 \in X$, where d denotes the distances of X and Y respectively.

It is easy to check that being roughly isometric is an equivalence relation, but it is also important to note that two roughly isometric metric spaces may have completely different topology, since φ is not assumed to be continuous. For example, an infinite cylinder is roughly isometric to an infinite cylinder with infinitely many identical handles attached at equal distance going off to the infinity. In addition, rough isometry neglects the compact factor of manifold. For example, an infinite cylinder is also roughly isometric to an infinite line.

Nonetheless, it is a remarkable fact that Kanai managed to prove that if a Riemannian manifold M is roughly isometric to $\mathbb{R}^n$, then the Harnack inequality for positive harmonic function on M is valid, hence the Liouville theorem for positive harmonic function holds [4]. The gist of Kanai's argument is his analysis of how the volume growth, the Sobolev inequality, and the Poincaré inequality are preserved under his conditions.

To handle the rough isometry properly, one needs to assume that a complete Riemannian manifold M satisfies the following conditions:

(R) the Ricci curvature of manifold is bounded below by a constant;

and the injectivity radius is positive, i.e., $inj(M) > 0$.

From now, we will also assume that all manifolds satisfy above condition (R).

We now collect relevant definitions and results concerning rough isometry which we need in this paper. They are all in Kanai's papers [3,4], so we supply them without proofs.

One of the key tools in combinatorially approximating a Riemannian manifold M is the concept of net as defined below: Let d be the distance function on M. A subset P of M is called an ε-separated subset for $\varepsilon > 0$, if $d(p, q) \geq \varepsilon$ for any distinct points p and q of P. An ε-separated subset is called maximal ε-separated if it is maximal with respect to the order relation of inclusion. Let P be a maximal ε-separated subset of M, then we can define a net structure $\mathcal{N} = \{N_p \mid p \in P\}$ by setting $N_p = \{q \in P \mid 0 < d(p, q) \leq 2\varepsilon\}$. Note that this family $\mathcal{N}$ satisfies that for all p, $q \in P$,

(i) N_p is a finite subset of P,
(ii) $q \in N_p$ if and only if $p \in N_q$.

A maximal ε-separated subset P of M with the net structure described above is called the ε-net in M.

A sequence $\mathbf{p} = (p_0, \cdots, p_s)$ of points in P is called a path from p_0 to p_s length s if each p_k is an element of $N_{p_{k-1}}$. Then for points p and q of P, we can define $\delta(p, q) =$ the minimum of the lengths of paths from p to q. It is easy to check that this δ defines a metric on P. In [4], Kanai proved that a net P, with this metric δ, is roughly isometric to M, i.e., there exist constants $\alpha \geq 1$ and $\beta \geq 0$ such that

$$\frac{1}{\alpha}\delta(x_1, x_2) - \beta \leq d(x_1, x_2) \leq \alpha\delta(x_1, x_2) + \beta$$

for all $x_1, x_2 \in P$.

A net P is said to be uniform if there exists a constant λ such that $\sharp N_p \leq \lambda < \infty$ for all $p \in P$, where $\sharp S$ denotes the cardinality of the set S. The condition (R) guarantees that an ε-net P on M is uniform and this uniformness plays a crucial role in the proof of the roughly isometric invariance of some analytic properties. In fact, the condition (R) implies that there exists a constant $\nu = \nu(r, \varepsilon, K, m)$ such that $\sharp \{p \in P \mid x \in B_r(p)\} \leq \nu$ for all $r > 0$ and for all $x \in M$, where K is a constant such that the Ricci curvature of M is bounded below by the constant $-(m-1)K^2$, and m is the dimension of M.

§2. Main Results

In this section, we prove the Harnack inequality and the Liouville theorem for positive harmonic function on a wider class of manifolds.

Our result in this section is a generalization of Kanai's in its method as well as its content. However, we modify his approach so that ours is more L^1 oriented than Kanai's L^2. Thus we could remove some minor dimension restriction of his. Also, we carefully choose various sets, especially those in (*) to fit our analysis. This is perhaps one of our main contributions in this section.

First of all, it is well known that the volume growth rate, the Sobolev inequality, and the Neumann type Poincaré inequality for balls imply the Harnack inequality, and hence the Liouville theorem for positive harmonic functions. This point was well exploited by Saloff-Coste [6], although the results in [6] are more on the parabolic case and the elliptic case is a corollary. Kanai's result also relies on the volume growth rate, the Sobolev inequality, and the Neumann type Poincaré inequality.

