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GENERALIZED VECTOR–VALUED VARIATIONAL INEQUALITIES AND FUZZY EXTENSIONS

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1. Introduction and Preliminaries

Recently, Giannessi [9] firstly introduced the vector-valued variational inequalities in a real Euclidean space. Later Chen et al. [5] intensively discussed vector-valued variational inequalities and vector-valued quasi variational inequalities in Banach spaces. They [4-8] proved some existence theorems for the solutions of vector-valued variational inequalities and vector-valued quasi-variational inequalities. Lee et al. [14] established the existence theorem for the solutions of vector-valued variational inequalities for multifunctions in reflexive Banach spaces.

On the other hand, Chang and Zhu [3] investigated the existence theorems of vector-valued variational inequalities for fuzzy mappings in locally convex Hausdorff topological vector spaces, which were the fuzzy extensions of some theorems in [12, 20, 22, 24]. Lee et al. [13] obtained the fuzzy generalizations of new results of Kim and Tan [11], and they [14] established the fuzzy extensions of their existence theorems. The noncompact cases of the existence theorems of vector-valued variational inequalities for multifunctions or fuzzy mappings in Banach spaces obtained by Lee et al. [14] was considered by Park et al. [19].

In this paper, we establish the existence theorems of the following more generalized vector-valued variational inequalities (GVVI) for multifunctions than inequalities in [14] by using variable dominated cones,

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and obtain the fuzzy extensions of our existence theorems. In section 2, we establish the existence theorems for (GVVI) under the upper-semicontinuity of the multifunction T, and obtain the existence theorems for (GVVI), by using the P-monotonicity and V-hemicontinuity of T, under the coercivity condition in Banach spaces. Also we obtain the existence theorems for (GVVI) in reflexive Banach spaces. In section 3, the fuzzy analogues of our results in section 2 are dealt with.

Let X and Y be two Banach spaces and D a nonempty convex subset of X. Let $T : X \to 2^{L(X,Y)}$ be a multifunction, where $L(X,Y)$ is the space of all continuous linear operators from X into Y. Let $\{C(x) | x \in D\}$ be a family of convex cones in Y such that $Int\, C(x) \neq \emptyset$, $\forall x \in D$, where Int denotes the interior.

Consider the following generalized vector-valued variational inequality :

(GVVI) Find $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in T(x_0)$ such that

$$\langle s_0, x - x_0 \rangle \notin -Int\, C(x_0),$$

where $\langle s_0, y \rangle$ denotes the evaluation of s_0 at y.

When T is an operator from X into $L(X,Y)$, (GVVI) reduces to the following vector-valued variational inequality (VVI) considered by Chen [4].

(VVI) Find $x_0 \in D$ such that $\langle T(x_0), x - x_0 \rangle \notin -Int\, C(x_0)$ for all $x \in D$.

When for every $x \in D$, $C(x) = C$, where C is a convex cone in Y with $Int\, C \neq \emptyset$, (GVVI) [respectively, (VVI)] reduces to the following vector-valued variational inequalities (GVVI)$'$ [resp., (VVI)$'$] investigated by Park et al. [19] and Lee et al. [14] [resp., Chen et al. [5, 7, 8], Yang [23]].

(GVVI)′ Find $x_0 \in D$ such that for each $x \in D$ there exists an $s_0 \in T(x_0)$ such that

$$\langle s_0, x - x_0 \rangle \notin -Int\ C.$$

(VVI)′ Find $x_0 \in D$ such that $\langle T(x_0), x - x_0 \rangle \notin -Int\ C$ for all $x \in D$.

The above inequality (VVI)′ is a generalization of the following classic scalar-valued variational inequality (VI). When $Y = \mathbb{R}$, $X = \mathbb{R}^n$, $C(x) = \mathbb{R}_+$, $\forall x \in D \subset \mathbb{R}^n$, then the (VVI)′ collapses to the (VI).

(VI) Find $x_0 \in D$ such that $\langle f(x_0), x - x_0 \rangle \geq 0$ for all $x \in D \subset \mathbb{R}^n$, where $f : D \to \mathbb{R}^n$ is a given operator.

Now we give the definition of a KKM multifunction.

DEFINITION 1.1. Let D be a subset of a topological vector space X. Then a multifunction $G : D \to 2^X$ is called *KKM* if for each nonempty finite subset N of D, $co\ N \subset G(N)$, where *co* denotes the convex hull and $G(N) = \bigcup\{Gx : x \in N\}$.

A *convex space* X is a nonempty convex set (in a vector space) with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Thus, a convex subset D of a topological vector space X with the relative topology is automatically a convex space. For details of the convex space, see Lassonde [12].

