GENERAL LOCAL COHOMOLOGY MODULES
AND COMPLEXES OF MODULES
OF GENERALIZED FRACTIONS

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0. Introduction

Throughout this paper, \( R \) will be a commutative ring (with non-zero identity) and \( M \) will denote an \( R \)-module.

The modules of generalized fractions were introduced by Sharp and Zakeri [16] and in [17, 3.5] they gave a relationship between modules of generalized fractions and local cohomology modules, that is,

\[
U_d[1]^{-d-1}M \cong H^d_m(M),
\]

where \( (R, \mathfrak{m}) \) is a Noetherian local ring of dimension \( d \), \( U_d[1] \) is the expansion (see [16, 3.2]) of \( \{(a_1, \ldots, a_d, 1) \in R^{d+1} : a_1, \ldots, a_d \text{ forms a system of parameters for } R \} \) and \( H^d_m(M) \) is the local cohomology module of \( M \).

In [5, 2.4], under the same ring as above, when \( U_{d'}[1] \) is the expansion of \( \{(a_1, \ldots, a_{d'}, 1) \in R^{d'+1} : a_1, \ldots, a_{d'} \text{ forms a system of parameters for } M \} \) where \( M \) is a finitely generated \( R \)-module of \( d' = \dim M \), we had a similar result

\[
U_{d'}[1]^{-d'-1}M \cong H^{d'}_m(M).
\]

In [3], Bijan-Zadeh studied a generalization of results of Sharp and Zakeri. He proved that, for a fixed sequence of elements \( x_1, \ldots, x_n \) of a ring \( R \),

\[
U(x)[1]^{-n-1}M \cong H^n_a(M),
\]

Received April 18, 1995.
1991 AMS Subject Classification: 13C05, 13D45.
Key words: Generalized fraction, complex, general local cohomology module
generalized ideal transform.
This research was supported by the Basic Science Research Institute Program,
Ministry of Education of Korea, Project No. BSRI-95-1427.
where \( a = (x_1, \ldots, x_n)R \) and \( U(x)[1] = \{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}, 1) \in R^{n+1} : \alpha_i \in \mathbb{N}\} \). Moreover, in [3, Theorem], for a given triangular subset \( U_n \) of \( R^n \), he obtained

\[
U_n[1]^{-n-1}M \cong H^n_{\Phi(U_n)}(M) \cong \lim_{\longrightarrow \, a \in \Phi(U_n)} H^n_a(M),
\]

where \( \Phi(U_n) = \{(a_1, \ldots, a_n)R : (a_1, \ldots, a_n) \in U_n \} \) and \( H^n_{\Phi(U_n)} \) is the \( n \)-th right derived functor of the general local cohomology functor \( L_{\Phi(U_n)} \) (see Definition 1.4 and Example 1.5(2)).

In [19, 5.2.3 and 15, 3.3], Sharp and Yassi established a relationship between the modules of generalized fractions and the generalized ideal transforms (see Definition 1.4), i.e.,

\[
U_n^{-n}M \cong \lim_{\longrightarrow \, a \in \Phi(U_n)} \text{Hom}_R(a, \text{Im } e^{n-1})
\]

where \( e^{n-1} : U_n^{-n+1}M \to U_n^{-n}M \) is the \( R \)-homomorphism for which

\[
e^{n-1}\left(\frac{m}{(a_1, \ldots, a_{n-1})}\right) = \frac{m}{(a_1, \ldots, a_{n-1}, 1)} \text{ for } m \in M \text{ and } (a_1, \ldots, a_{n-1}) \in U_{n-1}; \text{ and }
\]

\[
U_n^{-n}M \cong \lim_{\longrightarrow \, a \in \Phi(U_n)} \text{Hom}_R(a, U_{n-1}[1]^{-n}M)
\]

where \( R \) is Noetherian and \( U_{n-1}[1] = \{(a_1, \ldots, a_{n-1}, 1) \in R^n : \text{ there is } a_n \in R \text{ such that } (a_1, \ldots, a_n) \in U_n\} \).

Under an arbitrary ring, consider the complex \( C(U, M) \) (see Definition 1.3). In our main results (Theorem 2.2 and 2.4), we investigate the relationship between the modules of generalized fractions \( U_n^{-n}M, \ U_{n-1}[1]^{-n-1}M, \text{ Im } e^{n-1}, \text{ Ker } e^n \text{ and Ker } e^{n-1}/\text{Im } e^{n-2} \) and the general local cohomology modules of such modules.

