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ON INFERIOR LIMITS FOR THE INCREMENTS OF GAUSSIAN PROCESSES

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1. Introduction

Let $\{W(t); 0 \leq t < \infty\}$ be a standard Wiener process on a probability space $(\Omega, \Im, P)$. Let $a_T(0 < T < \infty)$ be a monotonically nondecreasing function of T for which

(i) $0 < a_T \leq T$,

(ii) T/a_T is monotonically nondecreasing,

(iii) $\lim_{T\to\infty} (T/a_T)/\log\log T = \infty$.

We denote

$$\gamma(T) = \left(2a_T(\log(T/a_T) - \log\log\log T)\right)^{-1/2}.$$

Shao [7] proved the following:

$$(*) \qquad \liminf_{T\to\infty} \gamma(T) \sup_{0\leq t\leq T-a_T} |W(t+a_T) - W(t)| = 1 \qquad \text{a.s.}$$

and

$$\liminf_{T\to\infty} \gamma(T) \sup_{0\leq t\leq T-a_T} \sup_{0\leq s\leq a_T} |W(t+s) - W(t)| = 1 \qquad \text{a.s.}$$

In this paper we are going to investigate the result $(*)$ on Gaussian processes. Let $\{X(t); 0 \leq t < \infty\}$ be an almost surely continuous Gaussian process with $X(0) = 0$, $E\{X(t)\} = 0$ and stationary increments $E\{X(t) - X(s)\}^2 = \sigma^2(|t-s|)$, where $\sigma(y)$ is a function of

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$y \geq 0$. Further we assume that $\sigma(t)$, $t > 0$, is a nondecreasing continuous concave, regularly varying function with exponent α $(0 < \alpha < 1)$ at infinity (e.g., if $\{X(t); 0 \leq t < \infty\}$ is a standard Wiener process, then $\sigma(t) = \sqrt{t}$). Let $a_T(0 < T < \infty)$ be a monotonically nondecreasing function of T for which

(i)′ $0 < a_T \leq T$,

(ii)′ T/a_T is monotonically nondecreasing,

(iii)′ there exists a large constant $r_0 > 0$ such that for $\epsilon > 0$

$$\lim_{T\to\infty} (T/a_T)/(\log\log T)^{1+\epsilon} = r, \quad r_0 \leq r < \infty.$$

For large $T > 0$, let us denote

$$\gamma_T = \left(2\sigma^2(a_T)(\log(T/a_T) - \log\log\log T)\right)^{-1/2}.$$

and define continuous parameter processes $X_1(T)$, $X_2(T), \cdots$, $X_6(T)$ by

$$X_1(T) = \sup_{0\leq s\leq a_T} \sup_{0\leq t\leq T-s} |X(t+s) - X(t)|,$$
$$X_2(T) = \sup_{0\leq s\leq a_T} \sup_{0\leq t\leq T-s} \left(X(t+s) - X(t)\right),$$
$$X_3(T) = \sup_{0\leq s\leq a_T} \sup_{0\leq t\leq T-a_T} |X(t+s) - X(t)|,$$
$$X_4(T) = \sup_{0\leq s\leq a_T} \sup_{0\leq t\leq T-a_T} \left(X(t+s) - X(t)\right),$$
$$X_5(T) = \sup_{0\leq t\leq T-a_T} |X(t+a_T) - X(t)|,$$
$$X_6(T) = \sup_{0\leq t\leq T-a_T} \left(X(t+a_T) - X(t)\right),$$

respectively. Clearly, $X_1(T)$ is the largest process and $X_6(T)$ is the smallest one of all $X_i(T)$, $i = 1, 2, \cdots, 6$. $X_i(T)$ are so-called the Csörgö-Révész increments (see [3], [4] and [6]).

In section 2 we shall investigate almost sure inferior limits of $X_i(T)$, $i = 1, 2, \cdots, 6$, and in section 3 we shall obtain a similar result as in section 2 for a stationary Gaussian sequence.