Kanai's result has one big advantage that his result is applicable even when the topology in the underlying manifold changes, but unfortunately, he had to assume that the manifold has to be roughly isometric to the Euclidean space $\mathbb{R}^n$. This limits the applicability of his result.

In this section, we prove that if a manifold is roughly isometric to a manifold in which appropriate volume growth rate, the Sobolev inequality, and the Neumann type Poincaré inequality are valid, then the Harnack inequality, and hence the Liouville theorem for positive harmonic functions hold. However our volume growth condition and the Sobolev inequality is slightly more general than usual. We now state our conditions on our manifold M:

(V) There exists a constant A such that for any $0 \in M$ and for all $R > 0$,

$$\frac{vol\, B_{2R}(0)}{vol\, B_R(0)} \le A ,$$

where $vol\, B_R(0)$ denotes the volume of the geodesic ball $B_R(0)$ of radius R with center at 0;

(I) there exist an integer $\ell \ge \dim M$ and a constant $C_0 > 0$ such that

$$\inf_{\Omega} \frac{(vol\, \partial\Omega)^{\frac{\ell}{\ell-1}}}{vol\, \Omega} \ge C_0$$

for any bounded domain Ω of M;

(P) there exists a constant $C_1 > 0$ such that for any $R > 0$ and for any point $0 \in M$,

$$\frac{vol\ H}{\min\{vol\ D_1, vol\ D_2\}} \geq \frac{C_1}{R}$$

where H is any hypersurface of $B_{R+\varepsilon}(0)$ dividing $B_{R+\varepsilon}(0)$ into D_1' and D_2', and $D_1 = D_1' \cap B_R(0)$ and $D_2 = D_2' \cap B_R(0)$, and ε is the one chosen for the net of M. Note this slightly general form of Poincaré inequality follows from the standard Poincaré inequality.

In [4], Kanai proved that two Riemannian manifolds which are roughly isometric to each other have the same volume growth rate. From this, it immediately follows that the condition (V) is a rough isometry invariant. Kanai proved that the condition (I) is also rough isometry invariant.

In what follows, M and N are complete Riemannian manifolds of dimensions m and n respectively, and let $\varphi : M \to N$ be a rough isometry between them. For sufficiently small ε and δ, P denotes an ε-net on M and Q a δ-net on N. We use the following notations:

$$S_R = P \cap B_R(0) = \{p \in P \mid d(p, 0) < R\},$$
$$T_R = Q \cap B_\rho(\varphi(S_R)) = \{q \in Q \mid d(q, \varphi(S_R)) < \rho\},$$
$$\Omega_R = \bigcup_{q \in T_R} B_\delta(q),$$

where $\rho > 0$ is a constant to be chosen suitably later.

THEOREM 2.1. *Let M be a complete Riemannian manifold, and let N be an another complete Riemannian manifold roughly isometric to M. Assume M and N satisfy condition (R). Suppose the conditions (V), (I) and (P) are valid for M, where $\ell \geq \max\{\dim M, \dim N\}$. Then with the above notations,*

(i) $\overline{\Omega}_{kR} \subset \Omega_{(k+1)R}$, *and* $N = \bigcup_{k \in \mathbb{N}} \Omega_{kR}$;

(ii) *there exists a constant $C > 0$ such that for any $R > 0$ and for any positive harmonic function u on N, the Harnack inequality holds:*

$$\sup_{\Omega_R} u \leq C \inf_{\Omega_R} u.$$

COROLLARY 2.2. *Let M be a complete Riemannian manifold, and let N be an another complete Riemannian manifold roughly isometric to M. Assume M and N satisfy condition (R). Suppose the conditions (V), (I) and (P) are valid for M, where $\ell \geq \max\{\dim M, \dim N\}$. Then any positive harmonic function on N is constant.*

REMARK. Our results above can be regarded as another vindication of the folklore which says that the polynomial volume growth rate, the Sobolev and Poincaré inequalities give the Harnack inequality. However, our conditions are slightly more general than those in most known results. Thus our result is new even without rough isometry assumption.

