ON PROJECTIVE REPRESENTATIONS OF A
FINITE GROUP AND ITS SUBGROUPS II

SEUNG AHN PARK AND EUNMI CHOI

1. Introduction

This is the sequel to our paper "On projective representations of
a group and its subgroups I" [4]. In Section 4 [4] we proved some
global properties on regularity condition. The purpose of this paper
is to study local properties, that is, we shall ask how the regularity
condition on subgroups is related to that on group. Throughout the
paper we use the same notations as in [4].

2. Regularity Conditions on Groups: Local condition

THEOREM 1. Let $G = H \times K$ with normal subgroups $H$ and $K$.
Let $\beta \in Z^2(H, F^*)$. If every $F$-class of $H$ is $(F, \beta)$-regular then every
$F$-class of $G$ is $(F, \text{Cor} \beta)$-regular, where $\text{Cor} = \text{Cor}_{H,G}$.

Proof. Choose integers $n$ and $m(\sigma)$ for $G$ satisfying the condition
(A) in [4]. Then these integers work for $H$, too. For $g \in G$, choose
$(\sigma, x) \in \mathcal{G} \times G$ such that $x^{-1} g^{m(\sigma^{-1})} x = g$, and write $m(\sigma^{-1}) = m$.
Choose $K$ as a transversal of $H$ in $G$ with respect to which $\text{Cor}_{H,G}$
is defined. Write $g = h\bar{g}$ and $x = z\bar{x}$ so that $z^{-1} h^{m} z = h \in H$ and
$\bar{x}^{-1} \bar{g}^{m} \bar{x} = \bar{g} \in K$, for $h, z \in H, g, \bar{x} \in K$. Then

$$v_H(h)^{\sigma^{-1}} v_H(h)^{-m} \prod_{i=1}^{m-1} \beta(h^i, h) \beta(h^{m}, z) = \beta(z, h)$$

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where \( v_H(h) \) is as in (A) with respect to \( \beta \). Since

\[
v_H(h)^\mu = \prod_{i=1}^{n-1} \beta \left( g^i g^{-1}, g g^{-1} \right)^\mu = \prod_{i=1}^{n-1} (\text{Cor} \beta)(g^i, g)
\]

(here we have used Theorem 5 [4]), we can choose an \( n \)-th root \( v(g) \) of \( \prod_{i=1}^{n-1} (\text{Cor} \beta)(g^i, g) \) such that \( v(g) = v_H(h)^\mu \). Then

\[
v(g)^{\sigma^{-1}} v(g)^{-m} \prod_{i=1}^{m-1} (\text{Cor} \beta)(g^i, g)(\text{Cor} \beta)(g^m, x)
\]

\[
= \left[ v_H(h)^{\sigma^{-1}} v_H(h)^{-m} \right]^\mu \cdot \left[ \prod_{i=1}^{m-1} \beta(h^i, h)(h^m, z) \right]^\mu = \beta(z, h)^\mu,
\]

which equals \( (\text{Cor} \beta)(x, g) \). This completes the proof.

**Corollary 2.** Suppose that \( G \) is a finite nilpotent group which is a direct product \( G = Y_{q_1} \times \cdots \times Y_{q_t} \) of normal Sylow subgroups \( Y_{q_i} \). If every \( F \)-class of \( Y_{q_i} \) is \((F, \alpha)-regular\) for \( \alpha \in Z^2(Y_{q_i}, F^*) \), then every \( F \)-class of \( G \) is \((F, \text{Cor}_{Y_{q_i}}, C \alpha)\)-regular.

Since \( G = Y_{q_1} \times (Y_{q_2} \times \cdots \times Y_{q_t}) \), it is clear by induction on \( t \).

For a partial answer of the main question, we assume that \( G \) is a nilpotent group or \( G \) is a central product of two subgroups.

**Theorem 3.** Let \( G \) be a nilpotent group as in Corollary 2, and let \( f \in Z^2(G, F^*) \). If every \( F \)-class of \( Y_q \) is \((F, f_{Y_q})\)-regular for all \( q \), then every \( F \)-class of \( G \) is \((F, f)\)-regular.

