LINEAR TRANSFORMATIONS THAT
PRESERVE THE ASSIGNMENT II

LEROY B. BEASLEY, GWANG-YEON LEE AND SANG-GU LEE

I. Introduction

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be vectors of positive integers, and let $\mathcal{U}(R, S)$ denote the class of all $m \times n$ matrices $A = [a_{ij}]$ of 0's and 1's such that

$$\sum_{k=1}^{n} a_{ik} = r_i \quad (i = 1, 2, \ldots, m),$$

$$\sum_{k=1}^{m} a_{kj} = s_j \quad (j = 1, 2, \ldots, n).$$

Thus $R$ is the row sum vector and $S$ is the column sum vector of each matrix in $\mathcal{U}(R, S)$. In [4] Brualdi, Hartfiel and Hwang introduced a class of functions generalizing the permanent function, which, like the permanent, are combinatorially significant as counting functions. We refer to matrices in $\mathcal{U}(R, S)$ as $(R, S)-assignments$, or as assignments when $R$ and $S$ are fixed in the discussion. For matrices $B = [b_{ij}]$ and $C = [c_{ij}]$ of the same order, write $B \leq C$ if $b_{ij} \leq c_{ij}$ for all $i$ and $j$. If $X = [x_{ij}]$ is an $m \times n$ matrix of 0's and 1's, then an assignment corresponds to an $m \times n$ matrix $A$ such that $A \in \mathcal{U}(R, S)$ and $A \leq X$. Thus, if we let

$$(1.1) \quad P_{R,S}(X) = \{ A \in \mathcal{U}(R, S) : A \leq X \},$$
then $P_{R,S}(X)$ counts the number of possible assignments. If we let $J_{m,n}$ be an $m \times n$ matrix whose entries are all ones, then

\begin{equation}
P_{R,S}(J_{m,n}) = |\mathcal{U}(R,S)|.
\end{equation}

We call $P_{R,S}(\cdot)$ the $(R,S) - assignment$ function or an assignment function.

A well-known special case of an assignment function occurs when $m = n$ and $R = S = (1,1,\cdots,1)$. In this case, $P_{R,S}(X)$ counts the number of permutation matrices $P$ with $P \leq X$ and hence $P_{R,S}(X)$ is the permanent of $X$, $\text{per}(X)$.

More generally, let $X = [x_{ij}]$ be an $m \times n$ matrix. We define the support of $X$ to be the set $\text{supp}(X) = \{(i,j) : x_{ij} \neq 0\}$. The $(R,S)$-assignment function $P_{R,S}(\cdot)$ is defined by

\begin{equation}
P_{R,S}(X) = \sum_{A \in \mathcal{U}(R,S)} \prod_{(i,j) \in \text{supp}(A)} x_{ij}.
\end{equation}

The preservers of the permanent were first determined by Marcus and May [5] and later Botta [3] gave a proof valid over any field. In this paper, we characterize the linear operators on the real matrices which preserve the value of an assignment function of each $m \times n$ matrix.

II. Results

Let $M_{m \times n}(R)$ be the vector space of $m \times n$ matrices. We assume throughout that $R = (r_1,\cdots,r_m)$ and $S = (s_1,s_2,\cdots,s_n)$ are vectors of positive integers with $1 \leq r_1 \leq \cdots \leq r_m < n$ and $1 \leq s_1 \leq \cdots \leq s_n < m$. If $\sum_{i=1}^{m} r_i \neq \sum_{j=1}^{n} s_j$, then $\mathcal{U}(R,S) = \emptyset$. So we assume throughout that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j = k$ for $n + 1 \leq k \leq mn$ and $0 < r_i, s_j \leq m$ for each $i,j$ where $m \leq n$, i.e., $\mathcal{U}(R,S) \neq \emptyset$. We have shown that for the case $m = n$ in [1]. Let $T : M_{m \times n}(R) \rightarrow M_{m \times n}(R)$ be a linear transformation such that