COROLLARY 2.3. *Let M be an m-dimensional complete Riemannian manifold with non-negative Ricci curvature. Let N be an n-dimensional complete Riemannian manifold which is roughly isometric to M. Assume $m \geq n$, and M and N satisfy condition (R). Then the Harnack inequality holds on N. Hence any positive harmonic function on N is constant.*

It is perhaps appropriate to comment on the method of proof. Kanai proved that conditions (V) and (I) are rough isometry invariants [4], but condition (P) is more subtle. Since the Neumann eigenvalue is sensitive to the perturbation of the domain and the rough isometry inevitably makes wild distortion of the balls, it is not clear if condition (P) is actually rough isometry invariant. If it were the case, our theorem easily follows from the standard Moser iteration procedure. To cope with this problem, Kanai devised a clever argument to circumvent the problem. Namely, he used an abstract version of John-Nirenberg inequality due to Bombieri and Guisti [1], which does not require the estimate of the eigenvalue of the balls. So we also adopt Kanai's method of proof. In what follows, Lemma 2.1 through Lemma 2.4 can be proved by adopting Kanai's argument. So we omit the proofs except for Lemma 2.2 which merits some explanation. The proof of Lemma 2.5 is substantially different, while this step is the easiest in Kanai's case due to special geometry of $\mathbb{R}^n$. The whole point of our proof is to choose various sets in (*) below to make various quantities fit together.

The following lemma is a consequence of Bombieri and Giusti's abstract John-Nirenberg inequality. To prove it, we need to check that for a positive harmonic function u and for some integer $q \in \mathbb{N}$,

$$\sup_{B_s(\Omega_R)} u^q \le \frac{C}{(r-s)^\ell} \int_{B_r(\Omega_R)} u^q \, dx\,, \quad \text{and}$$

$$\sup_{B_s(\Omega_R)} u^{-q} \le \frac{C}{(r-s)^\ell} \int_{B_r(\Omega_R)} u^{-q} \, dx,$$

where $\delta \ge r > s \ge 0$. But these follow easily by the Moser iteration argument.

LEMMA 2.1. *There exists a constant $C > 0$ such that for any $R > 0$, and for any positive harmonic function u on N,*

$$\sup_{\Omega_R} u \le e^{Cg(u)} \inf_{\Omega_R} u,$$

where

$$g(u) = \sup_{0 \le r \le \delta} \inf_{\alpha \in \mathbb{R}} \frac{1}{vol\, B_r(\Omega_R)} \int_{B_r(\Omega_R)} |\log u - \alpha| \, dx.$$

Thus to prove Theorem 2.1, it is sufficient to show that $g(u)$ is bounded independent of u or R. First, we need to prove the following isoperimetric inequality (2.1) on N, i.e., there exists a positive constant $C = C(\varepsilon, \delta, \rho, K, \dim M, \dim N)$ such that, for any hypersurface H in $B_\delta(\Omega_R)$ which divides $B_\delta(\Omega_R)$ into two domains D_1' and D_2',

$$\frac{vol\ H}{\min\{vol\, D_1, vol\, D_2\}} \ge \frac{C}{R}\,, \tag{2.1}$$

where $D_1 = D_1' \cap \Omega_R$ and $D_2 = D_2' \cap \Omega_R$. Then, by slightly modifying the standard argument using the coarea formula, (2.1) implies that there is a constant $C > 0$ such that for any $R > 0$ and for any $v \in C^\infty(N)$,

$$\int_{B_\delta(\Omega_{2R})} |\nabla v| \, dx \ge \frac{C}{R} \inf_{\alpha \in \mathbb{R}} \int_{\Omega_{2R}} |v - \alpha| \, dx. \tag{2.1$'$}$$

It is obvious that each of $vol\, B_R(0)$, $\sharp S_R$, $\sharp \varphi(S_R)$, $\sharp T_R$, and $vol\, \Omega_R$ has the same order of growth in term of R, which means that the ratios of any two of the above are bounded above and below by constants independent of R.

The following lemma follows from (2.1$'$).