**Proof.** Let \( K_j = Y_{q_1} \times \cdots \times Y_{q_{j-1}} \times Y_{q_{j+1}} \times Y_{q_t} \) with \( |K_j| = \mu_j \), and take \( K_j \) as a transversal of \( Y_{q_j} \) in \( G \) where \( \text{Cor}_{Y_{q_j}, C} \) (write \( \text{Cor} \) for it) is defined. If every \( F \)-class of \( Y_{q_j} \) is \((F, f_{Y_{q_j}})\)-regular, then every \( F \)-class of \( G \) is \((F, \text{Cor}_j(f_{Y_{q_j}}))\)-regular (by Theorem 1) thus every \( F \)-class of \( G \) is \((F, f^{\mu_j})\)-regular, since \( \text{Cor}_j(f_{Y_{q_j}}) \) is cohomologous to \( f^{\mu_j} \) (by Lemma 2 and 3 [4]). Choose \( n \) and \( m(\sigma) \) for \( G \) as in (A), and write \( m = m(\sigma^{-1}) \). Then these integers work for subgroups of \( G \).

For \( g \in G \), choose \( (\sigma, x) \in G \times G \) with \( x^{-1} g^m(\sigma^{-1}) x = g \). With \( v_j(g) \) which is analogues of \( v(g) \) for \( f^{\mu_j} \),

\[
v_j(g)^{\sigma^{-1}} v_j(g)^{-m} \prod_{i=1}^{m-1} f^{\mu_j}(g^i, g) f^{\mu_j}(g^m, x) = f^{\mu_j}(x, g).
\]
Since $v_j(g)^n = \prod_{i=1}^{n-1} f(g^i, g) = v(g)^{n \mu_j}$, there is $v_j'(g)$ such that $v_j'(g) = v(g)^{\mu_j}$, so we may let $v_j(g) = v_j'(g)$. Using integers $c_1, \ldots, c_t$ with $\sum_{j=1}^t \mu_j c_j = 1$, $\prod_{j=1}^t v_j(g)^{c_j} = v(g)^{\sum_{j=1}^t \mu_j c_j} = v(g)$, hence

$$f(x, g) = \left[f \sum_{j=1}^t \mu_j c_j \right](x, g) = \prod_{j=1}^t f^{\mu_j c_j}(x, g)$$

$$= \left[\prod_{j=1}^t v_j(g)^{c_j} \right]^{\sigma^{-1}} \left[\prod_{j=1}^t v_j(g)^{c_j} \right]^{-m} \prod_{i=1}^{m-1} \left[\prod_{j=1}^t f^{\mu_j c_j}(g^i, g) \right] \prod_{j=1}^t f^{\mu_j c_j}(g^m, x)$$

$$= v(g)^{\sigma^{-1}} v(g)^{-m} \prod_{i=1}^{m-1} f(g^i, g) f(g^m, x);$$

thus every $F$-class of $G$ is $(F, f)$-regular.

Theorem 3 implies that when $G$ is a nilpotent group and $f \in Z^2(G, F^*)$, if the number of irreducible representations of each Sylow subgroup $Y_q$ of $G$ (over $F$) equals that of irreducible $f_{Y_q}$-representations of $Y_q$ then the number of irreducible representations of $G$ equals that of irreducible $f$-representations of $G$.

**Corollary 4.** Suppose that $G = HK$ is a central product of two subgroups $H$ and $K$, and suppose that we choose a transversal $S$ of $H$ in $G$ as a subgroup of $K$ with respect to which $\text{Cor}_{H,G} = \text{Cor}$ is defined. If every $F$-class of $H$ is $(F, \alpha)$-regular for $\alpha \in Z^2(H, F^*)$ then every $F$-class of $G$ is $(F, \text{Cor} \alpha)$-regular.

**Proof.** Since $S$ is a subgroup of $K$ and $H \cap K \subseteq Z(G)$, we may apply Theorem 5 [4] to this case.

### 3. Regularity Conditions on Semi-direct Product

For a general finite group $G$, it is not known whether the converse of Theorem 6 [4] is true, even when $G$ is a semi-direct product and
even when $F$ is the complex field $C$. In this section we shall consider the same question over certain semi-direct cases and over $C$; such as Frobenius groups and dihedral groups.