That is, we have

\[
\text{Ker } e^{n-1}/\text{Im } e^{n-2} \cong \bigcup_{(a_1, \ldots, a_n) \in U_n} \text{Ann}_{U_{n-1}[1]^{-n}M(a_1, \ldots, a_n)R}
\]

and

\[
U_n[1]^{-n-1}M \cong H^1_{\Phi(U_n)}(\text{Im } e^{n-1}).
\]
In particular, under a Noetherian ring, we have for $n \geq 1$

$$U_n^{-n-1}M \cong H^1_{\Phi(U_n)}(U_{n-1}^{-n}M) \cong H^1_{\Phi(U_n)}(\text{Im } e^{n-1});$$

$$U_n^{-n}M \cong \lim_{\longrightarrow \atop a \in \Phi(U_n)} \text{Hom}_R(a, U_n^{-n}M) \cong \lim_{\longrightarrow \atop a \in \Phi(U_n)} \text{Hom}_R(a, \text{Ker } e^n) \cong \lim_{\longrightarrow \atop a \in \Phi(U_n)} \text{Hom}_R(a, \text{Im } e^{n-1}) \cong \lim_{\longrightarrow \atop a \in \Phi(U_n)} \text{Hom}_R(a, U_{n-1}^{-n}M);$$

and

$$H^i_{\Phi(U_n)}(U_n^{-n}M) \cong H^i_{\Phi(U_n)}(U_{n-1}^{-n}M) \cong H^i_{\Phi(U_n)}(\text{Im } e^{n-1}) \cong H^i_{\Phi(U_n)}(\text{Ker } e^n) \text{ for all } i \geq 2.$$

The notation and terminology about the modules of generalized fractions follow [16].

1. Preliminaries

We use $^T$ to denote matrix transpose and $D_n(R)$ to denote the set of all $n \times n$ lower triangular matrices over $R$. For $H \in D_n(R)$, $|H|$ denotes the determinant of $H$. Let $N$ denote the set of positive integers.

**Definition 1.1.** [16, 2.1]. Let $n$ be a positive integer. A non-empty subset $U_n$ of $R^n$ is said to be triangular if

(i) whenever $(a_1, \ldots, a_n) \in U_n$, then $(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}) \in U_n$ for all choices of positive integers $\alpha_1, \ldots, \alpha_n$; and

(ii) whenever $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n) \in U_n$, then there exist $(c_1, \ldots, c_n) \in U_n$ and $H, K \in D_n(R)$ such that $H[a_1 \ldots a_n]^T = [c_1 \ldots c_n]^T = K[b_1 \ldots b_n]^T$.

**Lemma 1.2.** Let $R$ be a ring and $M$ an $R$-module. Let $U_n$ be a triangular subset of $R^n$. Suppose $(a_1, \ldots, c_n)$ and $(b_1, \ldots, b_n)$ are elements of $U_n$ such that $H[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T$ for some $H \in D_n(R)$. Then we have

(1) [16, 2.8] $\frac{|H|m}{(a_1, \ldots, a_n)} = \frac{|H|m}{(b_1, \ldots, b_n)}$ in $U_n^{-n}M$.

(2) [16, 3.3(ii) and 15, 2.2] If $m \in (a_1, \ldots, a_{n-1})M$ then $\frac{m}{(a_1, \ldots, a_n)} = 0$ in $U_n^{-n}M$. In particular, if each element of $U_n$ is a poor $M$-sequence, then the converse holds.
DEFINITION 1.3. [13, p. 52]. Let $R$ be a ring and $M$ an $R$-module. A family $\mathcal{U} = (U_i)_{i \geq 1}$ is called a chain of triangular subsets on $R$ if the following conditions are satisfied:

(i) $U_i$ is a triangular subset of $R^i$ for all $i \in \mathbb{N}$;
(ii) $(1) \in U_1$;
(iii) whenever $(a_1, \ldots, a_i) \in U_i$ with $i \in \mathbb{N}$, then $(a_1, \ldots, a_i, 1) \in U_{i+1}$; and
(iv) whenever $(a_1, \ldots, a_i) \in U_i$ with $1 < i \in \mathbb{N}$, then $(a_1, \ldots, a_{i-1}) \in U_{i-1}$.

Each $U_i$ leads to a module of generalized fractions $U_i^{-1}M$ and we can obtain a complex by Lemma 1.2(2);

$$
0 \xrightarrow{e^{-1}} M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} U_2^{-1}M \longrightarrow \cdots
\longrightarrow U_i^{-1}M \xrightarrow{e^i} U_{i+1}^{-1}M \longrightarrow \cdots
$$

for which $e^0(m) = \frac{m}{(1)}$ for all $m \in M$ and

$$
e^i \left( \frac{x}{(a_1, \ldots, a_i)} \right) = \frac{x}{(a_1, \ldots, a_i, 1)}
$$

for all $i \in \mathbb{N}$, $x \in M$ and $(a_1, \ldots, a_i) \in U_i$.

Let $C(\mathcal{U}, M)$ denote the above complex and $H_i^1(M)$ the $i$-th cohomology module of this complex. That is $H_i^1(M) = \text{Ker } e^i / \text{Im } e^{i-1}$.

For a given triangular subset $U_n$ of $R^n$, let

$U_n[1] = \{(a_1, \ldots, a_n, 1) \in R^{n+1} : (a_1, \ldots, a_n) \in U_n\}$ and

$U_{n-1}[1] = \{(a_1, \ldots, a_{n-1}, 1) \in R^n : \text{there is } a_n \in R \text{ such that } (a_1, \ldots, a_n) \in U_n\}$.

Then clearly $U_n[1]$ and $U_{n-1}[1]$ are triangular subsets of $R^{n+1}$ and $R^n$ respectively. We interpret $U_0[1]^{-1}M = M$ and $U_0^0M = M$.

DEFINITION 1.4. [1, 2.1 and 15, 1.2]. A non-empty set $\Phi$ of ideals of $R$ is called a system of ideals of $R$ if whenever $a, b \in \Phi$ there is $c \in \Phi$ such that $c \subset ab$.

