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THE GEOMETRY OF LEFT–SYMMETRIC ALGEBRA

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1. Introduction

In this paper, we are interested in left invariant flat affine structures on Lie groups. These structures has been studied by many authors in different contexts. One of the fundamental questions is the existence of complete affine structures for solvable Lie groups G, raised by Milnor [15]. But recently Benoist answered negatively even for the nilpotent case [1]. Also moduli space of such structures for lower dimensional cases has been studied by several authors, sometimes with compatible metrics [5, 10, 4, 12].

For a given left invariant affinely flat connection ∇ on G, if we define a product on its Lie algebra by $X \cdot Y = \nabla_X Y$, $X, Y \in \mathfrak{g}$, then this product satisfies the so-called left symmetry of the associator and $\mathfrak{g}$ becomes a *left-symmetric algebra* compatible with its Lie structure. (See Section 3 for the definitions and details). This term seems to be first used by Vinberg in his study of convex homogeneous cones in conjunction with the study of homogeneous bounded complex domains [20, 13]. To a left-symmetric algebra structure on $\mathfrak{g}$, we can naturally associate a canonical Lie algebra representation to the Lie algebra of affine transformations of $\mathfrak{g}$, and we study some basic geometry of this representation in this paper.

We will start with left invariant (A, X)-structure on G as a more general setting, where X is a model space on which a Lie group A acts so that a pair (A, X) defines a geometry in the sense of Klein's Erlanger program. (See Section 2 for the more precise definitions.) Once G has a such structure, then a devoloping map into X is defined and left invariance of the structure induces a group representation from

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G to A, and hence a Lie algebra representation. We will compare this setup with the left-symmetric algebra formulation and its canonical representation in the following section. And then we will make some basic investigations in Section 4 to understand the geometry of the canonical representation using both view points.

Especially we would like to interprete the algebraic objects or results geometrically and vice versa. Some of the results about the left invariant affine structures in Section 3 are studied in [10] for the complete case, but we are more interested in the general case (i.e., incomplete situation) in this paper. We intend to make the initial steps to lay the foundations on the subject of left invariant geometric structures on Lie groups, especially focusing on the interplay between algebra and geometry of affine structures on Lie groups.

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2. Left-invariant (A, X)-structures

Let us recall some basic facts about (A, X)-structures on manifolds. We will follow the basic setup and notations as given in [11]. See also [7], [14], [16] and [19] for more on (A, X)-structures. Let A be a Lie group acting smoothly on a smooth manifold X in such a way that the action is determined locally, i.e., for $a \in A$, viewed as a map of X, if $a|_U$ is identity for some open set U of X, then $a = e =$ the identity element of A. A smooth manifold M is called an (A, X)-*manifold* if it has a cover of coordinate charts mapped into X whose coordinate transition maps are restrictions of elements of A. A maximal atlas of such coordinate charts will be called an (A, X)-structure on M.

A map $f : M \to N$ between (A, X)-manifolds is an (A, X)-*map* if it can be represented locally through coordinate charts as a restriction of an element of A. Note that (A, X)-map is a local diffeomorphism from the definition.

Let M be a connected (A, X)-manifold and consider the universal covering $\tilde{M}$ of M with the pull-back (A, X)-structure. Then there exists an (A, X)-map $D : \tilde{M} \to X$, by the usual analytic continuation argument, unique up to composition with an element of A, i.e., if D' is another (A,X)-map from $\tilde{M}$ into X, then $D' = a \circ D$ for some $a \in A$. Such a map D is called a developing map. More generally, it is easy to show the following, which is the proposition 1.2 of [11].

PROPOSITION 2.1. *Let M and M' be simply connected (A,X)-manifolds with developing maps D and D', respectively. Then a map $f : M \to M'$ is an (A,X)-map if and only if there exists $a \in A$ such that $a \circ D = D' \circ f$. Such a is, of course, unique.*

Now consider a Lie group G with a left-invariant (A,X)-structure, i.e., G has an (A,X)-structure such that all the left translations l_g, $g \in G$ are (A,X)-maps. By passing to the universal covering group, we will assume G is simply connected, if necessary, so that the existence of developing map is guaranteed. Then G has a developing map $D : G \to X$. Since the (A,X)-structure is left-invariant, for each l_g, $g \in G$, there exists a unique $\phi(g) \in A$ such that $D \circ l_g = \phi(g) \circ D$ by 2.1. Since $\phi(gh) \circ D = D \circ l_{gh} = D \circ l_g \circ l_h = \phi(g) \circ D \circ l_h = \phi(g)\phi(h) \circ D$, $\phi(gh)$ and $\phi(g) \circ \phi(h)$ agree on an open set $D(G)$ and hence are equal. Hence we obtain a Lie group homomorphism $\phi_D = \phi : G \to A$.