We need the following particular form of the generalized KKM theorems due to Park [16-18], which will be used in the proof of our main results.

THEOREM 1. *Let X be a convex space, K a nonempty compact subset of X, and $G : X \to 2^X$ a KKM multifunction. Suppose that*

(1) for each $y \in X$, $G(y)$ is closed ; and

(2) for each nonempty finite subset N of X, there exists a compact convex subset L_N of X such that $N \subset L_N$ and $L_N \cap \bigcap\{G(y) : y \in L_N\} \subset K$.

Then we have

$$K \cap \bigcap\{G(y) : y \in X\} \neq \emptyset.$$

In particular, if $X = K$, that is, X is a compact convex space, then the condition (2) is obviously held, and hence in this case, without the condition (2), it is true that $\bigcap\{G(y) : y \in X\} \neq \emptyset$.

2. Existence Theorems

First, we give the following definitions for the existence theorems for (GVVI).

DEFINITION 2.1. Let F be a multifunction from a topological space X into a topological space Y.

1. F is said to be *closed* at $x \in X$ if for each sequences $\{x_n\}_{n=1}^{\infty}$ converging to x and $\{y_n\}_{n=1}^{\infty}$ converging to y such that $y_n \in F(x_n)$ for all n, we have $y \in F(x)$. F is said to be *closed* if it is closed at every $x \in X$.
2. F is said to be *upper semi-continuous* at $x \in X$ if for every open set V in Y containing $F(x)$, there exists a neighborhood $N(x)$ of x such that $F(z) \subset V$ for all $z \in N(x)$. F is said to be *upper semi-continuous* if it is upper semi-continuous at every $x \in X$.

DEFINITION 2.2. Let X and Y be two normed spaces, $T : X \to L(X, Y)$ an operator, and P a nonempty closed convex cone in Y with $Int\ P \neq \emptyset$.

1. T is said to be *P-monotone* if for any $x, y \in X$, $\langle T(x) - T(y), x - y\rangle \in P$.
2. T is said to be *P-pseudomonotone* if for any $x, y \in X$, $\langle T(x), y - x\rangle \notin -Int\ P$ implies that $\langle T(y), y - x\rangle \notin -Int\ P$.
3. T is said to be *V-hemicontinuous* if for any $x, y, z \in X$, the mapping $\alpha \mapsto \langle T(x + \alpha y), z\rangle$ is continuous at 0^+.

DEFINITION 2.3. Let X and Y be two normed spaces, $T : X \to 2^{L(X,Y)}$ a multifunction, and P a nonempty convex cone in Y with $Int\ P \neq \emptyset$.

1. T is said to be *P-monotone* if for any $x, y \in X$, $s \in T(x)$ and $t \in T(y)$, $\langle s - t, x - y\rangle \in P$.
2. T is said to be *P-pseudomonotone* if for any $x, y \in X$, $\langle s, y - x\rangle \notin -Int\ P$ for some $s \in T(x)$ implies that $\langle t, y - x\rangle \notin -Int\ P$ for some $t \in T(y)$.

3. T is said to be *V-hemicontinuous* if for any $x, y \in X, \alpha > 0$ and $t_\alpha \in T(x + \alpha y)$, there exists a $t_0 \in T(x)$ such that for any $z \in X, \langle t_\alpha, z\rangle \mapsto \langle t_0, z\rangle$ as $\alpha \to 0^+$.

REMARK. 1. Definition 2.3 is a generalization of Definition 2.2.

2. We can easily prove that the P -monotonicity implies the P -pseudomonotonicity.

Now we prove the following existence theorems for (GVVI) in Banach spaces.

THEOREM 2. *Let X and Y be Banach spaces, D a nonempty closed convex subset of X, and $C : D \to 2^Y$ a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with Int $C(x) \neq \emptyset$ and $C(x) \neq Y$. Let $W : D \to 2^Y$ be a closed multifunction defined by $W(x) = Y \setminus (-Int\ C(x))$ for any $x \in D$.*

Let $T : X \to 2^{L(X,Y)}$ is upper semi-continuous and compact-valued. Suppose that $T(D)$ is contained in a compact subset of $L(X, Y)$.

Then (GVVI) is solvable.

Furthermore, the solution set of (GVVI) is a compact subset of D.

Proof. Define a multifunction $F_1 : D \to 2^D$ by

$$F_1(y) = \{x \in D : \langle s, y - x\rangle \notin -Int\ C(x) \text{ for some } s \in T(x)\}$$

for $y \in D$. Then F_1 is a KKM multifunction on D.