Given such a system of ideals $\Phi$, for every $R$-module $M$, we define

$$
L_\Phi(M) = \{m \in M : ma = 0 \text{ for some } a \in \Phi\} = \bigcup_{a \in \Phi} (0 :_M a)
$$
and

\[ G_\Phi(M) = \lim_{\longrightarrow} \text{Hom}_R(a, M). \]

Then \( L_\Phi \) and \( G_\Phi \) are additive, left exact functors from the category of all \( R \)-modules and \( R \)-homomorphisms to itself. The functor \( L_\Phi \) is called the general local cohomology functor with respect to \( \Phi \) and \( G_\Phi \) the generalized ideal transform determined by \( \Phi \), or, more briefly, the \( \Phi \)-transform.

For any \( R \)-module \( M \), the modules \( H^i_\Phi(M) \) are called general local cohomology modules of \( M \), where \( H^i_\Phi \) is the \( i \)-th right derived functor of \( L_\Phi \). That is, by [1, 2.3 and 2, 2.1] we have

\[ H^i_\Phi(\ ) = \lim_{\longrightarrow} \text{Ext}^i_R(R/a, \ ) = \lim_{\longrightarrow} H^i_a(\ ). \]

We say that an \( R \)-module \( M \) is a \( \Phi \)-torsion module if \( L_\Phi(M) = M \) [14, 1.4(i)].

**Example 1.5.** (1) \( \Phi = \{ a^i : a \text{ is an ideal of } R \text{ and } i \in \mathbb{N} \} \) is a system of ideals of \( R \).

(2) [3, Theorem] \( \Phi(U_n) = \{(a_1, \ldots, a_n)R : (a_1, \ldots, a_n) \in U_n \} \) is a system of ideals of \( R \), where \( U_n \) is a triangular subset of \( R^n \).

**Lemma 1.6.** Let \( R \) be Noetherian and \( M \) an \( R \)-module. Then we have the following.

(1) [19, 3.1.6 and 14, 1.4] If \( M \) is a \( \Phi \)-torsion module, then \( H^i_\Phi(M) = 0 \) for all \( i > 0 \).

(2) [1, 2.7] If \( \dim M = d \), then \( H^i_\Phi(M) = 0 \) for all \( i > d \).

**Proposition 1.7.** Let \( R \) be Noetherian and \( M \) an \( R \)-module. Then we have the following.

(1) \( \text{Supp}(H^i_\Phi(M)) \subset \bigcup_{a \in \Phi} V(a) \).

(2) If \( \text{Supp}(M) \subset \bigcup_{a \in \Phi} V(a) \), then \( H^i_\Phi(M) = 0 \) for all \( i > 0 \).

**Proof.** (1) By [12, p.85 3.13] and [9, 35.5] we have

\[ \text{Supp}(H^i_\Phi(M)) = \text{Supp}(\lim_{\longrightarrow} H^i_a(M)) \subset \bigcup_{a \in \Phi} \text{Supp}(H^i_a(M)) \subset \bigcup_{a \in \Phi} V(a). \]

(2) Since \( M \) is \( \Phi \)-torsion module, this follows from Lemma 1.6(1). □
Lemma 1.8. Let \( R \) be a ring and \( M \) an \( R \)-module. Then, in the complex \( C(\mathcal{U}, M) \), for \( n \geq 0 \) we have the following.

1. \([6, 2.4]\) \( \text{Supp}(U_{n+1}^{-n-1}M) \subset \text{Supp}(U_n[1]^{-n-1}M) \subset \{ p \in \text{Supp}(M) : ht_M p \geq n \} \).

2. \([6, 2.8]\) \( \text{Ass}(U_{n+1}^{-n-1}M) = \text{Ass}(\text{Im } e^n) = \text{Ass}(\text{Ker } e^{n+1}) \).

3. \([6, 2.7]\) For each \( \frac{m}{(a_1, \ldots, a_n)} + \text{Im } e^{n-1} \in H^n(M) \), there are \((b_1, \ldots, b_{n+1}) \in U_{n+1} \) and \( H \in D_n(R) \) such that \( H[a_1 \ldots a_n]^T = [b_1 \ldots b_n]^T \), and

\[(b_1, \ldots, b_{n+1})R \subset \left( \frac{m}{(a_1, \ldots, a_n)} \right) . \]

Proposition 1.9. Let \( R \) be Noetherian and \( M \) an \( R \)-module. Let \( \Phi \) be a system of ideals of \( R \) and \( d = \dim M \). Then, in the complex \( C(\mathcal{U}, M) \), for \( n \geq 0 \) we have the following.

1. \( H^1_\Phi(\text{Ker } e^n/\text{Im } e^{n-1}) = 0 \) for all \( i > d - n \).

2. \( H^1_\Phi(U_{n+1}^{-n-1}M) = H^1_\Phi(U_n[1]^{-n-1}M) = H^1_\Phi(\text{Ker } e^{n+1}) = H^1_\Phi(\text{Im } e^n) = 0 \) for all \( i > d - n \).