\begin{equation}
P_{R,S}(X) = P_{R,S}(T(X))
\end{equation}

for any $X \in M_{m \times n}(R)$. 
Let $E_{ij}$ denote the $(0,1)$-matrix whose only nonzero entry is in the $(i, j)$ position. A \textit{weighted cell} is a scalar multiple of $E_{ij}$ for some $(i, j)$, so that the set of cells is the set $\{\alpha_{ij}E_{ij} | \alpha_{ij} \in \mathbb{R}, 1 \leq j \leq m \text{ and } 1 \leq i \leq n\}$. We say that the two vectors $R$ and $S$ are \textit{compatible} if given any two positive integers, $1 \leq i, j \leq n$ there are two integers $k, l$ and some $A \in \mathcal{U}(R, S)$ such that $a_{ij} = a_{kl} = 1$ and $a_{il} = a_{kj} = 0$ with $i \neq k, j \neq l$, and $1 \leq k \leq m, 1 \leq l \leq n$. We may have to consider the following condition; for any pair $(i, j)$, there is some element of $\mathcal{U}(R, S)$ whose $(i, j)$ entry is nonzero. That is,

\begin{equation}
\{(i, j) : a_{ij} = 1 \text{ for some } x \in \mathcal{U}(R, S)\}
= \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}.
\end{equation}

Notice that if $R$ and $S$ are compatible then $r_i, s_j < n$ and condition (2.2) is satisfied. But we can easily show that (2.2) implies \textit{compatibility} if we allow row and column permutations. This will be possible because of our final theorem allows it. That means even if we make a weaker assumption, this does not effect our theorem.

Therefore, throughout, we assume that the two vectors $R$ and $S$ are compatible.

\textbf{Lemma 1.} $T$ is nonsingular.

\textbf{Proof.} Let $B \in \mathcal{U}(R, S)$ with $b_{pq} \neq 0$, and let $A(z) = z(B - E_{pq})$. Then, $P_{R, S}(A(z)) = 0$ for all $z$. However, the coefficient of $z^{k-1}$ in $P_{R, S}(A(z) + X)$ is $x_{pq}$ which is nonzero. Thus, $P_{R, S}(X + A(z))$ is a nonzero polynomial in $z$, and hence, is nonzero for some choice of $z$, say $z_0$. But then, since $T$ preserves $P_{R, S}$,

\begin{align*}
0 &= P_{R, S}(A(z_0)) \\
&= P_{R, S}(T(A(z_0))) \\
&= P_{R, S}(T(A(z_0)) + T(X)) \\
&= P_{R, S}(T(A(z_0) + X)) \\
&= P_{R, S}(A(z_0) + X) \neq 0.
\end{align*}

This contradiction establishes the lemma. \hfill $\blacksquare$

Let $R_i = \{X \in M_{mn}(\mathbb{R}) : x_{kl} = 0 \text{ for all } k \neq i, \text{ for all } l\}$ and $C_j = \{X \in M_{mn}(\mathbb{R}) : x_{kl} = 0 \text{ for all } l \neq j, \text{ for all } k\}$. Suppose $r_i = 1$ and $s_j = 1$ for all $i \leq q$ and $j \leq p$. 

LEMMA 2. If \(i \leq q\), then there exist \(k\) such that \(T(R_i) \subseteq R_k\) and \(k \leq q\), or \(T(R_i) \subseteq C_k\) and \(k \leq p\). If \(i \leq p\), then there exist \(l\) such that \(T(C_j) \subseteq C_l\) and \(l \leq p\), or \(T(C_j) \subseteq R_l\) and \(l \leq q\).

Proof. Since the column case is parallel to the row case, we consider \(T(R_i)\). If \(T(R_i) \not\subseteq R_k\) for all \(k \leq q\) and \(T(R_i) \not\subseteq C_k\) for all \(k \leq p\), then there are three possible cases:

Case 1. If the term rank of \(T(R_i)\) is greater than or equal 2, then there is \(X \in R_i\) such that the term rank of \(T(X)\) is at least 2. Say \(T(X) = L\) with \(l_{rs} \neq 0, l_{uv} \neq 0\) and \(r \neq u, s \neq v\). Choose \(A \in U(R, S)\) with \(a_{rs} = a_{uv} = 1\) and if possible with \(a_{rv} = 1\) or \(a_{us} = 1\). Let \(B = A - E_{rs} - E_{uv}\). Then \(P_{R,S}(tT(X) + B)\) is a polynomial of degree at least 2 since the coefficient of \(t^2\) is \(l_{rs}l_{uv} \neq 0\). But the polynomial \(P_{R,S}(tX + T^{-1}(B))\) is of degree at most 1. This contradicts that \(T\) preserves \(P_{R,S}(\cdot)\).

