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A REGULARITY THEOREM FOR THE INITIAL TRACES OF THE SOLUTIONS OF THE HEAT EQUATION

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0. Introduction

In the theory of partial differential equations with given initial values and boundary values one usually investigates to examine the well-posedness, that is, the unique existence of the solution as well as its continuous dependence on the data. This theory is strong enough for us to determine the situation anywhere and anytime provided that the initial data are actually given. However, in many cases the data are not completely known for us. Then in those situations arise the new problem to determine the unknown initial data by taking other conditions for the solution.

From this point of view, in this paper we discuss a very simple problem for the heat equation $(\partial_t - \Delta)U(x,t) = 0$ with the initial data whose regularity is unknown. The main theorem states that if $U(x,t)$ is a heat solution satisfying

$$\int |\partial^\alpha U(x,t)|^p dx < M$$

for $0 < t < T$, $|\alpha| \le s$ and $p > 1$ then its initial value $U(x, 0^+)$ must belong to the Sobolev space $W^{p,s}$ (see Theorem 2.4). Thus in view of Sobolev imbedding theorem we can obtain the regularity of the initial condition by considering the growth of solution. At a first glance, it is easily expected result. But, nevertheless, we can see that it is no longer true when $p = 1$.

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§1. The Cauchy problem

We use the multi-index notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers; $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, \cdots, n$. Also, we denote by C^∞ the set of all infinitely differentiable functions in $\mathbb{R}^n$ and by C_0^∞ the set of all C^∞ functions with compact support.

We recall the definition of Sobolev spaces. Let s be a nonnegative integer and let $1 \le p < +\infty$.

DEFINITION 1.1. We denote by $W^{p,s}$ the space of all distributions u such that

$$\partial^\alpha u \in L^p, \quad |\alpha| \le s$$

equipped with the norm

$$\|u\|_{p,s} = \left[\sum_{|\alpha| \le s} \|\partial^\alpha u\|_p^p \right]^{1/p}$$

where $\|\cdot\|_p$ denotes L^p-norm on $\mathbb{R}^n$.

Let $E(x,t)$ be the heat kernel defined by

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right), & t > 0 \\ 0, & t \le 0. \end{cases}$$

First, we present a direct problem which is, in fact, an initial value problem for the heat equation with initial data in $W^{p,s}$.

THEOREM 1.2. *Suppose that $T > 0$, $S \ge 0$ and $1 \le p < +\infty$. Then for every $u \in W^{p,s}$ $U(x,t) = E * u$ is well defined and a C^∞ function in $\mathbb{R}^n \times (0,T)$ satisfying that*

(1.1) $(\partial_t - \Delta)U(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T)$

(1.2) *There exists a constant $M > 0$ such that*

$$\int |\partial^\alpha U(x,t)|^p dx < M, \quad 0 < t < T, \ |\alpha| \le s$$

(1.3) *$U(x,t) \to u$ in $W^{p,s}$ as $t \to 0$.*

where $*$ denotes the convolution with respect to the space variable x.

Proof. Since E is exponentially decreasing at infinity with respect to the space variable, the convolution $E * u = U(x,t)$ is well defined and a C^∞ function in $\mathbb{R}^n \times (0,T)$ satisfying the heat equation (1.1). Moreover, considering $E(\cdot,t)$ as an approximation identity we have

$$\begin{aligned}\|\partial^\alpha U(x,t)\|_p &\leq \|E * \partial^\alpha u\|_p \\ &\leq \|E(\cdot,t)\|_1 \|\partial^\alpha u\|_p = \|\partial^\alpha u\|, \quad 0 < t < T,\end{aligned}$$

and

$$\|\partial^\alpha U(x,t) - \partial^\alpha u\|_p \leq \|E * \partial^\alpha u - \partial^\alpha u\|_p \to 0$$

as $t \to 0^+$ for all $|\alpha| \leq s$, since $\partial^\alpha u \in L^p$, $|\alpha| \leq s$. Thus it follows that

$$\|U(\cdot,t)\|_{p,s} \leq \|u\|_{p,s} \tag{1.4}$$

and

$$U(\cdot,t) \to u \quad \text{in} \quad W^{p,s}, \tag{1.5}$$

which proves the theorem.

REMARK. (i). The above theorem implies that every solution $U(x,t) = E * u$ of the initial value problem

$$\begin{cases} (\partial_t - \Delta)U(x,t) = 0 \\ U(x,0^+) = u \in W^{p,s} \end{cases} \tag{1.6}$$

is uniquely determined in the category of (1.2) and continuously depends on the initial data in view of (1.4).

(ii). In general, if $f \in L^p$ then we can obtain the similar inequality

$$\int |\partial^\alpha (E * f)|^p dx < +\infty$$

as (1.2) for every $\alpha \in \mathbb{N}_0^n$, since E is exponentially decreasing at infinity. But we should note that the upper bound may not be independent of the time variable t.