For a given point $x \in X$, let $Ev_x : A \to X$ be the evaluation map given by $Ev_x(a) = a(x) = a \cdot x$, and $ev_x : G \to X$ be the composition given by $ev_x = Ev_x \circ \phi$. Now observe that $D = ev_x$, where $x = D(e)$. Indeed, $D(g) = D(g \cdot e) = \phi(g) \cdot D(e) = \phi(g) \cdot x = Ev_x(\phi(g)) = ev_x(g)$. Furthermore, $d(ev_x)|_e$ should be non-singular since D is a local diffeomorphism. From this observation, we immediately obtain:

PROPOSITION 2.2. *$D : G \to \Omega = D(G) \subset X$ is a covering map.*

Proof. G acts on X as (A,X)-map through the representation $\phi : G \to A$. Since $D = ev_x$ where $x = D(e)$, $\Omega = D(G) = ev_x(G) = G \cdot x =$ the orbit of x, and Ω can be canonically identified with G/G_x, where $G_x = \{g \in G | g \cdot x = x\}$ is discrete since $D = ev_x$ is a local diffeomorphism. This shows that the projection $p : G \to G/G_x$ is a covering map and so is $D : G \to G/G_x \cong \Omega$. □

In general, for a given $y \in X$, $ev_y : G \to X$ does not have to be a covering map. But if $y = g \cdot x$, $x = D(e)$, then $ev_y = ev_{g \cdot x} = ev_x \circ r_g$ for a right translation r_g and ev_y becomes a covering map for $y \in \Omega$. This identity also shows that $\{r_g | g \in G_x\}$ is a deck transformation group of the covering $D : G \to \Omega$.

Let's examine what happens if we choose a different developing map $D' = a \circ D$, $a \in A$. If we denote the corresponding representation by $\phi' = \phi_{D'}$, then $\phi'(g) \cdot D' = D' \circ l_g$. From $\phi'(g) \circ a \circ D = a \circ D \circ l_g = a \circ \phi(g) \circ D$, we have $\phi'(g) = a \circ \phi(g) \circ a^{-1}$ and hence $\phi' = c_a \circ \phi$, where c_a is the conjugation by a. If we denote $x' = D'(e) = aD(e) = a(x)$, then $D' = ev_{x'}$

We can now conclude that a left-invariant (A, X)-structure on G gives rise to a class of representations $[\phi] \in A \backslash \mathrm{Hom}(G, A)$, where A acts on $\mathrm{Hom}(G, A)$ by composition with conjugation. And the representation ϕ has the property that $d(ev_x)|_e$ is a linear isomorphism for some $x \in X$.

Conversely, suppose we have a Lie group homomorphism $\phi : G \to A$ with the property that $d(ev_x)|_e$ is a linear isomorphism for some $x \in X$. Observe that $ev_x \circ l_g = \phi(g) \circ ev_x$ for all $g \in G$. Taking differential at $e \in G$ on both sides, we see that $d(ev_x)|_g$ is a linear isomorphism for all $g \in G$ and hence that ev_x is an immersion. In fact, non-singularity of $d(ev_x)|_e$ implies that the isotropy group G_x is discrete and the orbit map ev_x is a covering map onto its image. Now the pull back (A, X)-structure under $D = ev_x$ is left-invariant since $D \circ l_g = ev_x \circ l_g = \phi(g) \circ ev_x = \phi(g) \circ D$ and so l_g, $g \in G$, becomes an (A, X)-map by 2.1. Alternatively we can see as follows: Since ev_x is a covering projection and $ev_x \circ l_g = \phi(g) \circ ev_x$ holds, l_g is a lifting of $\phi(g)$ and hence l_g should be an (A, X)-map. Notice that if ϕ has the property that $d(ev_x)|_e$ has rank k for some $x \in X$, then the above argument shows that $d(ev_x)|_g$ also has rank k for all $g \in G$, and hence the orbit map ev_x becomes a submersion onto its image which is just a quotient map: $G \to G/G_x \cong G \cdot x$.

A left-invariant (A, X)-structure in G will be said to be *complete* if a developing map $D : G \to X$ is onto. A complete structure can be characterized as follows.