Proof. By Lemma 1.8 we have \( \text{Supp}(\text{Im } e^n) = \text{Supp}(\text{Ker } e^{n+1}) = \text{Supp}(U_{n+1}^{-n-1}M) \subset \text{Supp}(U_n[1]^{-n-1}M) \subset \{ p \in \text{Supp}(M) : ht_M p \geq n \} \) and \( \text{Supp}(\text{Ker } e^n/\text{Im } e^{n-1}) \subset \text{Supp}(U_n[1]^{-n-1}M) \) by \([6, 2.8(5)]\). Therefore the results follow from Lemma 1.6(2). \( \square \)

Remark. Let \( R \) be Noetherian and \( M \) a finitely generated \( R \)-module. In the complex \( C(\mathcal{U}, M) \), assume that \( \mathcal{U} = ((U_a)i_{i \geq 1}, ((U_h)i_{i \geq 1}, ((U_r)i_{i \geq 1}) or ((U_f)i_{i \geq 1}) \), where \( R \) is local in the case \(((U_f)i_{i \geq 1}) \) see \([6, \text{Example 1.3}]\). Then we have

\( H^1_\Phi(\text{Ker } e^n/\text{Im } e^{n-1}) = 0 \) for all \( i \geq d - n \).

For, from Lemma 1.8(3) we have easily \( \dim(\text{Ker } e^n/\text{Im } e^{n-1}) < d - n \).

Proposition 1.10. Let \( R \) be a ring and \( M \) an \( R \)-module. For a fixed positive integer \( n \), assume \( U_n \) is a triangular subsets of \( R^n \). Let \( \text{Ass}_f(U_n^{-n}M) = \{ q \in \text{Supp}(M) : q \text{ is a weakly associated prime ideal of } U_n^{-n}M \text{ in the sense of } [4, \text{p.289 Exercise 17}] \} \) and \( p \in \text{Ass}_f(U_n^{-n}M) \),
that is, \( \mathfrak{p} \) is a minimal prime over \((0 : x)\) for some \(0 \neq x \in U_n^{-}M\). Then we have, for all \((a_1, \ldots, a_n) \in U_n,\)

\[
(a_1, \ldots, a_n)R \not\subset \mathfrak{p}.
\]

In particular, for all \(0 \neq y \in U_n^{-}M\), we have \((a_1, \ldots, a_n)R \not\subset (0 : y)\).

**Proof.** By assumption there is \(\frac{m}{(b_1, \ldots, b_n)} \in U_n^{-}M\) such that

\[
\left(0 : \frac{m}{(b_1, \ldots, b_n)}\right) \subset \mathfrak{p} \quad \text{and} \quad \sqrt{\left(0 : \frac{m}{(b_1, \ldots, b_n)}\right)} = \mathfrak{p}R_{\mathfrak{p}}.
\]

Suppose that \((a_1, \ldots, a_n)R \subset \mathfrak{p}\) for some \((a_1, \ldots, a_n) \in U_n\). Then by the definition of the triangular subsets, there are \((c_1, \ldots, c_n) \in U_n\) and \(H, K \in D_n(R)\) such that \(H[a_1 \ldots a_n]^T = [c_1 \ldots c_n]^T = K[b_1 \ldots b_n]^T\). Hence we obtain \((c_1, \ldots, c_n)R \subset \mathfrak{p}\). By Lemma 1.2(1) we have

\[
\sqrt{\left(0 : R_{\mathfrak{p}} \frac{m}{(b_1, \ldots, b_n)}\right)} = \sqrt{\left(0 : R_{\mathfrak{p}} \frac{|K|m}{(c_1, \ldots, c_n)}\right)} = \mathfrak{p}R_{\mathfrak{p}},
\]

where \(\frac{m}{(b_1, \ldots, b_n)} = \frac{|K|m}{(c_1, \ldots, c_n)}\) is regarded as the canonical image in the \(R_{\mathfrak{p}}\)-module. Since \(c_n \in \mathfrak{p}\), there are \(r \in R \setminus \mathfrak{p}\) and \(t \in N\) such that

\[
\frac{c_n \cdot |K|m}{(c_1, \ldots, c_n)} = 0.
\]

Then by [17, 2.1] we get \(\frac{r|K|m}{(c_1, \ldots, c_n)} = 0\). That is, we have the following contradiction.

\[
r \in \left(0 : \frac{|K|m}{(c_1, \ldots, c_n)}\right) = \left(0 : \frac{m}{(b_1, \ldots, b_n)}\right) \subset \mathfrak{p}.
\]

From now on, let \(\Phi_U = (\Phi(U_i))_{i \geq 1}\) be the family of systems of ideals of \(R\) induced by a chain \(U = (U_i)_{i \geq 1}\) of triangular subsets on \(R\) as in Example 1.5(2).
REMARK. In [6, 2.8], using Proposition 1.10, we have the same results with $\text{Ass}_f(U_{n+1}^{-n-1}M)$ instead of $\text{Ass}(U_{n+1}^{-n-1}M)$.

**Lemma 1.11.** Let $R$ be a ring and $M$ an $R$-module. Let $\Phi_U = \langle \Phi(U_i) \rangle_{i \geq 1}$ be as above. Then, in the complex $C(U, M)$, we have the following.

1. For $1 \leq n \leq i$, we have $H^0_{\Phi(U_i)}(U_n^{-n}M) = H^0_{\Phi(U_i)}(\text{Ker } e^n) = H^0_{\Phi(U_i)}(\text{Im } e^{n-1}) = 0$.