Let $G$ be a Frobenius group with Frobenius kernel $N$ and complement $H$. Then $G = NH$, $N \cap H = \{1\}$ and $H$ acts on the normal subgroup $N$ without fixed points, i.e., $H \cap H^x = 1$ for $x \in G - H$ (here $H^x = x^{-1}Hx$). And we have $C_G(n) \subseteq N$ and $C_G(h) \subseteq H$ for $1 \neq n \in N$, $1 \neq h \in H$. If $|H|$ is even then $N$ is abelian and $H$ posseses a unique involution which is in $Z(H)$.

**Theorem 5.** Let $G = NH$ be a Frobenius group with Frobenius kernel $N$ and Frobenius complement $H$. Suppose that $|H| = 2$ and every class of $N$ is $f$-regular for $f \in Z^2(N, C^*)$. Then every class of $G$ is $\text{Cor}_{N,G,f}$-regular, where $\text{Cor}_{N,G}$ is defined by the transversal $H$ of $N$ in $G$.

**Proof.** Since $|H|$ is even, $N$ is abelian and so we have that $C_G(n) = N$ for all $n \in N - \{1\}$. Now let $H = \{1, h\}$. Since $G = N \cup (\bigcup_{x \in G} H^x)$, it follows that

$$G - N = \{h^x \mid x \in G\} = Nh = \{nh \mid n \in N\},$$

and $C_G(h^x) = H^x = \{1, h^x\}$ for $x \in G$. Thus the pair $(g_1, g_2)$ of elements in $G$ which commutes each other is one of the following:

$$(n, m), (h^x, 1), (h^x, h^x)$$

where $n, m \in N$. Hence $(\text{Cor}_f)(g_1, g_2) = (\text{Cor}_f)(g_2, g_1)$ when $g_1 g_2 = g_2 g_1$, thus every class of $G$ is $\text{Cor}_f$-regular.

**Theorem 6.** Let $D_n = \langle a, b \mid a^n = b^2 = 1, ab^{-1} = a^{-1} \rangle$ be a dihedral group of order $2^n$. If every class of $\langle a \rangle$ is $f$-regular for $f \in Z^2(\langle a \rangle, C^*)$, then every class of $G$ is $\text{Cor}_{\langle a \rangle,G,f}$-regular, where $\text{Cor}_{\langle a \rangle,G,f}$ is defined by a transversal $\langle b \rangle$ of $\langle a \rangle$ in $G$.

**Proof.** For odd $n$, $D_n$ is a Frobenius group with Frobenius kernel $\langle a \rangle$ and Frobenius complement $\langle b \rangle$ where $|\langle b \rangle| = 2$, and the assertion follows from Theorem 5. Suppose $n = 2m$ is even. Then $D_{2m} = \langle a \rangle \langle b \rangle$ is a semi-direct product but not a Frobenius group since $Z(D_{2m}) = \langle a^m \rangle = \{1, a^m\}$. It is easy to see that
On projective representations of a finite group and its subgroups

\[ C_G(a^i) = D_{2m} \text{ if } a = 0 \text{ or } i = m, \]
\[ C_G(a^i) = \langle a \rangle \text{ if } a^i \not\in Z(D_{2m}), \]
\[ C_G(b) = C_G(a^m b) = \{1, b, a^m, a^m b\}, \]
\[ C_G(ab) = C_G(a^{m+1} b) = \{1, ab, a^m, a^{m+1} b\}, \ldots, \]
\[ \text{and } C_G(a^{m-1} b) = C_G(a^{2m-1} b) = \{1, a^{m-1} b, a^m, a^{2m-1} b\}. \]

In the case that \( g = a^m, \ x = a^ib, \ 0 \leq i < 2m, \) we have
\( (\text{Corf})_i(g, x) = f(a^m, a^i)f(a^m, a^{-i}) = (\text{Corf})(x, g). \)

If \( g = a^ib, \ x = a^{m+i} b, \ 0 \leq i < m, \) we have
\( (\text{Corf})(g, x) = f(a^i, a^{m-i})f(a^{-i}, a^{m+i}) = (\text{Corf})(x, g). \)

Hence \( (\text{Corf})(g, x) = (\text{Corf})(x, g) \) for all \( g, x \in D_{2m} \) such that \( gx = xg. \) This completes the proof.