In fact, suppose that $N = \{x_1, \cdots, x_n\} \subset D$, $\sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0, i = 1, \cdots, n$ and $x = \sum_{i=1}^n \alpha_i x_i \notin F_1(N)$. Then for any $s \in T(x)$, we have $\langle s, x_i - x\rangle \in -Int\ C(x), i = 1, \cdots, n$. Thus we have

$$\begin{aligned}\langle s, x\rangle = \langle s, \sum_{i=1}^n \alpha_i x_i\rangle &= \sum_{i=1}^n \alpha_i \langle s, x_i\rangle \\ &\in \sum_{i=1}^n \alpha_i \langle s, x\rangle - Int\ C(x) \\ &= \langle s, x\rangle - Int\ C(x).\end{aligned}$$

Hence $0 \in$ *Int* $C(x)$, which contradicts the assumption $C(x) \neq Y$. Therefore, F_1 is a KKM multifunction on D. We claim that F_1 is closed-valued. In fact, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $F_1(y)$ converging to $x_* \in D$ for any fixed $y \in D$. Since $x_n \in F_1(y)$ for all n, there exists an $s_n \in T(x_n)$ such that

$$\langle s_n, y - x_n \rangle \in W(x_n) \qquad \text{for all } n. \tag{2.1}$$

On the other hand, by assumption $\overline{T(D)}$ is compact.

Hence without loss of generality, we can assume that there exists a $s_* \in L(X,Y)$ such that s_n converges to s_*. Since T is upper semi-continuous and compact-valued, T is closed[1], so $s_* \in T(x_*)$. Moreover we have

$$\begin{aligned}
&\|\langle s_n, y - x_n \rangle - \langle s_*, y - x_* \rangle\| \\
&\quad \le \|\langle s_n, x_* - x_n \rangle\| + \|\langle s_n - s_*, y - x_* \rangle\| \\
&\quad \le \|s_n\| \cdot \|x_* - x_n\| + \|s_n - s_*\| \cdot \|y - x_*\|.
\end{aligned}$$

Since $\{s_n\}$ is bounded in $L(X,Y)$, $\langle s_n, y - x_n \rangle$ converges to $\langle s_*, y - x_* \rangle$. By (2.1) and the closedness of W, we have $\langle s_*, y - x_* \rangle \in W(x_*)$. Consequently, there exists an $s_* \in T(x_*)$ such that $\langle s_*, y - x_* \rangle \notin -Int\ C(x)$.

Hence $F_1(y)$ is closed. Therefore, by Theorem **1** there exists an $x_0 \in \bigcap\{F_1(y) : y \in D\}$. Thus there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in T(x_0)$ such that $\langle s_0, x - x_0 \rangle \notin -Int\ C(x_0)$. It is clear that the solution set of (GVVI), $\bigcap\{F_1(y) : y \in D\}$ is compact.

The following theorem shows that (GVVI) is solvable under a coercivity condition in Banach spaces.

THEOREM 3. *Let X and Y be Banach spaces, D a nonempty convex subset of X and K a nonempty compact subset of X. Let $C : D \to 2^Y$ be a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with Int $C(x) \neq \emptyset$ and $C(x) \neq Y$, and $P := \bigcap_{x \in D} C(x)$ a nonempty convex cone in Y with Int $P \neq \emptyset$. Let $W : D \to 2^Y$ be a closed multifunction defined by $W(x) = Y \setminus (-Int\ C(x))$ for any $x \in D$, and $T : X \to 2^{L(X,Y)}$ a multifunction.*

Suppose that

(1) T is P-monotone, compact-valued and V-hemicontinuous.

(2) for each nonempty finite subset N of D, there exists a nonempty compact convex subset L_N of D such that $N \subset L_N$ and for each

$x \in L_N \setminus K$ there exists a $y \in L_N$ such that $\langle t, y - x \rangle \in -Int\ C(x)$ for all $t \in T(y)$.

Then (GVVI) *is solvable.*

Proof. Define a multifunction $F_1 : D \to 2^D$ by

$$F_1(y) = \{x \in D : \langle s, y - x \rangle \notin -Int\ C(x) \text{ for some } s \in T(x)\} \text{ for } y \in D.$$

Then by the same argument as the proof in Theorem 2, F_1 is a KKM multifunction on D. Define a multifunction $F_2 : D \to 2^D$ by

$$F_2(y) = \{x \in D : \langle t, y - x \rangle \notin -Int\ C(x) \text{ for some } t \in T(y)\}$$

for $y \in D$. Then F_2 is also a KKM multifunction on D. In fact, for any $x \in F_1(y)$, there exists an $s \in T(x)$ such that $\langle s, y - x \rangle \notin -Int\ C(x)$. By the P-monotonicity of T,