Case 2. If \(T(R_i) \subseteq R_k\) and \(r_k > 1\) \((k > q)\). Choose \(X \in R_i\) such that \(T(X) = L\) has \(l_{kl} \neq 0\) and \(l_{ks} \neq 0\). Choose \(A \in U(R, S)\) with \(a_{kl} = a_{ks} = 1\), and let \(B = A - E_{kl} - E_{ks}\). Then \(P_{R,S}(tX + T^{-1}(B))\) is of degree at most 1 while \(P_{R,S}(tT(X) + B)\) has degree at least 2 since the coefficient of \(t^2\) is \(l_{kl}l_{ks}\). This occurs a contradiction.

Case 3. If \(T(R_i) \subseteq C_k\) with \(k > p\). This is parallel to the Case 2. In any case we have arrived at a contradiction. Thus \(T(R_i) \subseteq R_k\) with \(k \leq q\), or \(T(R_i) \subseteq C_k\) with \(k \leq p\). Similarly, we have \(T(C_j) \subseteq C_l\) with \(l \leq p\), or \(T(C_j) \subseteq R_l\) with \(l \leq q\). ■

COROLLARY 1. If \(m \neq n\), then for each \(i \leq \alpha\), there is \(k \leq q\) such that \(T(R_i) \subseteq R_k\) and for each \(j \leq p\), there is \(l \leq p\) such that \(T(C_j) \subseteq C_l\).

Proof. This follows easily from the nonsingularity of \(T\). ■

From the fact that \(T\) is nonsingular and that \(T\), and hence \(T^{-1}\), preserves \(P_{R,S}(\cdot)\), we observe:

COROLLARY 2. For \(i, j < q\). If \(T(R_i) \subseteq R_k\) for some \(k\), then there exist \(l(\neq k)\) such that \(T(R_j) \subseteq R_l\). If \(T(R_i) \subseteq C_k\) for some \(k\), then \(m = n, q = p\) and there exist \(l\) such that \(T(R_i) \subseteq C_l\)
Lemma 3. If \( i > q \) and \( j > p \) then \( T(E_{ij}) \) has no entry in the first \( q \) rows or the first \( p \) columns.

Proof. Without loss of generality, we may assume \( r_1 = 1 \) and \( T(R_1) \subseteq R_k \) for some \( k \leq q \). Suppose \( i > q \) and \( j > p \) and \( T(E_{ij}) \) has a nonzero entry in row \( k \) for some \( k \leq q \) or in column \( l \) for some \( l \leq p \). Without loss of generality, suppose \( T(E_{ij}) = L \) and \( l_{kl} \neq 0 \) for some \( k \leq q \). Choose \( A \in \mathcal{U}(R, S) \) with \( a_{kr} = 1 \), and let \( B = A - E_{kr} \) then the coefficient of \( t \) in \( P_{R,S}(tT(E_{ij}) + B) \) is \( l_{kr} \neq 0 \). But \( tE_{ij} + T^{-1}(B) \) has no entry in rows where \( T(R_s) \subseteq R_k \) with \( s \leq q \). Thus \( P_{R,S}(tE_{ij} + T^{-1}(B)) = 0 \), a contradiction. \( \blacksquare \)

Henceforth we assume, without loss of generality, that \( T(R_i) = R_i, i \leq q \) and \( T(C_j) = C_j, j \leq p \). By Lemma 3, \( T^{-1}(R_i) = R_i, T^{-1}(C_j) = C_j \).

Lemma 4. If \( i > q \) and \( j > p \), then \( T(E_{ij}) \) is a weighted cell.

Proof. Since \( i > q \) and \( j > p \), if \( T(E_{ij}) = L \) and if \( l_{uv} \neq 0 \) then \( u > q \) and \( v > p \) by Lemma 3. Suppose \( l_{rs} \neq 0 \) and \( l_{uv} \neq 0 \). Choose \( A \in \mathcal{U}(R, S) \) with \( a_{rs} = a_{uv} = 1 \) and if possible with \( a_{rs} = a_{uv} = 1 \). Let \( B = A - E_{rs} - E_{uv} \). Then the coefficient of \( t^2 \) in \( P_{R,S}(tT(E_{ij}) + B) \) is \( l_{rs}l_{uv} \neq 0 \). But \( P_{R,S}(tE_{ij} + T^{-1}(B)) \) is a polynomial of degree at most 1, a contradiction. \( \blacksquare \)