2. Main results

THEOREM 1. *Let $\{X(t) : 0 \le t < \infty\}$ be a Gaussian processes as above. Let $a_T(0 < T < \infty)$ satisfy the conditions* (i)′ ∼(iii)′. *Then we have*

$$\liminf_{T\to\infty} \gamma_T X_i(T) = \sqrt{\frac{r}{1+r}} \quad i = 1, 2, \cdots, 6, \quad \text{a.s.}$$

In order to prove Theorem 1, we need some lemmas:

LEMMA 1. (Slepian [8], Adler [1]) *Let $\{X(t); t \in T\}$ and $\{Y(t); t \in T\}$ be centered Gaussian processes such that $EX^2(t) = EY^2(t)$ for all $t \in T$ and $E\{X(t)X(s)\} \le E\{Y(t)Y(s)\}$ for all $s, t \in T$. Then*

$$P\{\sup_{t\in T} X(t) \le u\} \le P\{\sup_{t\in T} Y(t) \le u\}$$

for any real number u.

LEMMA 2. *Let m and k be any positive number. Then for any small $\epsilon' > 0$ there exist large constants c, $C_{\epsilon'}$ (depending on ϵ') and $u_0 = u_0(k, m, \epsilon')$ such that for all $u \ge u_0$*

$$(1)\quad \begin{aligned} &\frac{1}{\sqrt{2\pi}} \frac{k}{m(u+c)} e^{-u^2/2} \le P\left\{ \sup_{0\le t\le k} \frac{X(t+m) - X(t)}{\sigma(m)} > u \right\} \\ &\le P\left\{ \sup_{0\le t\le k} \sup_{0\le s\le m} \frac{X(t+s) - X(t)}{\sigma(m)} > u \right\} \le C_{\epsilon'} \frac{k}{m} e^{-u^2/(2+\epsilon')}. \end{aligned}$$

Proof. Let us prove the first inequality. Let

$$\Phi(u) = \int_u^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx, \quad u \ge 0.$$

Then, by Fernique([5], p.71), the inequality

$$(2)\qquad \frac{1}{\sqrt{2\pi}(u+1)} e^{-u^2/2} \le \Phi(u) \le \frac{4}{3} \frac{1}{\sqrt{2\pi}(u+1)} e^{-u^2/2}$$

holds for all $u \ge 0$. For given k and m, we set $h = [k/m]$, where $[y]$ denotes the greatest integer not exceeding y. For $i = 0, 1, \cdots, h$, let us define the incremental random variable

$$Y(i) = X((i+1)m) - X(im).$$

From the concavity of $\sigma^2(t)$ it follows that, for $l = |i - j| \geq 1$,

$$\begin{aligned}
&\text{covariance}(Y(i),\ Y(j)) \\
&= E\{X(im)X(jm)\} - E\{X(im)X((j-1)m)\} \\
&\qquad - E\{X((i-1)m)X(jm)\} \\
&\qquad + E\{X((i-1)m)X((j-1)m)\} \\
&= \frac{1}{2}\{(\sigma^2((l+1)m) - \sigma^2(lm)) - (\sigma^2(lm) - \sigma^2((l-1)m)\} \leq 0.
\end{aligned}$$

Applying the inequality (2) and Lemma 1, we have

(3)
$$\begin{aligned}
&P\left\{\sup_{0\leq t\leq k} \frac{X(t+m)-X(t)}{\sigma(m)} \leq u\right\} \leq P\left\{\sup_{0\leq i\leq h} \frac{Y(i)}{\sigma(m)} \leq u\right\} \\
&\leq \{1-\Phi(u)\}^{h+1} \leq \exp(-(h+1)\Phi(u)) \\
&\leq \exp\left(-\frac{h+1}{\sqrt{2\pi}(u+1)}e^{-u^2/2}\right) \leq \exp\left(-\frac{k}{m}\frac{1}{\sqrt{2\pi}(u+1)}e^{-u^2/2}\right).
\end{aligned}$$