4. Examples

The present section is devoted to constructing explicit examples of groups for which the converse of Theorem 6 in [4] holds. Most of previously known examples of the situation have been nilpotent groups, the groups in this section are not nilpotent.

By a notation \([g, x]_\sigma = g^{-1}x^{-1}g^m(\sigma^{-1})x, \ \sigma \in G, \) we mean an \( F\)-commutator of \( g \) and \( x \) in \( G. \) A subgroup generated by all \( F\)-commutators is called \( F\)-commutator subgroup of \( G \) and denoted by \( G'(F). \) If \( F = E \) an algebraic closure over \( F \) then \([g, x]_\sigma = [g, x] \)
a commutator, and \( G'(F) = G' \) the commutator subgroup. Further for any subgroup \( H \) of \( G \) and \( \sigma \in \mathbb{G} = \text{Gal}(E/F) \) we write \([H, G]_\sigma \)
for \( ([h, g]_\sigma | h \in H, g \in G); \) if \( H = G \) then \([H, G]_\sigma = G'(F). \) The following lemma is important for application.

**Lemma 7.** ([2]) For a field \( F, \) the following are equivalent.

1. Finding a group \( G \) with \( f \in Z^2(G, F^*) \) such that \( f \) is not cohomologous to 1, \( o(f) < \infty, \) and every \( F\)-class of \( G \) is \( D_{\Gamma} \)-regular for \( \Gamma = F^f G. \)

2. Finding a group \( H \) with a \( p'\)-cyclic subgroup \( A \) such that \( \zeta_{|A|} \in F^*, \)
\( 1 \neq A \subseteq Z(H) \cap H'(F), \) and such that \( A \) contains no \( F\)-commutator of \( H \) except identity.

In case \( F = C, \) the lemma was studied in [5]. We construct an example of class of groups, which was studied while the second author
was in Tufts University. The author would like to express sincere
thank to Professor W.F. Reynolds. Let $F = C$ and
\[ H = \langle h_{1}, \ldots, h_{10} \mid [h_{2}, h_{1}] = h_{5}, \ [h_{3}, h_{1}] = h_{6}, \ [h_{4}, h_{1}] = h_{7}, \ [h_{3}, h_{2}] = h_{8}, \ [h_{4}, h_{2}] = h_{9}, \ [h_{4}, h_{3}] = h_{10}, \ \phi(h_{i}) = q \rangle. \]

This group has been introduced in [1, § 3], and studied in [5]. Let
$A = \langle h_{5}h_{10} \rangle$. Then $|H| = q^{10}$, $|A| = q$ and $A \subseteq Z(H) = H' = \langle h_{5}, h_{6}, h_{7}, h_{8}, h_{9}, h_{10} \rangle$. For the sake of clarity, we divide this example
into five steps.

STEP 1. A does not contain any commutator of $H$ except 1.

Take any elements $h, x$ in $H$. We may let, once and for all,
\[ (4.1) \quad h = h_{1}^{a_{1}} \cdots h_{10}^{a_{10}}, \quad x = h_{1}^{b_{1}} \cdots h_{10}^{b_{10}}, \quad 0 \leq a_{i}, b_{i} < q. \]

Let $k_{ij} = a_{i}b_{j} - a_{j}b_{i}$ for $1 \leq i < j \leq 4$. Since $H' \subseteq Z(H)$,
\[ (4.2) \quad [h, x] = h_{5}^{-k_{12}}h_{6}^{-k_{13}}h_{7}^{-k_{14}}h_{8}^{-k_{23}}h_{9}^{-k_{24}}h_{10}^{-k_{34}}. \]

If $1 \neq [h, x] \in A$ then $[h, x] = h_{5}^{i}h_{10}^{j}$ for some $0 < i < q$. Comparing
this with (4.2), we have (by modulo $q$) $k_{12} \equiv k_{13} \equiv -i$, $k_{13} \equiv k_{14} \equiv k_{23} \equiv k_{24} \equiv 0$. Thus $k_{12}k_{34} - k_{13}k_{24} + k_{14}k_{23} \equiv i^{2}$, whereas
$k_{12}k_{34} - k_{13}k_{24} + k_{14}k_{23} = 0$, hence $i^{2} \equiv 0$. However $q$ does not divide
$i^{2}$, this yields a contradiction.