By the above lemmas, we may now assume that \( T(E_{ij}) = E_{ij} \) if \( i \leq p \) and \( j \leq q \), \( T(E_{ij}) \) is a cell if \( i > p \) and \( j > q \) and \( T(E_{ij}) = \sum_{k=q+1}^{n} \alpha_k^{(i,j)} E_{ik} \) for some \( \alpha_k^{(i,j)} \)'s, for \( 1 \leq i \leq p \) and \( j > q \), and \( T(E_{ij}) = \sum_{k=p+1}^{m} \beta_k^{(i,j)} E_{kj} \) for \( i > p \) and \( 1 \leq j \leq q \).

Lemma 5. If \( 1 \leq i \leq p \) and \( j > q \), then \( T(E_{ij}) \) is a weighted cell.

Proof. Suppose \( T^{-1}(E_{ij}) \) is not a weighted cell for some \( 1 \leq i \leq p \) and \( j > p \). By permuting we may assume \( T^{-1}(E_{1,q+1}) = aE_{1r} + \)
\[ bE_{1s} + \ldots, \text{for some } r, s > q, \text{ with } a, b \neq 0, \text{ and } S_r \leq S_s. \text{ Let} \]

\[
X = \begin{bmatrix}
0 & \cdots & a & \cdots & b & \cdots \\
1 & & & & & \\
\vdots & & & & & \\
1 & & & & & \\
\vdots & & & & & \\
D & & & & & 
\end{bmatrix}
\]

where \( a \) is in the \((1, r)\) position and \( b \) is in the \((1, s + 1)\) position and

\[
T = \begin{bmatrix}
0 & \cdots & 0 & \cdots & a & \cdots & b & \cdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[ = E_{1q+1}. \]

We further require that \( \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \) is a \((0,1)\) matrix and has column sums \( s_1 - 1 = 0 = \ldots = 0 = s_q - 1, s_{q+1}, s_{q+2}, \ldots, s_{r-1}, s_r - 1, s_{r+1}, \ldots, s_n \) and row sums \( r_1 - 1 = 0 = \ldots = 0 = r_p - 1, r_{p+1} - 1, \ldots, r_q - 1, r_{q+1} - 1, r_{q+2}, \ldots, r_m \). So that \( P_{R,S}(X) = a \). Let \( Z = X - \sum_{i=1}^{n} x_i E_{1i} + E_{1r} \), then \( Z \in \mathcal{U}(R, S) \). Now,

\[
T(X) = \begin{bmatrix}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
1 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & & & x & y & \cdots & z \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\end{bmatrix}
\]

and since \( P_{R,S}(T(X)) = P_{R,S}(X) \neq 0 \), the column sums of \( \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} \) must be \( 0, \ldots, 0, s_{q+1} - 1, s_{q+2}, \ldots, s_n \).
Since \( s_r \leq s_s \), and \( Z \in \mathcal{U}(R, S) \), there must be some \( k > p \) such that \( x_{kr} = 0 \) and \( x_{ks} = 1 \). Let \( Y = X - E_{ks} + E_{kr} \). Then \( P_{R,S}(Y) = b \neq 0 \), so \( P_{R,S}(T(Y)) \) must nonzero. Further

\[
T(Y) = \begin{bmatrix}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 1 & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & x & y & \cdots & z & H
\end{bmatrix}
\]

where \( \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} - F + G \) when \( F = T(E_{ks}) \) and \( G = T(E_{kr}) \) are weighted cells. Now the number of nonzero entries in each columns of \( \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix} \) must be the same as those of \( \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} \) in order that \( P_{R,S}(T(Y)) \) be nonzero. Now, since \( Z \in \mathcal{U}(R, S) \), \( P_{R,S}(T(Z)) = P_{R,S}(Z) = 1 \) and

\[
T(Z) = \begin{bmatrix}
0 & \cdots & 0 & \alpha & \beta & \cdots & \gamma \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 1 & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & x & y & \cdots & z & E
\end{bmatrix}
\]

where \( T(E_{1,q+1}) = \begin{bmatrix} 0 & \cdots & 0 & \alpha & \beta & \cdots & \gamma \\ 0 & \cdots & 0 & \alpha & \beta & \cdots & \gamma \end{bmatrix} \).