By (3), we obtain

$$\begin{aligned}
P\left\{\sup_{0\leq t\leq k} \frac{X(t+m)-X(t)}{\sigma(m)} > u\right\} &\geq 1 - \exp\left(-\frac{k}{m}\frac{1}{\sqrt{2\pi}(u+1)}e^{-u^2/2}\right) \\
&\geq \frac{1}{\sqrt{2\pi}}\frac{k}{m(u+c)}e^{-u^2/2}.
\end{aligned}$$

The last inequality of (1) is easily shown from Choi([2], p.202).

Proof of Theorem 1. We first prove that

$$\liminf_{T\to\infty} \gamma_T\, X_i(T) \leq \sqrt{\frac{r}{1+r}} \quad \text{a.s.}$$

By using Lemma 2 and condition (iii)$'$, we have, for any small $\epsilon > 0$,

$$\begin{aligned}
&P\left\{\gamma_T X_1(T) \leq \sqrt{\frac{r}{1+r}}(1+\epsilon)\right\} \\
&\qquad \geq \exp\left(-C_{\epsilon''} r^{-\left(\frac{\epsilon}{2+\epsilon}\right)\left(\frac{r}{1+r}\right)} (\log\log T)^{-(1+\epsilon)\left(\frac{r-1}{r+1}\right)}\right)
\end{aligned}$$

for all large T, where $0 < \epsilon'' < \epsilon$. We set $T_k = \exp(k^b)$, $k = 1, 2, \cdots$, where b is chosen such that $0 < b < 1$. Then, for $T_k \leq T \leq T_{k+1}$, we have

$$P\left\{\gamma_{T_{k+1}} X_1(T_{k+1}) \leq \sqrt{\frac{r}{1+r}}(1+\epsilon)\right\} \geq \exp\left(-C_{\epsilon''} r^{-\left(\frac{\epsilon}{2+\epsilon}\right)\left(\frac{r}{1+r}\right)} (b \log k)^{-(1+\epsilon)\left(\frac{r-1}{r+1}\right)}\right). \tag{5}$$

Let

$$A_k = \left\{\gamma_{T_{k+1}} X_1(T_{k+1}) \leq \sqrt{\frac{r}{1+r}}(1+\epsilon)\right\}.$$

Then, by (5), we get

$$\sum_k P(A_k) = \infty.$$

Setting

$$A'_k = \left\{\gamma_{T_{k+1}} X_2(T_{k+1}) \leq \sqrt{\frac{r}{1+r}}(1+\epsilon)\right\}.$$

and

$$A''_k = \left\{\gamma_{T_{k+1}} X_2(T_{k+1}) \geq -\sqrt{\frac{r}{1+r}}(1+\epsilon)\right\},$$

we have

$$\sum_k P(A'_k) = \infty \quad \text{and} \quad \sum_k P(A''_k) = \infty.$$

From the concavity of $\sigma^2(t)$ and using Lemma 1, we obtain

$$P(A'_k \cap A'_l) \leq P(A'_k)P(A'_l) \quad \text{and} \quad P(A''_k \cap A''_l) \leq P(A''_k)P(A''_l)$$

where $k \neq l$. It follows from the second Borel-Cantelli lemma that

$$-\sqrt{\frac{r}{1+r}}(1+\epsilon) \leq \limsup_{k\to\infty} \gamma_{T_{k+1}} X_2(T_{k+1}) \leq \sqrt{\frac{r}{1+r}}(1+\epsilon) \quad \text{a.s.}$$

So, we have

$$\liminf_{k\to\infty} \gamma_{T_{k+1}} X_1(T_{k+1}) \leq \sqrt{\frac{r}{1+r}} \quad \text{a.s.}$$