$$\langle s - t, y - x \rangle \in -P \subset -C(x)$$

for any $t \in T(y)$. Hence for any $t \in T(y)$ $\langle t, y - x \rangle \notin -Int\ C(x)$ and thus $x \in F_2(y)$. Hence $F_1(y) \subset F_2(y)$ for any $y \in D$. Therefore F_2 is also a KKM multifunction on D. We claim that F_2 is closed-valued. Indeed, for any fixed $y \in D$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $F_2(y)$ which converges to $x_* \in D$. Since $x_n \in F_2(y)$ for each n, there exists a $t_n \in T(y)$ such that

$$\langle t_n, y - x_n \rangle \in W(x_n) \quad \text{for all } n. \tag{2.2}$$

Since $T(y)$ is compact, we may assume that $\{t_n\}_{n=1}^{\infty}$ converges to some $t_* \in T(y)$. Note that

$$\begin{aligned}\|\langle t_n, y - x_n \rangle - \langle t_*, y - x_* \rangle\| &= \|\langle t_n, x_* - x_n \rangle + \langle t_n - t_*, y - x_* \rangle\| \\ &\le \|\langle t_n, x_* - x_n \rangle\| + \|\langle t_n - t_*, y - x_* \rangle\| \\ &\le \|t_n\| \cdot \|x_* - x_n\| + \|t_n - t_*\| \cdot \|y - x_*\|.\end{aligned}$$

Since $\{t_n\}_{n=1}^{\infty}$ is bounded in $L(X, Y)$, $\langle t_n, y - x_n \rangle$ converges to $\langle t_*, y - x_* \rangle$. By (2.2) and the closedness of W we have $\langle t_*, y - x_* \rangle \in W(x_*)$. Hence $\langle t_*, y - x_* \rangle \notin -Int\ C(x_*)$, whence we have $x_* \in F_2(y)$.

Further, note that assumption (2) implies that, for each $x \in L_N \backslash K$ there exists a $y \in L_N$ such that $x \notin F_2(y)$. Hence $L_N \cap \bigcap\{F_2(y) : y \in L_N\} \subset K$. Therefore, the condition (2) of Theorem **1** holds. Thus, by Theorem **1**, there exists an $x \in K \cap \bigcap\{F_2(y) : y \in D\}$. Then for any $y \in D$, there exists a $t_y \in T(y)$ such that $\langle t_y, y - x\rangle \notin -Int\ C(x)$. By the convexity of D, for any $\alpha \in (0,1)$, there exists a $t_\alpha \in T(\alpha y + (1-\alpha)x)$ such that $\langle t_\alpha, \alpha(y-x)\rangle \notin -Int\ C(x)$. Dividing by α, we have $\langle t_\alpha, y - x\rangle \notin -Int\ C(x)$. By the V-hemicontinuity of T, there exists a $t_0 \in T(x)$ such that $\langle t_0, y - x\rangle \notin -Int\ C(x)$. Hence $x \in \bigcap\{F_1(y) : y \in D\}$. Thus $\bigcap\{F_1(y) : y \in D\} \neq \emptyset$. Consequently, there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in T(x_0)$ such that $\langle s_0, x - x_0\rangle \notin -Int\ C(x_0)$.

COROLLARY 2.1. *In Theorem 3, if D is closed, then the coercivity (2) can be replaced by the following without affecting its conclusion :*

(2)′ there exist a nonempty compact subset K of D and a $y \in K$ such that $\langle t, y - x\rangle \in -Int\ C(x)$ for $x \in D\backslash K$ and $t \in T(y)$.

Proof. It sufficies to show that (2)′ implies (2). In fact, for any nonempty finite subset N of D, we let $L_N = co\ (N \cup (K \cap D)) \subset D$. By (2)′, for any $x \in L_N\backslash K \subset D\backslash K$, there exists a $y \in (K \cap D) \subset L_N$ such that $\langle t, y - x\rangle \in -Int\ C(x)$ for all $t \in T(y)$. Hence (2) holds.

For $D = K$, Theorem 3 reduces to the following corollary ;

COROLLARY 2.2. *Let X and Y be Banach spaces, D a nonempty compact convex subset of X, $C : D \to 2^Y$ a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with $Int\ C(x) \neq \emptyset$ and $C(x) \neq Y$, and $P := \bigcap_{x\in D} C(x)$ a nonempty convex cone in Y with $Int\ P \neq \emptyset$. Let $W : D \to 2^Y$ be a closed multifunction defined by $W(x) = Y\backslash(-Int\ C(x))$ for any $x \in X$, and $T\ \ X \to 2^{L(X,Y)}$ a multifunction. If $T : X \to 2^{L(X,Y)}$ is P-monotone, compact-valued and V-hemicontinuous, then (GVVI) is solvable.*

Now we prove the following existence theorem for (GVVI) in reflexive Banach spaces.