PROPOSITION 2.3. *Let X be connected and let G be a Lie group with a left- invariant (A, X)-structure determined by a developing map D. Let $\phi_D : G \to A$ be the associated representation with $ev_x = Ev_x \circ \phi_D$, $x \in X$. Then the followings are equivalent.*

(1) *$D : G \to X$ is onto.*

(2) *G action on X determined by ϕ_D is transitive*

(3) *$d(ev_x)|_e$ is a linear isomorphism for all $x \in X$.*

Proof. The equivalence of (1) and (2) follows immediately from that $D = ev_o$, where $o = De$. Suppose that G action on X is transitive. Then for any $x \in X$, there is $g \in G$ such that $x = g \cdot o$, $o = De$. Since $ev_{g\cdot o} = ev_o \circ r_g$ and $d(ev_o)|_g = dD|_g$ is an isomorphism, $d(ev_x)|_e = d(ev_{g\cdot o})|_e$ is an isomorphism. Conversely, if $d(ev_x)|_e$ is an isomorphism, the orbit containing x is open since ev_x is a covering map. This implies that each orbit is open and hence it is closed, being a complement of the union of other orbits. Since X is connected, it consists of a single orbit. □

As we already saw, the study of left-invariant (A, X)-structures on a Lie group G is equivalent to that of Lie group homomorphisms $\phi : G \to A$ with the property that $d(ev_x)|_e$ is an isomorphism for some $x \in X$. This immediately suggests us to look at Lie algebra homomorphisms with the corresponding property. Since $d(ev_x)|_e = d(Ev_x)|_e \cdot d\phi|_e$ is an isomorphism, $\dim X = \dim G$, $d(Ev_x)|_e$ is surjective and $d\phi|_e$ is injective. If we identify the Lie algebra $\mathfrak{g}$ of G with the tangent space T_eG of G at e, and similarly the Lie algebra $\mathfrak{a}$ of A with T_eA, then the Lie algebra homomorphism $d\phi : \mathfrak{g} \to \mathfrak{a}$ is injective. Let A_x be the isotropy subgroup of A at $x \in X$ and $\mathfrak{a}_x$ be its Lie algebra. Since $d(Ev_x)|_e$ is surjective and its kernel is $\mathfrak{a}_x$, $d(ev_x)|_e$ is an isomorphism if and only if $d\phi(\mathfrak{g})$ is transversal to $\mathfrak{a}_x$. Let us summarize these discussions as follows.

THEOREM 2.4. *Let G be a simply connected (hence connected) Lie group. Then the followings are equivalent.*

(1) *G admits a left-invariant (A, X)-structure.*

(2) *There is a Lie group homomorphism $\phi : G \to A$ such that $d(ev_x)|_e$ is an isomorphism for some $x \in X$.*

(3) *There is a Lie algebra homomorphism $r : \mathfrak{g} \to \mathfrak{a}$ which is transversal to $\mathfrak{a}_x$ for some $x \in X$, where $\mathfrak{a}_x$ is the Lie algebra of the isotropy subgroup A_x of A.*

Note that to talk about left-invariant (A, X)-structure on a Lie group G, it is necessary from the start that A acts on X locally transitively.

If A acts on a connected X transitively as in the most interesting cases, then any $x \in X$ can be written as $x = a \cdot o$ for some $a \in A$ with respect to a fixed base point $o \in X$. And the isotropy group at x would become $A_x = aA_o a^{-1}$ so that $\mathfrak{a}_x = Ad_a(\mathfrak{a}_o)$. Thus we have

COROLLARY 2.5. *Suppose that A acts on a connected X transitively. Then G admits a complete left invariant (A, X)-structure if and only if there is a Lie algebra homomorphism r $\mathfrak{g} \to \mathfrak{a}$ which is transversal to each of $Ad_a(\mathfrak{a}_o)$, $a \in A$.*

3. Left-invariant affine structures

A smooth manifold which admits a linear connection ∇ whose torsion and curvature tensor vanish will be called an *affinely flat* (or simply *affine* in short) manifold. By a well known theorem of differential geometry, such a manifold is locally equivalent to an open subset of Euclidean space with the standard connection, i.e., for each point of M, there is a neighborhood and a coordinate map into the Euclidean space which is an affine equivalence. In fact, the torsion and curvature are exactly the obstructions to the existence of such a map. A diffeomorphism $f : (M, \nabla) \to (M', \nabla')$ is an *affine equivalence* if $f^*\nabla' = \nabla$, i.e., $f_*(\nabla_X Y) = \nabla'_{f_* X} f_* Y$ for any vector fields X, Y on M. Moreover, if M is a simply connected complete affine manifold, then M is affinely equivalent to $\mathbb{E}^n$, the Euclidean space with the standard connection. Therefore, an affine manifold can be considered as an (A, X)-manifold where $X = \mathbb{E}^n$ and $A = \mathrm{Aff}(n)$, the group of affine transformations on $\mathbb{E}^n$, and its universal cover $\tilde{M}$ has a developing map D into $\mathbb{E}^n$ and D is an affine diffeomorphism if the affine structure of M is complete.

Now let us consider a Lie group with a left-invariant affine structure, i.e., a left-invariant $(\mathrm{Aff}(n), \mathbb{E}^n)$-structure. We can study this using two different approaches : One way is through a representation $\phi : G \to \mathrm{Aff}(n)$ as in the previous section and the other way is to use a left invariant flat connection ∇ on G. Each has its own advantage and it certainly would be more helpful if we can view the subject in many different ways.