Thus \( \alpha \neq 0 \) since \( 1 = P_{R,S}(T(Z)) = \alpha P_{R,S}(T(X)) = \alpha a \). Now, let...
\[ W = Z - E_{ks} + E_{kr} \text{ then} \]

\[ T(W) = T(Z) - F + G \]

\[ = \begin{bmatrix}
0 & \cdots & 0 & \alpha & \beta & \cdots \\
1 & & & & & \\
& \ddots & & & & \\
& & 1 & & & \\
& & & \cdots & & \\
0 & & & \cdots & x & y \cdots z \\
\end{bmatrix} H. \]

so that \( P_{R,S}(T(W)) = \alpha P_{R,S}(T(Y)) = \alpha P_{R,S}(Y) =: \alpha b \neq 0. \) But, the \( s - th \) column of \( W \) has \( s_s - 1 \) 1's and hence \( P_{R,S}(W) = 0 \), a contradiction to the fact that \( T \) preserves \( P_{R,S}(\cdot) \). It follows from this contradiction that \( T^{-1}(E_{1,q+1}) \) is a weighted cell, and hence that \( T(E_{ij}) \) is a weighted cell for all \( 1 \leq i \leq p \) and \( j > q \). ■

**Lemma 6.** If \( i > p \) and \( 1 \leq j \leq q \) then \( T(E_{ij}) \) is a weighted cell.

**Proof.** The proof is identical to that of lemma 5 with the roles of the rows and columns exchanged ■

We will now show that \( T \) preserves the term rank of any matrix.

**Theorem 1.** The operator \( T \) is bijective on the set of weighted cells.

**Lemma 7.** Suppose that \( 1 \leq r_i, s_j < n \) for all \( r_i \) and \( s_j \). If \( A \in \mathcal{U}(R, S) \) and \( a_{pq} = a_{uv} = 1 \) then \( A' = A - E_{pq} - E_{uv} + E_{ij} + E_{rs} \in \mathcal{U}(R, S) \) if and only if

i) \((i, j) = (p, q) \) and \((r, s) = (u, v)\);

ii) \((i, j) = (u, v) \) and \((r, s) = (p, q)\);

iii) \((i, j) = (p, v), (r, s) = (u, q)\), and \( a_{ij} = a_{rs} = 0 \) or

iv) \((i, j) = (u, q), (r, s) = (p, v)\), and \( a_{ij} = a_{rs} = 0 \).

**Proof.** Note that in cases i) and ii), \( A' = A \).

The sufficiency is easily checked. For the necessity, the only way to make \( A - E_{pq} - E_{uv} \) into a member of \( \mathcal{U}(R, S) \) by adding two cells is if those two cells have ones in rows \( p \) and \( u \) and in columns \( q \) and \( v \) (or two ones in row \( p \) if \( p = u \), etc). It then follows that \( i = p \), or
i = u, r = p or r = u, j = p or j = v, and s = q or s = v. Further, if i = p then we must have that r = u, and visa versa. Likewise, if j = q then we must have that s = v, and visa versa. Finally, if \( a_{ij} \) or \( a_{rs} \) were nonzero then \( A' \) would not be a (0 1) matrix. These facts establish the necessity.

Lemma 8. Suppose that \( 1 \leq r_i, s_j < n \) for all \( r_i \) and \( s_j \), and that \( R \) and \( S \) are compatible. If \( T \) preserves the assignment function \( P_{R,S} \) then \( T \) preserves the set of matrices of term rank 1.

Proof. Suppose that some matrix of term rank 1 is not mapped into a matrix of term rank 1. Then, since \( T \) is bijective on the cells, there is some pair of cells of term rank 1 whose images are not term rank 1. Without loss of generality, assume that \( T(E_{pq}) = xE_{ij} \) and \( T(E_{pv}) = yE_{rs} \). Now, choose \( A \in \mathcal{U}(R, S) \) with \( a_{pq} = a_{bv} = 1 \), and \( a_{pv} = a_{bp} = 0 \). This is always possible since \( R \) and \( S \) are compatible.