THEOREM 4. *Let X be a reflexive Banach space, Y a Banach space, and D a nonempty closed, bounded and convex subset of X. Let $C : D \to 2^Y$ be a multifunction such that for each $x \in D$, $C(x)$ is a convex* *cone in Y with Int $C(x) \neq \emptyset$ and $C(x) \neq Y$, and $P := \bigcap_{x\in D} C(x)$ a nonempty convex cone in Y with Int $P \neq \emptyset$ Let $W : D \to 2^Y$ be a weakly closed multifunction defined by $W(x) = Y\setminus(-Int\ C(x))$ for any $x \in D$, and $T : X \to 2^{L(X,Y)}$ a multifunction.*

If T is P-monotone, compact-valued and V-hemicontinuous, then (GVVI) is solvable.

Proof. Define a multifunction $F_1 : D \to 2^D$ by

$$F_1(y) = \{x \in D : \langle s, y-x\rangle \notin -Int\ C(x) \text{ for some } s \in T(x)\} \text{ for } y \in D.$$

Then by the same argument as the proof in Theorem 2, F_1 is a KKM multifunction on D. Define a multifunction $F_2 : D \to 2^D$ by

$$F_2(y) = \{x \in D : \langle t, y - x\rangle \notin -Int\ C(x) \text{ for some } t \in T(y)\}$$

for $y \in D$. Then by the same argument as the proof in Theorem 3, $F_1(y) \subset F_2(y)$ for any $y \in D$, and hence F_2 is also a KKM multifunction on D. Now we claim that F_2 is weakly closed-valued. Indeed, for any fixed $y \in D$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $F_2(y)$ which converges weakly to $x_* \in D$. Since $x_n \in F_2(y)$ for each n, there exists a $t_n \in T(y)$ such that

$$\langle t_n, y - x_n\rangle \in W(x_n) \quad \text{for all } n. \tag{2.3}$$

Since $T(y)$ is compact, we may assume that $\{t_n\}_{n=1}^{\infty}$ converges to some $t_* \in T(y)$. Note that for any $q \in Y^*$, where Y^* is the topological dual of Y,

$$\begin{aligned}
&|q(\langle t_n, y - x_n\rangle - \langle t_*, y - x_*\rangle)| \\
\le&|q(\langle t_n - t_*, y - x_n\rangle)| + |q(\langle t_*, x_* - x_n\rangle)| \\
\le&\|q\|\|t_n - t_*\|\|y - x_n\| + |(q \circ t_*)(x_* - x_n)| \\
\le&\|q\|\|t_n - t_*\|(\|y\| + \|x_n\|) + |(q \circ t_*)(x_* - x_n)|.
\end{aligned}$$

Since $x_n \in D$ for all n and D is bounded, $\{x_n\}_{n=1}^{\infty}$ is bounded. Since $\|t_n - t_*\| \to 0$,

$$\|q\|\|t_n - t_*\|(\|y\| + \|x_n\|) \to 0.$$

On the other hand, since $q \circ t_*$ is continuous and linear from X to $\mathbb{R}$, we have

$$|(q \circ t_*)(x_* - x_n)| \to 0.$$

Consequently,

$\{\langle t_n, y - x_n\rangle\}_{n=1}^{\infty}$ converges weakly to $\langle t_*, y - x_*\rangle$. By (2.3) and the weak closedness of W we have $\langle t_*, y - x_*\rangle \in W(x_*)$. Hence $\langle t_*, y - x_*\rangle \notin -Int\ C(x_*)$, whence we have $x_* \in F_2(y)$.

Since D is a closed, bounded and convex subset of a reflexive Banach space X, D is weakly compact. Thus, by Theorem **1**, there exists an $x \in \bigcap\{F_2(y) : y \in D\}$. It follows from the V-hemicontinuity of T that there exists a $t_0 \in T(x)$ such that $\langle t_0, y - x\rangle \notin -Int\ C(x)$. Hence $x \in \bigcap\{F_1(y) : y \in D\}$. Thus $\bigcap\{F_1(y) : y \in D\} \neq \emptyset$. Consequently, there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in T(x_0)$ such that $\langle s_0, x - x_0\rangle \notin -Int\ C(x_0)$.

REMARK. When $C(x)$ is a constant cone in Theorem 4, we can show that (GVVI)$'$ is solvable under the P-pseudomonotonicity of T [14].

When T is a single-valued mapping, we can obtain the following corollary from Theorem 4.