LEMMA 2.2. *There exists a constant $C = C(\varepsilon, \delta, \rho, K, \dim M, \dim N)$ such that for any sufficiently large $R > 0$, and for any positive harmonic function u on N,*

$$g(u) \leq C,$$

where

$$g(u) = \sup_{0 \leq r \leq \delta} \inf_{\alpha \in \mathbb{R}} \frac{1}{vol\, B_r(\Omega_R)} \int_{B_r(\Omega_R)} |\log u - \alpha|\, dx.$$

Proof. Put $v = \log u$. Choose $r \in [0, \delta]$ such that

$$\begin{aligned}
g(u) &= \inf_{\alpha \in \mathbb{R}} \frac{1}{vol\ B_r(\Omega_R)} \int_{B_r(\Omega_R)} |v - \alpha|\, dx \\
&\leq \inf_{\alpha \in \mathbb{R}} \frac{1}{vol\ B_r(\Omega_R)} \int_{\Omega_{2R}} |v - \alpha|\, dx \\
&\leq CR \frac{1}{vol\ B_r(\Omega_R)} \int_{B_\delta(\Omega_{2R})} |\nabla v|\, dx \\
&\leq CR \frac{(vol\ \Omega_{3R})^{\frac{1}{2}}}{vol\ B_r(\Omega_R)} \left(\int_{\Omega_{3R}} |\nabla v|^2 dx \right)^{\frac{1}{2}} \\
&\leq CR \frac{(vol\ \Omega_{3R})^{\frac{1}{2}}}{vol\ B_r(\Omega_R)} \left(\int_{B_R(\Omega_{3R})} |\nabla \eta|^2 dx \right)^{\frac{1}{2}} \\
&\leq \frac{CR}{R} \frac{vol\ B_R(\Omega_{3R})}{vol\ B_r(\Omega_R)} \\
&\leq C\,.
\end{aligned}$$

In the above inequalities, (2.1′), the Hölder inequality, and the volume comparison are used. And η is a Lipschitz function which is defined as below:

$$\eta(x) = \begin{cases} 1\,, & x \in \Omega_{3R} \\ 1 - \frac{1}{R} d(x,\, \Omega_{3R}), & x \in B_R(\Omega_{3R}) \setminus \Omega_{3R} \\ 0\,, & \text{otherwise.} \end{cases}$$

The third inequality from the last is proved as follows: Since $\Delta v = -|\nabla v|^2$,

$$\begin{aligned}\int_{B_R(\Omega_{3R})} \eta^2|\nabla v|^2 dx &= -\int_{B_R(\Omega_{3R})} \eta^2 \Delta v dx \\ &= \int_{B_R(\Omega_{3R})} 2\eta\nabla\eta\cdot\nabla v dx \\ &\le \frac{1}{2}\int_{B_R(\Omega_{3R})} \eta^2|\nabla v|^2 dx + 2\int_{B_R(\Omega_{3R})} |\nabla\eta|^2 dx\,.\end{aligned}$$

Thus

$$\int_{\Omega_{3R}} |\nabla v|^2 dx \le 4\int_{B_R(\Omega_{3R})} |\nabla\eta|^2 dx\,. \qquad \square$$

The only step not justified so far in the proof of Theorem 2.1 is the inequality (2.1′) which was used in the proof of Lemma 2.2. As (2.1′) follows immediately from (2.1), the rest of this section is devoted to the proof of (2.1), which is the objective of Lemmas 2.3, 2.4, and 2.5.

For technical reasons, we shall redefine φ. First, since P and Q are roughly isometric, choose a rough isometry $\psi : P \to Q$. Now since M is roughly isometric to P, there is a rough isometry $\pi : M \to P$. Then $i\circ\psi\circ\pi$ is a rough isometry of M into N, where $i : Q \hookrightarrow N$ is the inclusion map. For the sake of simplicity, we again reset $\varphi = i\circ\psi\circ\pi$. This redefinition has an advantage that the restriction of $\varphi : M \to N$ to P is automatically a rough isometry between P and Q. Suppose that H is a hypersurface in $B_\delta(\Omega_R)$ which divides $B_\delta(\Omega_R)$ into two disjoint domains D_1' and D_2'. And put $D_1 = D_1'\cap\Omega_R$ and $D_2 = D_2'\cap\Omega_R$, then $D_1\cup D_2 = \Omega_R\setminus H$. For such D_1 and D_2, define $T_1 = \left\{q\in T_R \,|\, vol\,(D_1\cap B_\delta(q)) > \frac{1}{2}vol\,B_\delta(q)\right\}$, and $T_2 = \left\{q\in T_R \,|\, vol\,(D_2\cap B_\delta(q)) \ge \frac{1}{2}vol\,B_\delta(q)\right\}$ on Q. Then check $T_1\cup T_2 = T_R$. And put $S_1 = \varphi^{-1}(T_1)\cap S_R$ and $S_2 = \varphi^{-1}(T_2)\cap S_R$ on P. Then $S_R = S_1\cup S_2$. Let us define