§2. Regularity problems

Here we restate a uniqueness theorem for the heat equation in a simple form which will be very useful later.

THEOREM 2.1. ([F], Theorem 1.16) *Let $U(x,t)$ be a continuous function on $\mathbb{R}^n \times [0,T)$ with the following property*

(i) $(\partial_t - \Delta)u(x,t) = 0$ *in* $\mathbb{R}^n \times (0,T)$

(ii) $\int_0^T \int_{\mathbb{R}^n} |u(x,t)| e^{-k|x|^2} dx\, dt < +\infty$ *for some* $k > 0$.

Then $u(x,0) = 0$ *implies that* $u(x,t) \equiv 0$ *in* $\mathbb{R}^n \times [0,T)$.

Actually the regularity problem given here is nothing but a converse part of Theorem 1.2. But that result will give a meaningful information as a corollary.

Now we are in a position to state and prove the main theorem in this paper. The idea of this proof is, so called, the heat kernel method which was introduced in [CK], [KCK], and [M].

THEOREM 2.2. *Suppose that $U(x,t)$ is a C^∞ function in $\mathbb{R}^n \times (0,T)$ satisfying*

(2.1) $(\partial_t - \Delta)U(x,t) = 0,\ (x,t) \in \mathbb{R}^n \times (0,T)$

(2.2) *there exists a constant $M > 0$ such that*

$$\int |\partial^\alpha U(x,t)|^p dx < M,\ 0 < t < T,\ |\alpha| \leq s,$$

for $T > 0$, $s \geq 0$ and $1 < p < \infty$. Then the initial value $U(x,0^+)$ exists in $W^{p,s}$ where the limit $U(x,0^+) = \lim_{t\to 0^+} U(x,t)$ is taken in the topology of $W^{p,s}$. Furthermore, $U(x,t)$ can be uniquely expressed by $U(x,t) = U(x,0+) * E$.

Proof. Consider a function

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Multiplying f by a suitable cut off function we obtain a function $v(t)$ such that

$$v'(t) = \delta(t) + w(t)$$

where $v(t) = f(t)$ for $t \le T/4$, $v(t) = 0$ for $t \ge T/2$ and $w(t) \in C^\infty(\mathbb{R})$ with $\operatorname{supp} w \subset [T/4, T/2]$. Define

$$G(x,t) = -\int_0^\infty U(x, t+s)v(s)ds \tag{2.3}$$

Then $G(x,t)$ is well defined and continuous on $\mathbb{R}^n \times [0, T/2)$. Moreover, we have

$$\begin{aligned} &(\partial_t - \Delta)G(x,t) = 0, \quad 0 < t < T/2 \\ \text{and} \quad &\partial_t G = U(x,t) + \int_0^\infty U(x,t+s)w(s)ds, \quad 0 < t < T/2 \end{aligned} \tag{2.4}$$

Putting

$$H(x,t) = -\int_0^\infty U(x,t+s)w(s)ds \tag{2.5}$$

we have

$$\begin{aligned} U(x,t) &= \partial_t G(x,t) + H(x,t) \\ &= \Delta G(x,t) + H(x,t), \end{aligned} \tag{2.6}$$

where $H(x,t)$ is a continuous functions on $\mathbb{R}^n \times [0, T/2)$. Now we estimate $G(x,t)$ and $H(x,t)$ more accurately. Let $q = p/(p-1)$ and $\phi \in L^q$. Then applying Hölder inequality we have, for $0 \le t < T/2$, $|\alpha| \le s$,

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^n} |\partial^\alpha U(x,t+s)\phi(x)v(s)|dx\, ds \\ &\le \int_0^\infty \|\partial^\alpha U(x,t+s)\|_p \|\phi\|_q |v(s)|ds \\ &\le CM^{1/p}\|\phi\|_p \end{aligned}$$

for some $C > 0$. From the Fubini theorem it follows that $G(x,t)$ is a bounded linear functional on L^q. Then $\partial^\alpha G(x,t) \in L^p$ and

$$\int |\partial^\alpha G(x,t)|^p dx \le C^p M. \tag{2.7}$$

By the similar argument we obtain that $\partial^\alpha H(x,t) \in L^p$ for $0 \leq t < T/2$, $|\alpha| \leq s$ and

$$\int |\partial^\alpha H(x,t)|^p dx < M_1, \tag{2.8}$$

for some $M_1 > 0$. Thus if we put

$$g(x) = G(x,0), \quad h(x) = H(x,0)$$

then $g(x)$ and $h(x)$ are continuous on $\mathbb{R}^n$. Moreover, they belong to $W^{p,s}$. If we put $G_1(x,t) = G(x,t) - g * E$ and $H_1(x,t) = H(x,t) - h * E$ then $G_1(x,t)$ and $H_1(x,t)$ satisfy the conditions of Theorem 2.1. Thus we have

$$G(x,t) = g * E, \quad H(x,t) = h * E$$

on $\mathbb{R}^n \times [0, T/2)$.