Let $L \cong H/A$ and consider the central cyclic group extension
\[ (4.3) \quad 1 \to A \to H \to \frac{H}{A} \to 1. \]

Then by Lemma 7, there is a noncobounding $f \in Z^{2}(L, C^{*})$ such that
every class of $L$ is $f$-regular.

STEP 2. Any permutation $\tau$ of $\{h_{1}, h_{2}, h_{3}, h_{4}\}$ extends to an
automorphism of $H$.

Define $\tau : H \to H$ by $\tau(h)$ equals
\[ \tau(h_{1})^{a_{1}}\tau(h_{2})^{a_{2}}\tau(h_{3})^{a_{3}}\tau(h_{4})^{a_{4}} \cdot [\tau(h_{2}), \tau(h_{1})]^{a_{5}}[\tau(h_{3}), \tau(h_{1})]^{a_{6}} \]
\[ \cdot [\tau(h_{4}), \tau(h_{1})]^{a_{7}}[\tau(h_{3}), \tau(h_{2})]^{a_{8}}[\tau(h_{4}), \tau(h_{2})]^{a_{9}}[\tau(h_{4}), \tau(h_{3})]^{a_{10}}. \]
Clearly $\tilde{\tau}$ is a well-defined bijection, and for $x \in H$,

\begin{equation}
\tilde{\tau}(hx) = h^1_{a+b_1} h^2_{a+b_2} h^3_{a+b_3} h^4_{a+b_4} h^5_{a+b_5} h^6_{a+b_6} h^7_{a+b_7} h^8_{a+b_8} h^9_{a+b_9} h^{10}_{a+b_{10}},
\end{equation}

Hence by applying $\tilde{\tau}$ to $hx$, we have

\[ \tilde{\tau}(hx) = \tau(h_1)^{a+b_1} \tau(h_2)^{a+b_2} \tau(h_3)^{a+b_3} \tau(h_4)^{a+b_4} \cdot \left[ \tau(h_2), \tau(h_1) \right]^{a+b_5+b_6} \cdot \left[ \tau(h_3), \tau(h_1) \right]^{a+b_6+b_7} \cdot \left[ \tau(h_4), \tau(h_3) \right]^{a+b_7+b_8} \cdot \left[ \tau(h_4), \tau(h_2) \right]^{a+b_8+b_9} \cdot \left[ \tau(h_4), \tau(h_3) \right]^{a+b_9+b_{10}}. \]

On the other hand, using the calculation similar to the one that yielded (4.4), we have $\tilde{\tau}(hx) = \tilde{\tau}(h)\tilde{\tau}(x)$.

Due to STEP 2, we define an automorphism $e$ of $H$ such that

\begin{equation}
e(h_1) = h_3, \ e(h_2) = h_4, \ e(h_3) = h_1, \ e(h_4) = h_2.\end{equation}

Then $e(h_5) = h_{10}, \ e(h_6) = h_6^{-1}, \ e(h_7) = h_8^{-1}, \ e(h_8) = h_8^{-1}, \ e(h_9) = h_9^{-1}$ and $e(h_{10}) = h_5$, hence $e^2 = 1, \ e(h_5 h_{10}) = h_5 h_{10}$.

Let $T = H \langle \epsilon \rangle$ be the semi-direct product with multiplication $h u \cdot x y = h x^u \cdot u y$ for $h, x \in H, u, y \in \langle \epsilon \rangle$, where $x^u$ means $u^{-1} x u$. Then $(h u)^{-1} = (h^{-1})^u u$, and we may consider $H$ and $\langle \epsilon \rangle$ as subgroups of $T$. $T$ is not nilpotent, $A \subseteq Z(T)$, and $e$ fixes $h_5 h_{10}$.