Now, let \( A' = A - E_{pq} - E_{bv} + E_{pv} + E_{bp} \). By lemma 3, \( P_{R,S}(A') = 1 \). Thus \( P_{R,S}(T(A')) = 1 \). Since \( T \) is bijective on the cells, we must have that the pattern \( T(A') \) of \( T(A') \) is in \( \mathcal{U}(R, S) \). But the pattern \( \overline{T(A)} \) of \( T(A) \) differs from that of \( T(A') \) only by changing two ones to zeros and two zeros to ones. That is,

\[
\overline{T(A')} = \frac{T(A - E_{pq} - E_{bv} + E_{pv} + E_{bp})}{T(A)} - \overline{T(E_{pq})} - \overline{T(E_{bv})} + \overline{T(E_{pv})} + \overline{T(E_{bp})} = \overline{T(A)} - E_{ij} - E_{gh} + E_{rs} + E_{kl}
\]

for some \((g, h)\) and \((k, l)\).

By lemma 7, and the fact that \( T \) is bijective on the set of cells we have that \( r = i \) or \( s = j \), a contradiction. Thus \( T \) preserves term rank 1.

We now obtain some of the structure of assignment preserves from the following lemma.

Lemma 9. [2, Beasley and Pullman, Corollary 3.1.2] Suppose that \( T \) is a nonsingular linear operator on \( M_{m \times n}(R) \). The linear operator
$T$ preserves the set of matrices of term rank 1 if and only if $T$ is one of or a composition of some of the following operators:

(i) $X \rightarrow X^t$
(ii) $X \rightarrow PXQ$ for fixed but arbitrary $n \times n$ permutation matrices $P$ and $Q$.
(iii) $X \rightarrow X \circ A$ for some fixed but arbitrary matrix $A$ with no zero entries.

In order to complete our characterization of operators which preserve assignment functions when $1 \leq r_i, s_j < n$ for all $r_i$ and $s_j$ we show that the three types of operators in Lemma 9 which also preserve the assignment function are the types specified in the theorem.

**Lemma 10.** Let $P$ and $Q$ are permutation matrices. Then

\begin{equation}
P_{R,S}(X) = P_{R,S}(PXQ)
\end{equation}

if and only if $PR^t = R^t$ and $SQ = S$.

**Proof.** For each $A \in \mathcal{U}(R, S), PAQ \in \mathcal{U}(R, S)$ only if $PR^t = R^t$ and $SQ = S$. This establishes the necessity. Now, suppose that $PR^t = R^t$ and $SQ = S$. Then,

$$
\begin{align*}
P_{R,S}(X) &= \sum_{A \in \mathcal{U}(R,S)} \prod_{(i,j) \in \text{supp}(A)} x_{ij} \\
&= \sum_{PAQ \in \mathcal{U}(R,S)} \prod_{(i,j) \in \text{supp}(PAQ)} x_{ij} \\
&= \sum_{A \in \mathcal{U}(R,S)} \prod_{(i,j) \in \text{supp}(A)} (PXQ)_{ij} \\
&= P_{R,S}(PXQ).
\end{align*}
$$

**Remark.** We note that the assignment is not invariant under permutations of rows and columns and under transposition. For example, if $R = (2, 2, 2)$ and $S = (3, 2, 1)$, then $\mathcal{U}(R, S) = \{A_1, A_2, A_3\}$ where

$$
A_1 = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1 
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 0 
\end{bmatrix}.
$$
Then $P_{R,S}(A_1) = 1$. But $P_{R,S}(A_1^t) = 0$. And let

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

then

$$A_1Q = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$ 

So, $P_{R,S}(A_1) = 1$ and $P_{R,S}(A_1Q) = 0$.

Note that if $R = S$ then $U(R, S) = U(S, R) = U(R, R)$ and hence, in this case, $P_{R,S}(A) = P_{R,S}(A^t)$ for all $A$. We have thus established the following lemma.

**Lemma 11.** The transpose operator preserves the assignment function $P_{R,S}$ if and only if $R = S$.

**Lemma 12.** Suppose $1 \leq r_i, s_j < n$ for all $r_i$ and $s_j$, and that $R$ and $S$ are compatible. If $T$ preserves the assignment function $P_{R,S}$ and if $T(X) = X \circ M$, then there exist diagonal matrices $D_1$ and $D_2$ such that $M = D_1JD_2$, where $J$ is the matrix of all ones; thus, $T(X) = D_1XD_2$.