Also we obtain

$$\liminf_{T\to\infty} \gamma_T X_1(T) \le \liminf_{k\to\infty} \gamma_{T_{k+1}} X_1(T_{k+1}),$$

because

$$\gamma_T X_1(T) \le \frac{\sigma(a_{T_{k+1}})}{\sigma(a_{T_k})} \Big(\frac{\log((T_{k+1}/a_{T_{k+1}})/\log\log T_{k+1})}{\log((T_k/a_{T_k})/\log\log T_k)}\Big)^{1/2} \times \gamma_{T_{k+1}} X_1(T_{k+1})$$

and

$$1 \le \frac{\sigma(a_{T_{k+1}})}{\sigma(a_{T_k})} \le \frac{\sigma(\exp(bk^{b-1})a_{T_k})}{\sigma(a_{T_k})} \le \exp(b(\alpha+\epsilon)k^{b-1}) \longrightarrow 1 \quad \text{as} \quad k\to\infty.$$

Next we prove that

$$\liminf_{T\to\infty} \gamma_T X_6(T) \ge \sqrt{\frac{r}{1+r}} \quad \text{a.s.}$$

By using the inequality (3) and condition (iii)$'$, we have, for given small $\epsilon > 0$,

$$\begin{aligned} P\Big\{\gamma_T X_6(T) \le \sqrt{\frac{r}{1+r}-\epsilon}\Big\} &= P\Big\{\sup_{0\le t\le T-a_T} \frac{X(t+a_T)-X(t)}{\sigma(a_T)} < \sqrt{\frac{r}{1+r}-\epsilon}\Big(2\log\frac{T/a_T}{\log\log T}\Big)^{1/2}\Big\} \\ &\le \exp\Big(-\frac{T}{a_T}\frac{1}{\sqrt{2\pi}}\Big(\sqrt{\frac{r}{1+r}-\epsilon}\Big(2\log\frac{T/a_T}{\log\log T}\Big)^{1/2}+1\Big)^{-1} \\ &\qquad \times \Big(\frac{T/a_T}{\log\log T}\Big)^{-((r/(1+r))-\epsilon)}\Big) \\ &\le \exp\Big(-\frac{1}{4\sqrt{\pi}((r/(1+r))-\epsilon)^{1/2}} r^{(1/(1+r))+\epsilon'} \log\log T\Big), \end{aligned}$$

for all large T, where $0 < \epsilon' < \epsilon$. Putting $T_k = \exp(k^A)$ where $A = \sqrt{(r/(1+r))-\epsilon}$, we get

$$\begin{aligned} P\Big\{\gamma_{T_k} X_6(T_k) \le \sqrt{\frac{r}{1+r}-\epsilon}\Big\} &\le \exp\Big(-\frac{1}{4\sqrt{\pi}} r^{(1/(1+r))+\epsilon'} \log k\Big) =: \exp(-B\log k) = k^{-B} \end{aligned}$$

where $B > 1$ for large $r > 0$. Using the Borel-Cantelli lemma, we have

$$\liminf_{k\to\infty} \gamma_{T_k} X_6(T_k) \geq \sqrt{\frac{r}{1+r}} \quad \text{a.s.}$$

Let $T_k \leq T \leq T_{k+1}$ and $T_k = \exp(k^A)$, $0 < A < 1$. It suffices to prove that

$$\liminf_{T\to\infty} \gamma_T X_6(T) \geq \liminf_{k\to\infty} \gamma_{T_k} X_6(T_k) \qquad \text{a.s.}$$