Consider another central cyclic group extension

\begin{equation}1 \rightarrow A \rightarrow T \rightarrow \frac{T}{A} \rightarrow 1.\end{equation}

Since $\langle \epsilon \rangle$ is an operator group on $H$ and since $A$ is an $\langle \epsilon \rangle$-invariant normal subgroup of $H$, $T/A$ is isomorphic to the semi-direct product of $H/A$ and $\langle \epsilon \rangle$ with respect to the induced action on $H/A$. Hence the exact sequence (4.6) is isomorphic to

\[ 1 \rightarrow A \rightarrow H \langle \epsilon \rangle \rightarrow \frac{H}{A} \langle \epsilon \rangle \rightarrow 1. \]
STEP 3. A does not contain any commutators of \( T \) of the form \([h, t] \) or \([t, h] \) except the identity, for \( h \in H, t \in T \).

Let \( t = xy \) (\( x \in H, y \in \langle e \rangle \)), then \( y \) must be either 1 or \( e \), and

\[ [h, t] = h^{-1}((x^{-1})^y y) h(x y) = h^{-1} h y [h, x] y \in H. \]

If \( y = 1 \) then \([h, t] = [h, x] \) a commutator of \( H \); this is reduced to STEP 1. Let \( y = e \). Then for any \( h \) and \( x \) as in (4.1),

\[ h^{-1} h^e = h_1 h_2 h_3 h_4 h_5 h_6 h_7 h_8 h_9 h_{10}. \]

Moreover by (4.2),

\[ [h, x]^e = h_1^{-k_{12}} h_2 h_3 h_4 h_{13} h_{14} h_{23} h_{24} h_9 h_5^{-k_{34}}. \]

Multiplying the above two equations, we have

\[ [h, t] = h^{-1} h^e [h, x]^e = h_1 a_3 - a_1 h_2 a_2 - a_2 h_3 a_1 - a_3 h_4 a_2 a_1 - a_4 \]
\[ h_5 h_6 h_7 a_5 - a_5 + a_1 - a_3 - 2a_6 h_8 a_4 a_1 - a_3 - 2a_8 h_9 a_4 h_10. \]

Suppose that A contains a nonidentity generator of \([H, T]\), i.e., \( 1 \neq [h, t] = h_1 h_{10} \in A \). Comparing exponents, we have

(i) \( a_1 - a_3 \equiv a_2 - a_4 \equiv 0 \)

(ii) \( a_5 - a_{10} - k_{12} \equiv i, \quad -a_5 + a_{10} - k_{34} \equiv i \)

(iii) \( a_1 a_3 - 2a_6 + k_{13} \equiv 0, \quad a_2 a_4 - 2a_9 + k_{24} \equiv 0 \)

(iv) \( a_1 a_4 - a_7 - a_8 + k_{14} \equiv 0, \quad a_2 a_3 - a_7 - a_8 + k_{23} \equiv 0 \)

Here, (i) has been used to simplify the other congruences. The definition of \( k_{ij} \), and (i), (ii) and (iv) give rise to

\[ -2i \equiv k_{12} + k_{34} \equiv a_1 b_2 - a_2 b_1 + a_1 b_4 - a_2 b_3 \]
\[ 0 \equiv k_{14} - k_{23} \equiv a_1 b_4 - a_2 b_1 - a_2 b_3 + a_1 b_2. \]

hence \( 2i \equiv 0 \mod(q) \). This is a contradiction since \( q \) is odd. Thus A does not contain any commutators of \( T \) of the form \([h, t] \) except 1. Also
since \([t, h] = [h, t]^{-1}\), the fact that \(1 \neq [h, t] \notin A\) implies \(1 \neq [t, h] \notin A\). Moreover \(A\) contains no generators of \([H, T]\) or \([T, H]\) except 1.