COROLLARY 2.3 [4]. *Let X be a reflexive Banach space, Y a Banach space, D a nonempty bounded, closed and convex subset of X. $C : D \to 2^Y$ a multifunction such that for each $x \in D$. $C(x)$ is a convex cone in Y with $Int\ C(x) \neq \emptyset$ and $C(x) \neq Y$, and $P := \bigcap_{x \in D} C(x)$ a nonempty convex cone in Y with $Int\ P \neq \emptyset$. Let W be a weakly closed multifunction defined by $W(x) = Y \backslash (-Int\ C(x))$ for any $x \in X$, and $T : X \to L(X, Y)$ an operator. If $T : X \to L(X, Y)$ is P-monotone, and V-hemicontinuous, then* (VVI) *is solvable.*

REMARK. Note that for $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, corollaries extend or reduce to the well-known scalar valued variational inequalities due to Hartman and Stampacchia [10], Browder [2], Stampacchia [21], Mosco [15] and many others.

3. Fuzzy Extensions

Let X and Y be two normed spaces and $\mathcal{F}(L(X,Y))$ the collection of all fuzzy sets on $L(X,Y)$. A mapping F from X into $\mathcal{F}(L(X,Y))$ is called a fuzzy mapping.

If $F : X \to \mathcal{F}(L(X,Y))$ is a fuzzy mapping, then $F(x), x \in X$ (denoted by F_x), is a fuzzy set in $\mathcal{F}(L(X,Y))$ and $F_x(s), s \in L(X,Y)$, is the degree of membership of s in F_x. Let $A \in \mathcal{F}(L(X,Y))$ and $\beta \in [0,1]$. Then the set $(A)_\beta = \{s \in L(X,Y) : A(s) \geq \beta\}$ is said to be a *β-cut* of A.

DEFINITION 3.1 [25]. A fuzzy set A on $L(X,Y)$ is *compact* if for each $\beta \in (0,1]$, $(A)_\beta$ is compact in $L(X,Y)$.

DEFINITION 3.2. Let X and Y be two normed spaces, $F : X \to \mathcal{F}(L(X,Y))$ a fuzzy mapping, and P a nonempty convex cone in Y with *Int* $P \neq \emptyset$.

1. F is said to be *P-monotone* if for any $x, y \in X$ and $s, t \in L(X,Y)$ with $F_x(s) > 0$ and $F_y(t) > 0$, $\langle s - t, x - y\rangle \in P$.
2. F is said to be *P-pseudomonotone* if for any $x, y \in X$ and $\beta \in (0,1]$, $\langle s, y - x\rangle \notin -Int\ P$ for some $s \in L(X,Y)$ with $F_x(s) \geq \beta$ implies that $\langle t, y - x\rangle \notin -Int\ P$ for some $t \in L(X,Y)$ with $F_y(t) \geq \beta$.
3. F is said to be *hemicontinuous* if for any $x, y \in X$ and $t_\alpha \in L(X,Y)$ with $F_{x+\alpha y}(t_\alpha) \geq \beta$ where $\alpha, \beta \in (0,1]$, there exists $t_0 \in L(X,Y)$ with $F_x(t_0) \geq \beta$ such that for any $z \in X$, $\langle t_\alpha, z\rangle \to \langle t_0, z\rangle$ as $\alpha \to 0^+$.
4. F is said to be *closed* at $x_0 \in X$ if for each open subset V of $L(X,Y)$ such that if $F_{x_0}(s) \geq \beta$ where $\beta \in (0,1]$, then $s \in V$, there exists a neighborhood $N(x_0)$ of x_0 such that if $x \in N(x_0)$ and $F_x(s) \geq \beta$, then $s \in V$. F is called *closed* if it is closed at each point of X.

Now we can easily obtain fuzzy analogues of Theorem 2 and Theorem 3 respectively.

THEOREM 5. *Let X and Y be Banach spaces, D a nonempty closed convex subset of X. Let $C : D \to 2^Y$ be a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with Int $C(x) \neq \emptyset$ and $C(x) \neq Y$. Let $W : D \to 2^Y$ be a closed multifunction defined by $W(x) = Y \backslash (-Int\ C(x))$ for any $x \in D$, and $F : X \to \mathcal{F}(L(X,Y))$ a* *fuzzy mapping such that there exists a real number* $\beta \in (0,1]$ *such that for each* $x \in X, (F_x)_\beta$ *is a nonempty subset of* $L(X,Y)$. *Suppose that* F *is closed, and for each* $x \in X, F_x$ *is a compact fuzzy set on* $L(X,Y)$.

If $\bigcup_{x\in D}(F_x)_\beta$ *is contained in a compact subset of* $L(X,Y)$, *then there exists an* $x_0 \in D$ *such that for each* $x \in D$, *there exists an* $s_0 \in L(X,Y)$ *with* $F_{x_0}(s_0) \geq \beta$ *such that* $\langle s_0, x - x_0\rangle \notin -Int\ C(x_0)$.