On the other hand, it follows from (2.2), (2.6) and (2.8) that there exists $M_2 > 0$ such that for $0 < t < T$, $|\alpha| \leq s$,

$$\int |\partial^\alpha \partial_t G(x,t)|^p dx = \int |\partial^\alpha \Delta G(x,t)|^p dx < M_2. \tag{2.9}$$

Since $G(x,t)$ converges to g in $W^{p,s}$ by Theorem 1.2, $\partial^\beta G(x,t)$ converges to $\partial^\beta g(x)$ in the distribution sense for all $\beta \in \mathbb{N}_0^n$. Applying (2.9) we obtain that for each $\phi \in C_0^\infty$ and $|\alpha| \leq s$,

(2.10)
$$\begin{aligned} \left| \int \partial^\alpha \Delta g(x) \phi(x) dx \right| &= \left| \lim_{t \to 0^+} \int \partial^\alpha \Delta G(x,t) \phi(x) dx \right| \\ &\leq \lim_{t \to 0+} \|\partial^\alpha \Delta G(x,t)\|_p \|\phi\|_q \\ &\leq C \|\phi\|_q \end{aligned}$$

with some constant C and $q = p/(p-1)$. But since C_0^∞ is dense in L^q the inequality (2.10) holds for every $\phi \in L^q$, which means that $\partial^\alpha \Delta g(x)$ is a bounded linear functional on L^q. Thus, $\partial^\alpha \Delta g$ belongs to L^p for $|\alpha| \leq s$, which implies that $\Delta g \in W^{p,s}$.

Now we define $u \in W^{p,s}$ by

$$u = \Delta g + h \tag{2.11}$$

Then we have

$$\begin{aligned} u * E &= (\Delta g + h) * E \\ &= \Delta(g * E) + h * E \\ &= \Delta G(x,t) + H(x,t) \\ &= U(x,t), \end{aligned}$$

and $U(x,t) \to u$ in $W^{p,s}$ as $t \to 0^+$, which completes the proof.

REMARK. If $p = 1$ then this theorem may not be true. To see this consider the heat kernel $E(x,t)$. This satisfies all the conditions but $E(x,0^+)$ becomes $\delta(x)$ which does not belong to L^1.

Now we will give some corollaries of the above result. Using the Sobolev imbedding theorem we can directly obtain;

COROLLARY 2.3. *If $s > \frac{n}{2} + k$ and if $U(x,t)$ is a C^∞ function in $\mathbb{R}^n \times (0,T)$ satisfying*

$$(\partial_t - \Delta)U(x,t) = 0, \quad 0 < t < T$$

and

$$\int |\partial^\alpha U(x,t)|^2 dx < M, \quad 0 < t < T$$

then the initial value $U(x,0+)$ belongs to $C^k(\mathbb{R}^n)$, i.e., k-times differentiable function in $\mathbb{R}^n$.

Finally we give an another integral representation of heat solutions.

COROLLARY 2.4. *$U(x,t)$ is a heat solution satisfying that*

$$\int |U(x,t)|^2 dx < M, \quad 0 < t < T$$

if and only if there exists a function $f \in L^2$ such that

$$U(x,t) = \int e^{ixy - ty^2} f(y)dy, \quad 0 < t < T.$$

Proof. Since $L^2 = W^{2,0}$, applying Theorem 2.2 there exists a functions $g \in L^2$ such that $U(x,t) = g * E$. If we set $f(x) = (2\pi)^n \hat{g}(x)$ then Parseval's identity gives

$$\begin{aligned} U(x,t) &= g(x) * E \\ &= (2\pi)^n (\hat{g}, \hat{E}(x - \cdot, t)) \\ &= (2\pi)^n \int e^{ixy - ty^2} \hat{g}(y) dy \\ &= \int e^{ixy - ty^2} f(y) dy. \end{aligned}$$

where $\hat{g}$ is the Fourier transform of g. The converse part is easily obtained.

References

[CK] S. -Y. Chung and D. Kim,, *Representation of quasianalytic ultradistributions*, Ark. Mat. **31**, 51–60.

[F] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Inc., 1963.

[KCK] K. W. Kim, S. -Y. Chung and D. Kim,, *Fourier hyperfunctions as the boundary values of smooth solutions heat equations*, Publ. RIMS, Kyoto Univ. **29** (1993), 289-300.

[M] T. Matsuzawa, *A calculus approach to hyperfunctions III*, Nagoya Math. Jour. **118** (1990).

[W] D. V. Widder, *The heat equation*, Academic Press, New York, 1975.

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