2. For $1 \leq i < n$, we have $U_n^{-n}M, \text{Im } e^{n-1}, \text{Ker } e^n, U_{n-1}[1]^{-n}M$ and $H^{n-2}_U(M)$ are $\Phi(U_i)$-torsion modules.

**Proof.** (1) Since $\text{Im } e^{n-1} \subset \text{Ker } e^n \subset U_n^{-n}M$, the results follow immediately from Proposition 1.10.

(2) Since by Lemma 1.2(2) for all $\frac{m}{(a_1, \ldots, a_n)} \in U_n^{-n}M$

$$(a_1, \ldots, a_i)R \cdot \frac{m}{(a_1, \ldots, a_i, \ldots, a_n)} = 0$$

where the ideal $(a_1, \ldots, a_i)R \in \Phi(U_i)$ is induced from $(a_1, \ldots, a_n) \in U_n$, we get $U_n^{-n}M, \text{Im } e^{n-1}$ and $\text{Ker } e^n$ are $\Phi(U_i)$-torsion modules.

Next, using the same method, we have $U_{n-1}[1]^{-n}M$ is a $\Phi(U_i)$-torsion module.

For $H^{n-2}_U(M)$, by Lemma 1.8(3) for all $x \in H^{n-2}_U(M)$ we have

$$(a_1, \ldots, a_{n-1})R \subset (0 : x)$$

for some $(a_1, \ldots, a_{n-1})R \in \Phi(U_{n-1})$. Hence $H^{n-2}_U(M)$ is a $\Phi(U_i)$-torsion module for $i < n$. □

**Corollary 1.12.** Let $R$ be Noetherian and $M$ an $R$-module. Let $\Phi_U = \langle \Phi(U_i) \rangle_{i \geq 1}$ be as above. Then, in the complex $C(U, M)$, we have the following.

1. If $M$ is a $\Phi(U_i)$-torsion module, then $G_{\Phi(U_i)}(M) = 0$.

2. For $1 \leq i < n$, we have $G_{\Phi(U_i)}(U_n^{-n}M) = G_{\Phi(U_i)}(\text{Ker } e^n) = G_{\Phi(U_i)}(\text{Im } e^{n-1}) = G_{\Phi(U_i)}(U_{n-1}[1]^{-n}M) = G_{\Phi(U_i)}(H^{n-2}_U(M)) = 0$. 
Proof. (1) By \([19, 3.1.10]\) we have the following exact sequence;

\[ 0 \rightarrow M/L_{\Phi(U_i)}(M) \rightarrow G_{\Phi(U_i)}(M) \rightarrow H^1_{\Phi(U_i)}(M) \rightarrow 0. \]

Hence the assertion follows from Lemma 1.6(1).

(2) These immediately follow from (1) and Lemma 1.11(2). \(\square\)

2. Main results

**Lemma 2.1.** Let \(R\) be a ring and \(M\) an \(R\)-module. Let \(\Phi_U = (\Phi(U_i))_{i \geq 1}\) be the family of systems of ideals of \(R\) induced by a chain \(\mathcal{U} = (U_i)_{i \geq 1}\) of triangular subsets on \(R\). Then, in the complex \(C(\mathcal{U}, M)\), for \(n \geq 1\) we have the following.

1. [3, Theorem] \(U_n[1]^{-n-1}M \cong H^n_{\Phi(U_n)}(M)\).
2. \([7, 3.3]\) \(U_n[1]^{-n-1}M \cong U_n^{-n}M/\text{Im } e^{n-1}\).
3. \([19, 3.3.8]\) \(U_n^{-n}M \cong G_{\Phi(U_n)}(\text{Im } e^{n-1})\).

**Theorem 2.2.** Let \(R\) be a ring and \(M\) an \(R\)-module. Let \(\Phi_U = (\Phi(U_i))_{i \geq 1}\) be the family of systems of ideals of \(R\) induced by a chain \(\mathcal{U} = (U_i)_{i \geq 1}\) of triangular subsets on \(R\). Then, in the complex \(C(\mathcal{U}, M)\), for \(n \geq 1\) we have the following.

1. \(H^{n-1}_U(M) \cong \bigcup_{(a_1, \ldots, a_n) \in U_n} \text{Ann}_{U_{n-1}[1]^{-n}M}(a_1, \ldots, a_n)R\).
2. \(U_n[1]^{-n-1}M \cong H^1_{\Phi(U_n)}(\text{Im } e^{n-1})\).