**STEP 4.** A does not contain any commutators of \(T\) except 1.

Let \([r, t]\) for some \(r, t \in T\) be any commutator in \(T\) and let \(r = hu\) and \(t = xy\) for \(h, x \in H\), \(u, y \in \langle e \rangle\). Then

\[
[r, t] = (h^{-1}u(x^{-1})^yuhyxy \in H.,
\]

There are 4 cases for choosing \(y\) and \(u\). If either \(y\) or \(u\) is 1, then \([r, t]\) is \([r, x]\) or \([h, t]\); this is STEP 3. Assume \(y = u = e\). From (4.7) and \(H' \subseteq Z(H)\),

\[
[r, t] = (h^{-1}h^e)^e \cdot (x^{-1}x^e) \cdot [x, h]^{-1} = (h^{-1}h^e)^e \cdot (x^{-1}x^e) \cdot [h, x].
\]

We computed \([h, x]\), \(h^{-1}h^e\) and \(x^{-1}x^e\). Further we calculate

\[
\begin{align*}
(h^{-1}h^e)^e &= h_1^{-a_1-a_3}h_2^{-a_2-a_4}h_3^{-a_1-a_3}h_4^{-a_2-a_4}h_5^{-a_4(a_1-a_3)+a_5-a_10}h_6^{-a_2+2a_6} \\
&\quad \cdot h_7^{-a_1a_2+a_7+a_8}h_8^{-a_1a_2+a_7+a_8}h_9^{-a_2+2a_9}h_{10}^{-2(a_1-a_3)-a_5+a_10},
\end{align*}
\]

and

\[
\begin{align*}
(h^{-1}h^e)^e \cdot (x^{-1}x^e) &= h_1^{-a_1-a_3-b_1+b_3}h_2^{-a_2-a_4-b_2+b_4}h_3^{-a_1+a_3+b_1-b_3}h_4^{-a_2+a_4+b_2-b_4} \\
&\quad \cdot h_5^{-a_2-a_4}(b_1-b_3)-a_1a_2+a_7+a_8+b_4(b_1-b_3)+b_2b_3-b_7-b_8 \\
&\quad \cdot h_6^{-a_3(b_1-b_3)-a_1a_2+a_7+a_8+b_3(b_2-b_4)+b_1b_4-b_7-b_8 \\
&\quad \cdot h_7^{-a_2-a_4}(b_2-b_4)-a_3(a_1-a_3)-a_5+a_10-b_4(b_1-b_3)-b_5-b_10 \\
&\quad \cdot h_8^{-a_2-a_4}(b_2-b_4)-a_3(a_1-a_3)-a_5+a_10-b_4(b_1-b_3)-b_5-b_10.
\end{align*}
\]

Multiply this by \([h, x]\) as in (4.2), then we have

\[
[r, t] = (h^{-1}h^e)^e \cdot (x^{-1}x^e) \cdot [h, x]
\]

\[
= h_1^{-a_1-a_3-b_1+b_3}h_2^{-a_2-a_4-b_2+b_4}h_3^{-a_1+a_3+b_1-b_3}h_4^{-a_2+a_4+b_2-b_4} \\
&\quad \cdot h_5^{-a_2-a_4}(b_1-b_3)-a_1a_2+a_7+a_8+b_4(b_1-b_3)+b_2b_3-b_7-b_8 \\
&\quad \cdot h_6^{-a_3(b_1-b_3)-a_1a_2+a_7+a_8+b_3(b_2-b_4)+b_1b_4-b_7-b_8 \\
&\quad \cdot h_7^{-a_2-a_4}(b_2-b_4)-a_3(a_1-a_3)-a_5+a_10-b_4(b_1-b_3)-b_5-b_10 \\
&\quad \cdot h_8^{-a_2-a_4}(b_2-b_4)-a_3(a_1-a_3)-a_5+a_10-b_4(b_1-b_3)-b_5-b_10.
\]
Suppose that \( 1 \neq [r, t] = h_i^r h_i^{-1} \) for some \( i \). Then

1) \( a_1 - a_3 - b_1 + b_3 \equiv a_2 - a_4 - b_2 + b_4 \equiv 0 \)

2) \((a_4 - a_2)(b_1 - b_3) - a_4(a_1 - a_3) + a_5 - a_{10} + b_2(b_1 - b_3) + b_5 - b_{10} - k_{12} \equiv k_{12}
\)
\((a_2 - a_4)(b_1 - b_3) + a_4(a_1 - a_3) - a_5 - a_{10} - b_2(b_1 - b_3) + b_5 - b_{10} - k_{34} \equiv k_{34}\)