**Proof.** Let

$$D_1 = \text{diag}\{m_{11}, m_{21}, \cdots, m_{nn}\}$$

and

$$D_2 = \text{diag}\{1, m_{12}m_{11}^{-1}, \cdots, m_{nn}m_{11}^{-1}\},$$

and let $N = D_1^{-1}MD_2^{-1}$. Let $2 \leq i, j \leq n$ be fixed, and choose $A \in U(R, S)$ with $a_{11} = a_{ij} = 1$ and $a_{ij} = a_{1j} = 0$. Such an element always exist since $1 \leq r_i, s_j < n$ for all $r_i$ and $s_j$. Let $B = A - E_{11} - E_{ij} + E_{1j} + E_{i1}$ so $B \in U(R, S)$. Now, $P_{R,S}(D_1^{-1}AD_2^{-1}) = \prod_{i=1}^{n} m_{i1}^{-r_i} \cdot \prod_{j=2}^{n} (m_{1j}m_{11}^{-1})^{-s_j} = P_{R,S}(D_1^{-1}BD_2^{-1})$. Thus $P_{R,S}(D_1^{-1}AD_2^{-1}) = P_{R,S}(D_1^{-1}BD_2^{-1})$, and hence $P_{R,S}((D_1^{-1}AD_2^{-1}) \circ M) = P_{R,S}((D_1^{-1}BD_2^{-1}) \circ M)$ since $T$ preserves $P_{R,S}$. Since for diagonal matrices $D$ and $E$, $DXE \circ M = D(X \circ M)E = X \circ DME$, and since $T$ preserves $P_{R,S}$ we have that $P_{R,S}(A \circ N) = P_{R,S}(B \circ N)$. Now, $P_{R,S}(A \circ N) = n_{11} \cdot n_{1j} \cdot \beta$ and $P_{R,S}(B \circ N) = n_{ij} \cdot n_{11} \cdot \beta$ where $\beta$ is $\prod_{(k,l) : \text{supp}(A) \setminus \{(1,1),(i,j)\}} n_{kl}$.
It now follows that \( n_{ij} = 1 \) since \( n_{11} = n_{i1} = n_{1j} = 1 \). Since \( i \) and \( j \) were chosen arbitrarily, we have that \( N = J \), and hence \( T(X) = X \circ M = D_1 XD_2 \). ■

We now only have to describe the allowable diagonal equivalence operators.

**Lemma 13.** If \( T(X) = DXL \) for some diagonal matrices

\[
D = \text{diag}\{d_1, d_2, \ldots, d_m\}
\]

and

\[
L = \text{diag}\{l_1, l_2, \ldots, l_n\}
\]

in \( M_{m \times n}(R) \), then \( \prod_{i=1}^m d_i^{r_i} \cdot \prod_{j=1}^n l_j^{s_j} = 1 \).

**Proof.** Let \( A \in \mathcal{U}(R, S) \), then \( P_{R,S}(A) = 1 \), and hence \( P_{R,S}(T(A)) = 1 \). That is, \( P_{R,S}( DAL ) = 1 \). But \( P_{R,S}( DAL ) = \prod_{i=1}^m d_i^{r_i} \cdot \prod_{j=1}^n l_j^{s_j} \).

■

An immediate consequence of the above lemmas is the following theorem.

**Theorem 2.** If \( T \) is a linear operator on \( M_{m \times n}(F) \) and \( 1 \leq r_i, s_j < n \) for all \( r_i \) and \( s_j \), and \( R \) and \( S \) are compatible, then \( T \) preserves the assignment function \( P_{R,S} \) if and only if

\[
T(X) = PDXLQ \text{ for all } X \in M_{m \times n}(R),
\]
or

\[
T(X) = PDX^tLQ \text{ and } R = S \text{ for all } X \in M_n(R),
\]

where \( P \) and \( Q \) are permutation matrices such that \( PR^t = R^t \) and \( SQ = S \) and \( D = \text{diag}\{d_1, d_2, \ldots, d_m\} \) and \( L = \text{diag}\{l_1, l_2, \ldots, l_n\} \) are diagonal matrices such that \( \prod_{i=1}^m d_i^{r_i} \cdot \prod_{j=1}^n l_j^{s_j} = 1 \).

**References**


LeRoy B. Beasley  
Department of Mathematics  
Utah State University  
Logan, Utah 84322-3900 U.S.A

Gwang-Yeon Lee  
Department of mathematics  
Hanseo University  
Seosan 356-820, Korea

Sang-Gu Lee  
Department of Mathematics  
SungKyunKwan University  
Suwon 440-746, Korea