Since $T_k - a_{T_k} \leq T - a_T$, we have

$$\begin{aligned}
\gamma_T X_6(T) &\geq \Big(2\sigma^2(a_{T_{k+1}})\log \frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\Big)^{-1/2}\Big(X_6(T_k) \\
&\qquad - \sup_{0\leq t\leq T_k - a_{T_k}} \sup_{0\leq s\leq a_{T_{k+1}} - a_{T_k}} |X(t+s) - X(t)|\Big) \\
&=: D_k \gamma_{T_k} X_6(T_k) - \Big(2\sigma^2(a_{T_{k+1}})\log \frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\Big)^{-1/2} \\
&\qquad \times \sup_{0\leq t\leq T_k - a_{T_k}} \sup_{0\leq s\leq a_{T_{k+1}} - a_{T_k}} |X(t+s) - X(t)|
\end{aligned}$$

where

$$D_k = \Big(\frac{\sigma^2(a_{T_k})\log((T_k/a_{T_k})/\log\log T_k)}{\sigma^2(a_{T_{k+1}})\log((T_{k+1}/a_{T_{k+1}})/\log\log T_k)}\Big)^{1/2}$$

Now,

$$1 \geq D_k \geq \frac{\sigma(a_{T_k})}{\sigma(a_{T_{k+1}})}\Big(\frac{T_k}{T_{k+1}}\Big)^{1/2} \longrightarrow 1 \quad \text{as } k\to\infty.$$

The remainder of the proof is to show that

(6)

$$\begin{aligned}
&\limsup_{k\to\infty}\Big(2\sigma^2(a_{T_{k+1}})\log \frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\Big)^{-1/2} \\
&\qquad\qquad \times \sup_{0\leq t\leq T_k - a_{T_k}} \sup_{0\leq s\leq a_{T_{k+1}} - a_{T_k}} |X(t+s) - X(t)| \\
&\quad = 0.
\end{aligned}$$

In Choi ([2], p.203), we see that

$$\limsup_{k\to\infty}\{2\sigma^2(a_{T_k})(\log(T_k/a_{T_k})+\log\log T_k)\}^{-1/2}X_3(T_k)\le 1 \quad \text{a.s.}$$

So we have

$$\limsup_{k\to\infty}\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_{k+1}}-a_{T_k}}\gamma_1(T_k)|X(t+s)-X(t)|\le 1 \quad \text{a.s.}$$

where

$$\gamma_1(T_k) = \left\{2\sigma^2(a_{T_{k+1}}-a_{T_k})\Big(\log\frac{T_k+a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+\log\log(T_k+a_{T_{k+1}})\Big)\right\}^{-1/2}$$

For large k, we have also

$$a_{T_{k+1}}-a_{T_k}\le a_{T_{k+1}}\Big(1-\frac{T_k}{T_{k+1}}\Big)\le Aa_{T_{k+1}}k^{A-1},$$

which implies

$$\begin{aligned}
&\gamma_1^{-2}(T_k)\Big(2\sigma^2(a_{T_{k+1}})\log\frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\Big)^{-1}\\
&=(a_{T_{k+1}}-a_{T_k})^{2\alpha}s^2(a_{T_{k+1}}-a_{T_k})\Big(\log\frac{T_k+a_{T_{k+1}}}{a_{T_{k+1}}-a_{T_k}}+\log\log(T_k+a_{T_{k+1}})\Big)\\
&\qquad\times\Big((a_{T_{k+1}})^{2\alpha}s^2(a_{T_{k+1}})\log\frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\Big)^{-1}\\
&\le(Aa_{T_{k+1}}k^{A-1})^{2\alpha}s^2(Aa_{T_{k+1}}k^{A-1})\Big(\log\frac{T_k+a_{T_{k+1}}}{Aa_{T_{k+1}}k^{A-1}}+\log\log(T_k+a_{T_{k+1}})\Big)\\
&\qquad\times\Big((a_{T_{k+1}})^{2\alpha}s^2(a_{T_{k+1}})\log\frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\Big)^{-1}\\
&\le k^{2\alpha(A-1)}\Big\{\log[\frac{1}{A}k^{1-A}((r+\epsilon)A\log(k+1)+1)]+(\log\log 2+A\log(k+1))\Big\}\\
&\qquad\times\{\log[(r+\epsilon)A\log(k+1)]-\log\log(k+1)^A\}^{-1}\\
&\longrightarrow 0 \quad \text{as} \quad k\to\infty.
\end{aligned}$$