**Proof.** (1) Consider the following exact sequence

\[ 0 \rightarrow \ker e^{n-1}/\text{Im } e^{n-2} \rightarrow U_{n-1}[1]^{-n+1}M/\text{Im } e^{n-2} \rightarrow U_{n-1}[1]^{-n+1}M/\ker e^{n-1} \rightarrow 0. \]

Since \(\ker e^{n-1}/\text{Im } e^{n-2} \cong H^{n-1}_U(M)\), \(U_{n-1}[1]^{-n+1}M/\text{Im } e^{n-2} \cong U_{n-1}[1]^{-n}M\) by Lemma 2.1(2) and \(U_{n-1}[1]^{-n+1}M/\ker e^{n-1} \cong \text{Im } e^{n-1}\), we have the following long exact sequence

\[ (*) \]

\[ 0 \rightarrow H^0_{\Phi(U_n)}(H^{n-1}_U(M)) \rightarrow H^0_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \]

\[ \rightarrow H^0_{\Phi(U_n)}(\text{Im } e^{n-1}) \rightarrow H^1_{\Phi(U_n)}(H^{n-1}_U(M)) \]

\[ \rightarrow H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \rightarrow H^1_{\Phi(U_n)}(\text{Im } e^{n-1}) \rightarrow \ldots. \]
Since $H_{\Phi(U_n)}^{n-1}(M)$ is a $\Phi(U_n)$-torsion module and $H_{\Phi(U_n)^0}^0(\text{Im } e^{n-1}) = 0$ by Lemma 1.11, we easily have the conclusion.

(2) From [19, 3.1.10], we have the following exact sequence

$$0 \longrightarrow \text{Im } e^{n-1}/H_{\Phi(U_n)}^0(\text{Im } e^{n-1}) \longrightarrow G_{\Phi(U_n)}(\text{Im } e^{n-1})$$
$$\longrightarrow H_{\Phi(U_n)}^1(\text{Im } e^{n-1}) \longrightarrow 0.$$ 

Since $H_{\Phi(U_n)}^0(\text{Im } e^{n-1}) = 0$ by Lemma 1.11(1) and $G_{\Phi(U_n)}(\text{Im } e^{n-1}) \cong U_{-n}M$ by Lemma 2.1(3), we have

$$H_{\Phi(U_n)}^1(\text{Im } e^{n-1}) \cong U_{-n}M/\text{Im } e^{n-1} \cong U_n[1]^{-n-1}M.$$ 

by Lemma 2.1(2). □

The next Exactness theorem was proved by Sharp and Zakeri [18, 3.3] under the condition of a Noetherian ring and O’carroll [11, 3.1] gave a simple proof for an arbitrary ring. We had shown a refinement of the result of O’carroll [6, 2.13]. We describe another proof of this Exactness theorem using Theorem 2.2(1).

**Corollary 2.3.** Let $R$ be a ring and $M$ an $R$-module. Let $U = (U_i)_{i \geq 1}$ be a chain of triangular subsets on $R$. Then $C(U, M)$ is exact if and only if for all $i \geq 1$ each element of $U_i$ is a poor $M$-sequence.

**Proof.** ($\Rightarrow$) We prove by induction on $i$. In case $i = 1$, by the hypothesis and Theorem 2.2(1), we have $H_{U_1}^0(M) \cong \bigcup_{(a_1) \in U_1} \text{Ann } U_{-i-1}^{-1}M(a_1) = \bigcup_{(a_1) \in U_1} \text{Ann } M(a_1) = 0$. Hence each element of $U_1$ is a poor $M$-sequence.

Suppose that each element of $U_{i-1}$ is a poor $M$-sequence and hence each element of $U_{i-1}[1]$ is a poor $M$-sequence. Note that by Lemma 1.2(2) for all $(b_1, \ldots, b_{i-1}) \in U_{i-1}$

$$\frac{m}{(b_1, \ldots, b_{i-1}, 1)} \neq 0 \in U_{i-1}[1]^{-1}M \iff m \notin (b_1, \ldots, b_{i-1})M.$$

Let $(a_1, \ldots, a_i) \in U_i$. Then we may assume that $\{a_1, \ldots, a_{i-1}\}$ is an $M$-sequence. Therefore it is sufficient to show that if $m \notin (a_1, \ldots, a_{i-1})M$, then $a_im \notin (a_1, \ldots, a_{i-1})M$.
Assume that \( m \not\in (a_1, \ldots, a_{i-1})M \). Hence by the above note we have \( \frac{m}{(a_1, \ldots, a_{i-1}, 1)} \neq 0 \) in \( U_{i-1}[1]^{-1}M \).

On the other hand, by the hypothesis and Theorem 2.2(1) we have

\[
H_U^{i-1}(M) \cong \bigcup_{(b_1, \ldots, b_i) \in U_i} \text{Ann}_{U_{i-1}[1]^{-1}}(b_1, \ldots, b_i)R = 0.
\]

Hence we have \((a_1, \ldots, a_i)R \cdot \frac{m}{(b_1, \ldots, b_{i-1}, 1)} \neq 0\) for all non-zero element \( \frac{m}{(b_1, \ldots, b_{i-1}, 1)} \) of \( U_{i-1}[1]^{-1}M \). In particular, we have

\[
\frac{a_im}{(a_1, \ldots, a_{i-1}, 1)} \neq 0 \quad \text{for} \quad \frac{m}{(a_1, \ldots, a_{i-1}, 1)} \in U_{i-1}[1]^{-1}M,
\]

since \((a_1, \ldots, a_{i-1})R \cdot \frac{m}{(a_1, \ldots, a_{i-1}, 1)} = 0\) by Lemma 1.2(2). Hence we obtain \( a_im \not\in (a_1, \ldots, a_{i-1})M \) by the above note.