$$(*)\qquad \begin{aligned} S_1' &= N_k(S_1)\setminus(N_1(S_2)\setminus S_1),\\ S_2' &= S_{R+2\varepsilon}\setminus S_1',\\ U_1 &= B_\varepsilon(S_1')\cap B_{R+\varepsilon}(0),\\ U_2 &= B_{R+\varepsilon}(0)\setminus\overline{U}_1,\\ \tilde H &= \partial B_\varepsilon(S_1')\cap B_{R+\varepsilon}(0),\end{aligned}$$

where k is an integer such that $\alpha(\rho + 2\varepsilon + \beta) \le k$, and $N_j(S) = \{p \in P \mid \delta(p, S) \le j\}$ for any $j \in \mathbb{N}$. It is easy to check that

$$B_\varepsilon(S_1' \cup S_2') \supset B_{R+\varepsilon}(0), \quad \text{and} \quad B_{R+\varepsilon}(0) \setminus \tilde{H} = U_1 \cup U_2.$$

LEMMA 2.3. *With the above notations, there exists a constant $C = C(\varepsilon, \delta, \rho, K, \dim N)$ satisfying*

$$\min\left\{1,\ \max\left\{\frac{\sharp\, \partial T_1 \cap T_R}{\sharp\, T_1}, \frac{\sharp\, \partial T_2 \cap T_R}{\sharp\, T_2}\right\}\right\} \le C\, \frac{vol\ H}{\min\{vol\ D_1, vol\ D_2\}} .$$

LEMMA 2.4. *Let T_1 and T_2 be non-empty disjoint subsets of T_R. Then there exists a constant $C = C(\varepsilon, \delta, \rho, k, K, \dim M, \dim N)$ such that*

$$\begin{aligned} &\min\left\{1,\ \max\left\{\frac{\sharp\, \partial S_1 \cap S_R}{\sharp\, S_1}, \frac{\sharp\, \partial S_2 \cap S_R}{\sharp\, S_2}\right\}\right\} \\ &\le C \max\left\{\frac{\sharp\, \partial T_1 \cap T_R}{\sharp\, T_1}, \frac{\sharp\, \partial T_2 \cap T_R}{\sharp\, T_2}\right\} \end{aligned}$$

where $S_i = \varphi^{-1}(T_i) \cap S_R$ for $i = 1, 2$.

LEMMA 2.5. *Let the following sets be given as in (*). Then there exists a constant $C = C(\varepsilon, K, \dim M)$ such that*

$$\begin{aligned} &\min\left\{1,\ \max\left\{\frac{vol\, \tilde{H}}{\min\{vol\, U_1, vol\, U_2\}}\right\}\right\} \\ &\le C \max\left\{\frac{\sharp\, \partial S_1 \cap S_R}{\sharp\, S_1}, \frac{\sharp\, \partial S_2 \cap S_R}{\sharp\, S_2}\right\} \end{aligned}$$

Proof. The proof is divided into two cases.

Case 1. $S_2 \setminus \partial S_1 = \emptyset$ and $S_1 \setminus \partial S_2 = \emptyset$. In this case, $S_1 = \partial S_2 \cap S_R$ and $S_2 = \partial S_1 \cap S_R$, and these imply that

$$\frac{\sharp\, \partial S_1 \cap S_R}{\sharp\, S_1} = \frac{\sharp\, S_2}{\sharp\, S_1} \quad \text{and} \quad \frac{\sharp\, \partial S_2 \cap S_R}{\sharp\, S_2} = \frac{\sharp\, S_1}{\sharp\, S_2}.$$

Thus we obtain easily

$$\max\left\{\frac{\sharp\,\partial S_1\cap S_R}{\sharp\,S_1},\ \frac{\sharp\,\partial S_2\cap S_R}{\sharp\,S_2}\right\}\geq 1.$$