where s(·) is a slowly varying function. Hence we obtain

$$\begin{aligned}
0 &\le \limsup_{k\to\infty}\Big(2\sigma^2(a_{T_{k+1}})\log\frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\Big)^{-1/2} \\
&\qquad\times \sup_{0\le t\le T_k-a_{T_k}}\ \sup_{0\le s\le a_{T_{k+1}}-a_{T_k}} |X(t+s)-X(t)| \\
&= \limsup_{k\to\infty}\gamma_1(T_k) \sup_{0\le t\le T_k-a_{T_k}}\ \sup_{0\le s\le a_{T_{k+1}}-a_{T_k}} |X(t+s)-X(t)| \\
&\qquad\times \gamma_1^{-1}(T_k)\Big(2\sigma^2(a_{T_{k+1}})\log\frac{T_{k+1}/a_{T_{k+1}}}{\log\log T_k}\Big)^{-1/2} \\
&\le 0.
\end{aligned}$$

This proves (6) and the proof of Theorem 1 is completed.

3. A Parallel result for partial sums

Let $\{X_n; n = 1,2,\cdots\}$ be a stationary Gaussian sequence with $X_0 = 0$, $EX_1 = 0$, $EX_1^2 = 1$ and $E(X_1X_{1+n}) \le 0$ for all $n = 1,2,\cdots$. We define partial sums $S(0) = 0$ and $S(n) = \sum_{i=1}^n X_i$ and set $\sigma^2(n) = ES^2(n)$. Assume that $\sigma(n)$ can be extended to a continuous function $\sigma(t)$ of $t > 0$ which is nondecreasing and regularly varying with exponent α $(0 < \alpha < 1)$ at infinity. Suppose that $\{a_n : n = 1,2,\cdots\}$ is a nondecreasing sequence of positive integers such that

(i)″ $1 \le a_n \le n$,

(ii)″ n/a_n is nondecreasing,

(iii)″ there exists a large constant $r_0 > 0$ such that for $\epsilon > 0$ $\lim_{n\to\infty}(n/a_n)/(\log\log n)^{1+\epsilon} = r, \quad r_0 \le r < \infty.$

For large n, we define

$$\gamma_n = \big\{2\sigma^2(a_n)(\log(n/a_n) - \log\log\log n)\big\}^{-1/2}.$$

and

$$\begin{aligned}
X_1(n) &= \sup_{1\le j\le a_n}\ \sup_{0\le i\le n-j} |S(i+j)-S(i)|, \\
X_2(n) &= \sup_{1\le j\le a_n}\ \sup_{0\le i\le n-j} \big(S(i+j)-S(i)\big), \\
X_3(n) &= \sup_{1\le j\le a_n}\ \sup_{0\le i\le n-a_n} |S(i+j)-S(i)|,
\end{aligned}$$

$$X_4(n) = \sup_{1\le j\le a_n} \sup_{0\le i\le n-a_n} (S(i+j) - S(i)),$$
$$X_5(n) = \sup_{0\le i\le n-a_n} |S(i+a_n) - S(i)|,$$
$$X_6(n) = \sup_{0\le i\le n-a_n} (S(i+a_n) - S(i)),$$

respectively. By the similar way as the proof of Theorem 1, we have the following

THEOREM 2. *Let* $\{X_n : n = 1, 2\cdots\}$ *be a Gaussian sequence as above. Let* a_n *satisfy the conditions* (i)$''$ ~(iii)$''$. *Then*

$$\liminf_{n\to\infty} \gamma_n X_i(n) = \sqrt{\frac{r}{1+r}} \quad i = 1, 2, \cdots, 6, \quad \text{a.s.}.$$

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