(\( \Leftarrow \)) By Theorem 2.2(1), it is enough to show that

\[
\bigcup_{(a_1, \ldots, a_i) \in U_i} \text{Ann}_{U_{i-1}[1]^{-1}}(a_1, \ldots, a_i)R = 0 \quad \text{for all} \quad i \geq 1.
\]

Assume that, for some \( 0 \neq \frac{m}{(b_1, \ldots, b_{i-1}, 1)} \in U_{i-1}[1]^{-1}M \) and \((a_1, \ldots, a_i) \in U_i\),

\[
(a_1, \ldots, a_i)R \cdot \frac{m}{(b_1, \ldots, b_{i-1}, 1)} = 0 \quad \text{in} \quad U_{i-1}[1]^{-1}M.
\]

Then from Lemma 1.2(2) we have, in \( U_i^{-1}M \),

\[
\frac{m}{(b_1, \ldots, b_{i-1}, 1)} \neq 0 \quad \text{and} \quad (a_1, \ldots, a_i)R \frac{m}{(b_1, \ldots, b_{i-1}, 1)} = 0.
\]

On the other hand, by the definition of triangular subset there are \((c_1, \ldots, c_i) \in U_i\) and \( H, K \in D_i(R) \) such that \( H(a_1 \ldots a_i)^T = [c_1 \ldots c_i]^T = K[b_1 \ldots b_{i-1} 1]^T \), since \((b_1, \ldots, b_{i-1}, 1) \in U_i\). Hence we have \((c_1, \ldots, c_i)R \subset (a_1, \ldots, a_i)R \) and then

\[
(c_1, \ldots, c_i)R \cdot \frac{m}{(b_1, \ldots, b_{i-1}, 1)} = 0 \quad \text{in} \quad U_i^{-1}M.
\]
In particular,
\[
\frac{c_i m}{(b_1, \ldots, b_{i-1}, 1)} = \frac{c_i |K|m}{(c_1, \ldots, c_{i-1}, c_i)} = 0 \text{ in } U_i^{-i} M
\]
by Lemma 1.2(1). Therefore we get \(c_i |K|m \in (c_1, \ldots, c_{i-1})M\) by Lemma 1.2(2). Hence we obtain \(|K|m \in (c_1, \ldots, c_{i-1})M\), since \(\{c_1, \ldots, c_i\}\) is a poor \(M\)-sequence. This leads to the following contradiction:
\[
\frac{m}{(b_1, \ldots, b_{i-1}, 1)} = \frac{|K|m}{(c_1, \ldots, c_{i-1}, c_i)} = 0 \text{ in } U_i^{-i} M
\]
by Lemma 1.2(2).

**Theorem 2.4.** Let \(R\) be Noetherian and \(M\) an \(R\)-module. Let \(\Phi_U = (\Phi(U_i))_{i \geq 1}\) be the family of systems of ideals of \(R\) induced by a chain \(U = (U_i)_{i \geq 1}\) of triangular subsets on \(R\). Then, in the complex \(C(U, M)\), for \(n \geq 1\) we have the following.

1. \(U_n[1]^{-n-1}M \cong H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong H^1_{\Phi(U_n)}(\text{Im } e^{n-1}).\)

2. \(H^1_{\Phi(U_n)}(U_n^{-n}M) = 0.\)

3. \(\text{Im } e^n \cong H^1_{\Phi(U_n)}(\text{Ker } e^n).\)

4. \(U_n^{-n}M \cong G_{\Phi(U_n)}(U_n^{-n}M) \cong G_{\Phi(U_n)}(\text{Ker } e^n) \cong G_{\Phi(U_n)}(\text{Im } e^{n-1}) \cong G_{\Phi(U_n)}(U_{n-1}[1]^{-n}M).\)

5. \(H^i_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong H^i_{\Phi(U_n)}(\text{Im } e^{n-1}) \cong H^1_{\Phi(U_n)}(U_n^{-n}M) \cong H^i_{\Phi(U_n)}(\text{Ker } e^n), \text{ for all } i \geq 2.\)

6. \(0 \rightarrow H^n_U(M) \rightarrow H^1_{\Phi(U_{n+1})}(\text{Im } e^{n-1}) \rightarrow H^1_{\Phi(U_{n+1})}(\text{Ker } e^n) \rightarrow 0\) is a short exact sequence.

7. \(H^i_{\Phi(U_{n+1})}(\text{Im } e^{n-1}) \cong H^i_{\Phi(U_{n+1})}(\text{Ker } e^n), \text{ for all } i \geq 2.\)

**Proof.** ((1) and the first isomorphism of (5)) For (1), by Theorem 2.2(2), it is enough to show that

\[
H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong H^1_{\Phi(U_n)}(\text{Im } e^{n-1}).
\]

But this and the first isomorphism of (5) follow from the above long exact sequence (\(*\)) and Lemma 1.6(1), since \(H^{n-1}_U(M)\) is a \(\Phi(U_n)\)-torsion module.
Consider the following short exact sequence

\[
0 \longrightarrow \text{Im } e^{n-1} \longrightarrow U_n^{-n}M \longrightarrow U_n^{-n}M/\text{Im } e^{n-1} \longrightarrow 0
\]

where the isomorphism follows from Lemma 2.1(2) again. Then, since \(U_n[1]^{-n-1}M\) is a \(\Phi(U_n)\)-torsion module and \(H^0_{\Phi(U_n)}(U_n^{-n}M) = 0\) by Lemma 1.11, using the long exact sequence induced from the above short exact sequence, we have the second isomorphism induced from (5) and the following short exact sequence

\[
0 \longrightarrow U_n[1]^{-n-1}M \longrightarrow H^1_{\Phi(U_n)}(\text{Im } e^{n-1}) \longrightarrow H^1_{\Phi(U_n)}(U_n^{-n}M) \longrightarrow 0.
\]

Then, from (1) and the above short exact sequence, we have \(H^1_{\Phi(U_n)}(U_n^{-n}M) = 0\).