Proof. Define a multifunction $\tilde{F} : X \to 2^{L(X,Y)}$ by for any $x \in X, \tilde{F}(x) = (F_x)_\beta$. Let $x_1 \in X$ and V be any open set such that $\tilde{F}(x_1) \subset V$, then $s \in V$ for any $s \in L(X,Y)$ with $F_{x_1}(s) \geq \beta$. By the closedness of F, there exists a neighborhood $N(x_1)$ of x_1 such that if $x \in N(x_1)$ and $F_x(s) \geq \beta$, then $s \in V$, that is, there exists a neighborhood $N(x_1)$ of x_1 such that $x \in N(x_1)$ implies $\tilde{F}(x) \subset V$. Hence $\tilde{F}$ is upper semi-continuous. Since for each $x \in X$, F_x is a compact fuzzy set on $L(X,Y)$, then $\tilde{F}(x)$ is compact. By Theorem 2, we know that there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in \tilde{F}(x_0)$ such that $\langle s_0, x - x_0\rangle \notin -Int\ C(x_0)$. Hence there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in L(X,Y)$ with $F_{x_0}(s_0) \geq \beta$ such that $\langle s_0, x - x_0\rangle \notin -Int\ C(x_0)$.

THEOREM 6. *Let* X *and* Y *be Banach spaces,* D *a nonempty convex subset of* X, *and* K *a nonempty compact subset of* X. *Let* $C : D \to 2^Y$ *be a multifunction such that for each* $x \in D$, $C(x)$ *is a convex cone in* Y *with Int* $C(x) \neq \emptyset$ *and* $C(x) \neq Y$, *and* $P := \bigcap_{x\in D} C(x)$ *a nonempty convex cone in* Y *with Int* $P \neq \emptyset$. *Let* $W : D \to 2^Y$ *be a closed multifunction defined by* $W(x) = Y\setminus(-Int\ C(x))$ *for any* $x \in X$, *and* $F : X \to \mathcal{F}(L(X,Y))$ *a fuzzy mapping such that there exists a real munber* $\beta \in (0,1]$ *such that for each* $x \in X, (F_x)_\beta$ *is a nonempty subset of* $L(X,Y)$. *Suppose that*

(1) F *is* P *-monotone, hemicontinuous, and for each* $x \in X, F_x$ *is a compact fuzzy set on* $L(X,Y)$.

(2) *for each nonempty finite subset* N *of* D, *there exists a nonempty compact convex subset* L_N *of* D *such that* $N \subset L_N$ *and for each* $x \in L_N\setminus K$ *there exists a* $y \in L_N$ *such that* $\langle t, y - x\rangle \in -Int\ C(x)$ *for all* $t \in L(X,Y)$ *with* $F_y(t) \geq \beta$.

Then there exists an $x_0 \in D$ *such that for each* $x \in D$, *there exists an* $s_0 \in L(X,Y)$ *with* $F_{x_0}(s_0) \geq \beta$ *such that* $\langle s_0, x - x_0\rangle \notin -Int\ C(x_0)$.

Proof. Define a multifunction $\tilde{F} : X \to 2^{L(X,Y)}$ by $\tilde{F}(x) = (F_x)_\beta$

for any $x \in X$. It follows from the P-monotonicity of F that for any $x, y \in X$, for any $s \in \tilde{F}(x)$ and $t \in \tilde{F}(y)$, $\langle s-t, x-y\rangle \in P$. This implies that $\tilde{F}$ is P-monotone. The V-hemicontinuity of $\tilde{F}$ is easily proved and the compactness of $\tilde{F}(x)$ for each $x \in X$ is proved similarly as in the proof of Theorem 5. Condition (2) implies that assumption (2) in Theorem 3 is satisfied for the multifunction $\tilde{F}$ By Theorem 3 there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in \tilde{F}(x_0)$ such that $\langle s_0, x-x_0\rangle \notin -Int\ C(x_0)$. Hence there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in L(X, Y)$ with $F_{x_0}(s_0) \geq \beta$ such that $\langle s_0, x-x_0\rangle \notin -Int\ C(x_0)$.

COROLLARY 3.1. *In Theorem 6, if D is closed, then the coercivity (2) can be replaced by the following without affecting its conclusion :*

(2)″ there exist a nonempty compact subset K of X and a $y \in K \cap D$ such that $\langle t, y-x\rangle \in -Int\ C(x)$ for all $t \in L(X, Y)$ with $F_y(t) \geq \beta$ and $x \in D \backslash K$.

Proof. It is proved similarly as in the proof of Corollary 2.1.