Case 2. Consider the case $S_2\setminus\partial S_1\neq\emptyset$. By the standard volume comparison theorem and the uniformness of the net P, we have

$$\begin{aligned}\text{(2.2)}\qquad vol\ U_1 &\geq \sum_{p\in S_1} vol\ \ B_{\frac{\varepsilon}{2}}(p)\\ &\geq C\sharp\,S_1\end{aligned}$$

and

$$\begin{aligned}\text{(2.3)}\qquad vol\ U_2 &\geq \sum_{p\in S_2\setminus\partial S_1} vol\,B_{\frac{\varepsilon}{2}}(p)\\ &\geq C\,\sharp\,(S_2\setminus\partial S_1)\\ &\geq C\,\sharp\,N_1(S_2\setminus\partial S_1)\\ &\geq C\,\sharp\,S_2\,.\end{aligned}$$

Note that $B_\varepsilon(S_1)\subset U_1$ and $N_1(S_2)\setminus S_1\subset U_2$. But for $p\in S_2\cap\partial S_1$, we cannot guarantee that $B_{\frac{\varepsilon}{2}}(p)\subset U_2$. Thus in (2.3), we had to sum over $p\in S_2\setminus\partial S_1$ in the first inequality.

Let $x\in\partial B_\varepsilon(S_1')\cap B_{R+\varepsilon}(0)$. Then there exists $p_0\in S_1'$ such that $x\in\partial B_\varepsilon(p_0)$. Choose $\eta\in\mathbb{R}^+$ such that $B_\eta(x)\subset B_{R+\varepsilon}(0)$. Since $B_{R+\varepsilon}(0)\subset B_\varepsilon(S_1'\cup S_2')$, either every point in $B_\eta(x)$ belongs to $B_\varepsilon(p)$ for some $p\in S_1'$ or there exists $x_\eta\in B_\eta(x)$ such that $x_\eta\notin B_\varepsilon(S_1')$. The former case means that x is an interior point of $B_\varepsilon(S_1')$. Therefore the latter must be true: namely, $d(x_\eta,S_2')<\varepsilon$. Letting $\eta\to 0, x_\eta\to x$. By the compactness of S_2' and the continuity of the distance function from S_2', there exists $q_0\in S_2'$ such that $d(x,q_0)=d(x,S_2')\leq\varepsilon$. From these, $d(p_0,q_0)\leq 2\varepsilon$, i.e., $p_0\in\partial S_2'$. Therefore we have

$$\text{(2.4)}\qquad \partial B_\varepsilon(S_1')\cap B_{R+\varepsilon}(0)\subset(\cup_{p\in\partial S_2'\cap B_{R+2\varepsilon}(0)}\partial B_\varepsilon(p))\cap B_{R+\varepsilon}(0)\ .$$

On the other hand, for $p\in\partial S_2'\cap B_{R+2\varepsilon}(0)$, we can choose points $s\in S_1, r\in S_2$, and $q\in S_2'$ such that $\delta(p,s)\leq k, \delta(q,r)\leq k$, and $\delta(p,q)=1$.

We can also choose a point $u \in \partial S_2 \cap S_R$ such that $\delta(s,u) \leq \delta(s,r) - 1$. Thus $\delta(p,u) \leq 3k$. Therefore $p \in N_{3k}(u)$, i.e.,

$$\partial S_2' \cap B_{R+2\varepsilon}(0) \subset N_{3k}(\partial S_2 \cap S_R) \ . \tag{2.5}$$

By (2.4) and (2.5),

$$\begin{aligned} vol\ \tilde{H} &\leq \sum_{p \in \partial S_2' \cap B_{R+2\varepsilon}(0)} vol\ \partial B_\varepsilon(p) \\ &\leq C \sharp N_{3k}(\partial S_2 \cap S_R) \\ &\leq C \sharp (\partial S_2 \cap S_R). \end{aligned} \tag{2.6}$$

Combining (2.2), (2.3), and (2.6),

(2.7)

$$\begin{aligned} \frac{vol\ \tilde{H}}{\min\{vol\ U_1, vol\ U_2\}} &\leq C \frac{vol\ \tilde{H}}{\min\{\sharp S_1,\ \sharp S_2\}} \\ &\leq C \frac{\sharp \partial S_2 \cap S_R}{\min\{\sharp S_1,\ \sharp S_2\}} \\ &\leq C \max\left\{ \frac{\sharp \partial S_2 \cap S_R}{\sharp S_1},\ \frac{\sharp \partial S_2 \cap S_R}{\sharp S_2} \right\} . \end{aligned}$$