((3) and the third isomorphism of (5)) From (2) and the following short exact sequence

\[
0 \longrightarrow \text{Ker } e^n \longrightarrow U_n^{-n}M \longrightarrow \text{Im } e^n \longrightarrow 0,
\]

we have the results by means of the long exact sequence induced by this short exact sequence, since \(\text{Im } e^n\) is a \(\Phi(U_n)\)-torsion module (Lemma 1.11(2)) and \(H^0_{\Phi(U_n)}(U_n^{-n}M) = 0\) (Lemma 1.11(1)).

(4) By [15, p. 176], we have the following exact sequence

\[
(**) \quad 0 \longrightarrow L_{\Phi(U_n)}(U_n^{-n}M) \longrightarrow U_n^{-n}M
\]

\[
\quad \longrightarrow G_{\Phi(U_n)}(U_n^{-n}M) \longrightarrow H^1_{\Phi(U_n)}(U_n^{-n}M) \longrightarrow 0.
\]

From the above sequence, we have

\[U_n^{-n}M \cong G_{\Phi(U_n)}(U_n^{-n}M),\]

since \(L_{\Phi(U_n)}(U_n^{-n}M) = H^1_{\Phi(U_n)}(U_n^{-n}M) = 0\) by (2) and Lemma 1.11(1).
Next, since $L_{\Phi(U_n)}(\text{Ker } \epsilon^n) = 0$ and $H^1_{\Phi(U_n)}(\text{Ker } \epsilon^n) \cong \text{Im } \epsilon^n$ by (3), from the above sequence (***) with $U_{n-1}^n M$ replaced by $\text{Ker } \epsilon^n$ we have the following exact sequence

$$0 \longrightarrow \text{Ker } \epsilon^n \longrightarrow G_{\Phi(U_n)}(\text{Ker } \epsilon^n) \longrightarrow \text{Im } \epsilon^n \longrightarrow 0.$$ 

Hence we get

$$G_{\Phi(U_n)}(\text{Ker } \epsilon^n)/\text{Ker } \epsilon^n \cong \text{Im } \epsilon^n \cong U_{n-1}^n M/\text{Ker } \epsilon^n.$$ 

That is $U_{n-1}^n M \cong G_{\Phi(U_n)}(\text{Ker } \epsilon^n)$.

The third isomorphism is Lemma 2.1(3).

For the last isomorphism, using [19, 3.1.10] again, we have the following exact sequence

$$0 \longrightarrow U_{n-1}[1]^{-n} M/L_{\Phi(U_n)}(U_{n-1}[1]^{-n} M) \longrightarrow G_{\Phi(U_n)}(U_{n-1}[1]^{-n} M) \longrightarrow H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n} M) \longrightarrow 0.$$ 

On the other hand, we have

$$U_{n-1}[1]^{-n} M/L_{\Phi(U_n)}(U_{n-1}[1]^{-n} M) \cong \text{Im } \epsilon^{n-1}$$

by Lemma 2.1(2) and Theorem 2.2(1). We also get

$$H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n} M) \cong U_{n-1}[1]^{-n-1} M \cong U_{n}^{-n} M/\text{Im } \epsilon^{n-1}$$

by (1) and Lemma 2.1(2). Hence by the five lemma we have

$$U_{n}^{-n} M \cong G_{\Phi(U_n)}(U_{n-1}[1]^{-n} M).$$

((6) and (7)) From the following short exact sequence

$$0 \longrightarrow \text{Im } \epsilon^{n-1} \longrightarrow \text{Ker } \epsilon^n \longrightarrow H^1_{\Phi(U_n)}(M) \longrightarrow 0$$

we have the results, since $H^1_{\Phi(U_n)}(M)$ is a $\Phi(U_{n+1})$-torsion module and $H^0_{\Phi(U_{n+1})}(\text{Ker } \epsilon^n) = 0$ by Lemma 1.11. □

**Remark.** The last proof of Theorem 2.4(4) is another simple proof of [19, 5.2.3] and [15, 3.3].

**Corollary 2.5.** Let $R$ and $M$ be as above. Then for $n \geq 1$ we have the following.

$$H^n_{\Phi(U_n)}(M) = 0 \iff H^1_{\Phi(U_{n+1})}(\text{Im } \epsilon^{n-1}) \rightarrow H^1_{\Phi(U_{n+1})}(\text{Ker } \epsilon^n) \rightarrow H^1_{\Phi(U_{n+1})}(\text{Ker } \epsilon^n) \cong H^n_{\Phi(U_{n+1})}(\text{Ker } \epsilon^n)$$

**Proof.** These immediately follow from Theorem 2.4(6). □
References


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