For $D = K$, Theorem 6 reduces to the following corollary :

COROLLARY 3.2. *Let X and Y be Banach spaces, D a nonempty compact convex subset of X, $C : D \to 2^Y$ be a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with $Int\ C(x) \neq \emptyset$ and $C(x) \neq Y$, and $P := \bigcap_{x\in D} C(x)$ a nonempty convex cone in Y with $Int\ P \neq \emptyset$. Let $W : D \to 2^Y$ be a closed multifunction defined by $W(x) = Y \backslash (-Int\ C(x))$ for any $x \in D$, and $F : X \to \mathcal{F}(L(X, Y))$ a fuzzy mapping such that there exists a real number $\beta \in (0, 1]$ such that for each $x \in X$, $(F_x)_\beta$ is a nonempty subset of $L(X, Y)$. If F is P-monotone, hemicontinuous, and for each $x \in X, F_x$ is a compact fuzzy set on $L(X, Y)$, then there exists an $x_0 \in D$ such that for each $x \in D$ there exists an $s_0 \in L(X, Y)$ with $F_{x_0}(s_0) \geq \beta$ such that $\langle s_0, x-x_0\rangle \notin -Int\ C(x_0)$.*

Now we obtain the following fuzzy extension of Theorem 4.

THEOREM 7. *Let X be a reflexive Banach space, Y a Banach space, and D a nonempty closed, bounded and convex subset of X. Let $C : D \to 2^Y$ be a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with $Int\ C(x) \neq \emptyset$ and $C(x) \neq Y$, and $P := \bigcap_{x\in D} C(x)$*

a nonempty convex cone in Y with Int $P \neq \emptyset$. Let $W : D \to 2^Y$ be a weakly closed multifunction defined by $W(x) = Y \backslash (-Int\ C(x))$ for any $x \in X$, and $F : X \to \mathcal{F}(L(X,Y))$ a fuzzy mapping such that there exists a real munber $\beta \in (0,1]$ such that for each $x \in X, (F_x)_\beta$ is a nonempty subset of $L(X,Y)$. If F is P -monotone, hemicontinuous, and for each $x \in X, F_x$ is a compact fuzzy set on $L(X,Y)$, then there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in L(X,Y)$ with $F_{x_0}(s_0) \geq \beta$ such that $\langle s_0, x - x_0 \rangle \notin -Int\ C(x_0)$.

Proof. Define a multifunction $\tilde{F} : X \to 2^{L(X,Y)}$ by $\tilde{F}(x) = (F_x)_\beta$ for any $x \in X$. It follows from the P -monotonicity of F that for any $x, y \in X$, for any $s \in \tilde{F}(x)$ and $t \in \tilde{F}(y)$, $\langle s - t, x - y \rangle \in P$. This implies that $\tilde{F}$ is P -monotone. The V-hemicontinuity of $\tilde{F}$ is easily proved and the compactness of $\tilde{F}(x)$ for each $x \in X$ is proved similarly as in the proof of Theorem 5. Consequently by Theorem 4 there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in \tilde{F}(x_0)$ such that $\langle s_0, x - x_0 \rangle \notin -Int\ C(x_0)$. Hence there exists an $x_0 \in D$ such that for each $x \in D$, there exists an $s_0 \in L(X,Y)$ with $F_{x_0}(s_0) \geq \beta$ such that $\langle s_0, x - x_0 \rangle \notin -Int\ C(x_0)$.

COROLLARY 3.3. *Let X be a reflexive Banach space and Y a Banach space. Let D be a nonempty bounded, closed and convex subset of X, and $C : D \to 2^Y$ be a multifunction such that for each $x \in D$, $C(x)$ is a convex cone in Y with Int $C(x) \neq \emptyset$ and $C(x) \neq Y$, and $P := \bigcap_{x \in D} C(x)$ a nonempty convex cone in Y with Int $P \neq \emptyset$. Let $W : D \to 2^Y$ be a weakly closed multifunction defined by $W(x) = Y \backslash (-Int\ C(x))$ for any $x \in D$, and $F : X \to \mathcal{F}(L(X,Y))$ a fuzzy mapping such that there exists a real number $\beta \in (0,1]$ such that for each $x \in X$, $(F_x)_\beta$ is a nonempty subset of $L(X,Y)$. If F is P -monotone, hemicontinuous, and for each $x \in X, F_x$ is a compact fuzzy set on $L(X,Y)$, then there exists an $x_0 \in D$ such that for each $x \in D$ there exists an $s_0 \in L(X,Y)$ with $F_{x_0}(s_0) \geq \beta$ such that $\langle s_0, x - x_0 \rangle \notin -Int\ C(x_0)$.*

REMARK. When $C(x)$ is a constant cone in Corollary 3.3, we can show that the result of Corollary 3.3 holds under P-pseudomonotonicity of F [14].

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