Since $\partial S_2 \cap S_R \subset N_1(\partial S_1 \cap S_R)$, we have

$$\sharp \partial S_2 \cap S_R \leq C \sharp \partial S_1 \cap S_R . \tag{2.8}$$

By (2.7) and (2.8),

$$\frac{vol\ \tilde{H}}{\min\{vol\ U_1, vol\ U_2\}} \leq C \max\left\{ \frac{\sharp \partial S_1 \cap S_R}{\sharp S_1},\ \frac{\sharp \partial S_2 \cap S_R}{\sharp S_2} \right\} .$$

In case $S_1 \setminus \partial S_2 \neq \emptyset$, we reverse the role of S_1 and S_2. Then the proof is the same. □

Appendix to Section 2

We quoted an abstract version of John-Nirenberg lemma due to Bombieri and Giusti in §2. They have an inequality in p.38 of [1] as follows: for $0 \le s < r \le 1$,

(2.9)
$$w(s) \le \frac{15}{16} w(r) + \left(\frac{1}{\theta_0} + \frac{1}{\theta_1}\right) \log \frac{\mu(B_r)}{\mu(B_s)} + \left(\frac{1}{2p_0} + 4A\right) \frac{1}{Q^2 (r-s)^{2\sigma}}.$$

They chose a sequence

$$r_j = \left(1 - \frac{1}{j}\right) r_{j-1}, \quad j = 2, 3, \cdots$$
$$r_1 = 1$$

and applied this to (2.9) with $r = r_{j-1}$ and $s = r_j$. Then by induction,

$$w(r_{N+1}) \le \left(\frac{15}{16}\right)^N w(r_1) + \left(\frac{1}{2p_0} + 4A\right) \frac{1}{Q^2} \sum_{j=1}^{N} \left(\frac{15}{16}\right)^{N-j} j^{4\sigma}$$
$$+ \left(\frac{1}{\theta_0} + \frac{1}{\theta_1}\right) \sum_{j=1}^{N} \left(\frac{15}{16}\right)^{N-j} \log \frac{\mu(B_{r_j})}{\mu(B_{r_{j+1}})}.$$

But the second term blows up:

$$\sum_{j=1}^{N} \left(\frac{15}{16}\right)^{N-j} j^{4\sigma} = \left(\frac{15}{16}\right)^{N-1} 1^{4\sigma} + \left(\frac{15}{16}\right)^{N-2} 2^{4\sigma} + \cdots + \left(\frac{15}{16}\right)^{0} N^{4\sigma}$$
$$\ge N^{4\sigma} \to \infty \quad \text{as} \quad N \to \infty.$$

To correct the proof, we need to modify the setting slightly as follows: Choose a new sequence

$$r_j = 1 - \frac{1}{j}, \quad j = 1, 2, \cdots.$$

Applying this sequence to (2.9) and iterating j from 1 to $N+1$, we then get

$$w(r_1) \le \left(\frac{15}{16}\right)^N w(r_{N+1}) + \left(\frac{1}{2p_0} + 4A\right) \frac{1}{Q^2} \sum_{j=1}^{N} \left(\frac{15}{16}\right)^{j-1} (j+1)^{4\sigma}$$
$$+ \left(\frac{1}{\theta_0} + \frac{1}{\theta_1}\right) \sum_{j=1}^{N} \left(\frac{15}{16}\right)^{j-1} \log \frac{\mu(B_{r_{j+1}})}{\mu(B_{r_j})}.$$

Since

$$\sum_{j=1}^{\infty}\left(\frac{15}{16}\right)^{j-1}(j+1)^{4\sigma}<\infty,$$

we can take the limit of the second term. And since

$$\sum_{j=1}^{N}\left(\frac{15}{16}\right)^{j-1}\log\frac{\mu(B_{r_{j+1}})}{\mu(B_{r_j})}\leq\sum_{j=1}^{N}\left(\frac{15}{16}\right)^{j-1}\log\frac{\mu(B_1)}{\mu(B_0)}\ ,$$

the last term is bounded by $16\left(\frac{1}{\theta_0}+\frac{1}{\theta_1}\right)\log\frac{\mu(B_1)}{\mu(B_0)}$. From this, we have the conclusion:

$$w(0)\leq c\left(\frac{1}{2p_0}+4A\right)\frac{1}{Q^2}+16\left(\frac{1}{\theta_0}+\frac{1}{\theta_1}\right)\log\frac{\mu(B_1)}{\mu(B_0)}\ .$$

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