3) \((a_1 - a_3)(b_1 - b_3) - a_2^2 + 2a_9 + b_4(b_1 - b_3) + b_1b_3 - 2b_6 - k_{13} \equiv 0
\)
\((a_2 - a_4)(b_2 - b_4) - a_2^2 + 2a_9 + b_4(b_2 - b_4) + b_2b_4 - 2b_9 - k_{24} \equiv 0\)

4) \((a_2 - a_4)(b_1 - b_3) - a_1 a_2 + a_7 + a_8 + b_2b_3 + b_4(b_1 - b_3) + b_7 - b_8 - k_{14} \equiv 0
\)
\((a_1 - a_3)(b_2 - b_4) - a_1 a_2 + a_7 + a_8 + b_4 + b_3(b_2 - b_4) - b_7 - b_8 - k_{23} \equiv 0\)

From 1), we have \( a_1 - a_3 \equiv b_1 - b_3 \) and \( a_2 - a_4 \equiv b_2 - b_4 \) (mod \( q \)). By applying this to 2) and 4), we have

\[ 2i \equiv -2(a_2 - a_4)(a_1 - a_3) + (a_2 - a_4)(a_1 - a_3) + (b_2 - b_4)(a_1 - a_3) \equiv -k_{12} - k_{34}, \]
\[ 0 \equiv b_4(b_1 - b_3) + b_2b_3 - b_3(b_2 - b_4) - b_1b_4 - k_{14} + k_{23} \equiv -k_{14} + k_{23} \]

However since \( 0 \equiv (a_1 - a_3)(b_2 - b_4) - (a_2 - a_4)(b_1 - b_3) - k_{12} + k_{23} - k_{14} + k_{34} \) (by 1)), we have \( -k_{14} + k_{23} \equiv -k_{12} - k_{34} \), and so \( 2i \equiv 0 \) (mod \( q \)), which is contradict to \( q \geq 3 \). This means that \( A \) contains no generators of \( T \) except 1.

**STEP 5.** Finally, we have a situation of Theorem 3 for this group.

In (4.6), let \( T/A \cong G \). Then by Lemma 7, there is a noncobounding \( f' \in Z^2(G, C^*) \) where every class of \( G \) is \( f' \)-regular. Considering \( f' \) together with \( f \) as in below (4.3), define isomorphisms \( H/A \to L \) by \( h_i A \mapsto l \) (\( l \in L, h_i \in H \)) and \( T/A \to G \) by \( t_g A \mapsto g \) (\( g \in G, t_g \in T \)). Then the 2-cocycles \( f \) and \( f' \) are indeed constructed by \( f = \chi \alpha \) and \( f' = \chi \alpha' \), where \( \chi \) is a generator of \( \text{Hom}(A, C^*) \), and \( \alpha \) and \( \alpha' \) are 2-cocycles in \( Z^2(L, A) \) and \( Z^2(G, A) \) respectively, defined by \( \alpha(l, l') = h_l h_l ^{-1} h_l ^{-1} \), \( \alpha'(g, g') = t_g t_{g'} t_{g'} ^{-1} t_{g'} ^{-1} \) (refer to [2]). For any \( l \in L \), we may choose \( h_l = t_l \), so that \( \alpha = \alpha' \). Hence \( f = \chi \alpha = \chi (\alpha') = f' \). Therefore, we have the situation of Theorem 3 that for \( f' \in Z^2(G, C^*) \), every class of \( G \) is \( f' \)-regular and every class of \( L \) is \( f' \)-regular. While \( L \) is a Sylow \( q \)-subgroup of \( G \).
Remark. For any element \( t = xy \in T \) (\( x \in H, y \in \langle e \rangle \)), \( t^{2q} = (xy)^q = 1 \). Hence \( \exp(T) = 2q \). Therefore, if \( F \) is any field of characteristic 0 that contains \( \zeta_{2q} \) then by choosing integer \( n \) divisible by \( 2q \) and \( m(\sigma) \), \( t^{m(\sigma)} = t \) for all \( t \in T \), \( \sigma \in G \). Hence \( A \) contains no \( F \)-commutators of \( T \) except 1; this is a family of examples with the \( F \).

References


Seung Ahn Park
Department of Mathematics
Sogang University
Seoul 121-742, Korea

Eunmi Choi
Department of Mathematics
Hannam University
Ojungdong 133, Taejon, Korea