In general a connection ∇ on a Lie group G is completely determined by the action on the left invariant vector fields, i.e., by $\nabla_X Y$ for $X, Y \in \mathfrak{g}$, using the Leibniz rule. And ∇ is left invariant if and only if $\nabla_X Y \in \mathfrak{g}$ whenever $X, Y \in \mathfrak{g}$. Indeed, ∇ is left invariant iff

$l_{g*}(\nabla_X Y) = \nabla_{l_{g*}X} l_{g*}Y$ for all $g \in G$ and $l_{g*}X = X$ if $X \in \mathfrak{g}$. To perceive the problem algebraically, denote $\nabla_X Y$ by $X \cdot Y$ for a left invariant connection ∇ and vector fields $X, Y \in \mathfrak{g}$. Then having a left invariant connection on G is the same as having an algebra structure on $\mathfrak{g}$, and a Lie group isomorphism $\phi : (G, \nabla) \to (G', \nabla')$ is an affine equivalence if and only if $\phi_* = d\phi : (\mathfrak{g}, \cdot) \to (\mathfrak{g}', \cdot')$ is an algebra isomorphism. In this way, the geometric problems involving left invariant connection become algebraic ones. We will pursue this point of view in the study of left invariant affine structures on Lie groups.

A left invariant connection ∇ on G is said to be bi-invariant if it is also right invariant. As usual, this holds if and only if ∇ is adjoint invariant. We can characterize bi-invariant connections using the associated algebra structure $X \cdot Y = \nabla_X Y$ as follows.

PROPOSITION 3.1. *The following statements are equivalent.*

(1) *A left invariant connection ∇ on G is bi-invariant.*
(2) *Ad_g is an algebra automorphism on $(\mathfrak{g}, \cdot)$ for all $g \in G$.*
(3) *ad_X is an algebra derivation on $(\mathfrak{g}, \cdot)$ for all $X \in \mathfrak{g}$, i.e.,*

$$ad_X(Y \cdot Z) = ad_X(Y) \cdot Z + Y \cdot ad_X(Z)$$

or

$$[X, Y \cdot Z] = [X, Y] \cdot Z + Y \cdot [X, Z], \; X, Y, Z \in \mathfrak{g}.$$

Proof. Let ∇ be a left invariant connection on G. Then ∇ is bi-invariant iff ∇ is adjoint invariant, i.e., $a_g^*\nabla = \nabla$ for all $g \in G$, where $a_g(h) = ghg^{-1}$ for $h \in G$. As noted above, this holds iff $a_{g*} = Ad_g : (\mathfrak{g}, \cdot) \to (\mathfrak{g}, \cdot)$ is an algebra isomorphism for all $g \in G$. And this is equivalent to say that $ad_X : (\mathfrak{g}, \cdot) \to (\mathfrak{g}, \cdot)$ is an algebra derivation. □

If furthermore ∇ is torsion free, then the algebra automorphism becomes a Lie algebra automorphism since $[X, Y] = X \cdot Y - Y \cdot X$, $X, Y \in \mathfrak{g}$. Hence we obtain the following diagram for torsion free bi-invariant connection.

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\text{ad}} & \mathrm{Der}(\mathfrak{g}, \cdot) & \subset & \mathrm{Der}(\mathfrak{g}) \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\text{Ad}} & \mathrm{Aut}(\mathfrak{g}, \cdot) & \subset & \mathrm{Aut}(\mathfrak{g}) \end{array}$$

Now suppose further that a left invariant connection ∇ is affinely flat so that it has vainshing torsion and curvature tensor, then this condition becomes algebraically as follows. We will use small letters for element of $\mathfrak{g}$ and xy for $X \cdot Y = \nabla_X Y$ from now on.

$$xy - yx = [x, y] \tag{3.1}$$

$$x(yz) - y(xz) - [x, y]z = 0 \tag{3.2}$$

for all $x, y, z \in \mathfrak{g}$. Notice that (3.1) corresponds to the torsion-free condition and (3.2) to flatness of ∇. From these we obtain immediately that

$$(x, y, z) = (y, x, z), \tag{3.3}$$

where $(x, y, z) = (xy)z - x(yz)$, the associator of x, y, z. An algebra which satisfies (3.3) is called a *left-symmetric algebra.* (See [20], [9], [17].) Hence finding a left invariant affinely flat connection on G is the same as finding a left-symmetric algebra structure on $\mathfrak{g}$ which is *compatible* with Lie algebra structure of $\mathfrak{g}$ in the sense of (3.1). Also note that a left symmetric algebra is so called Lie-admissible, that is, if we define a bracket with the equation (3.1), then the first Bianchi identity $R(x, y)z + R(y, z)x + R(z, x)y = 0$, where $R(x, y)z = x(yz) - y(xz) - [x, y]z$, with (3.1) and (3.2) gives the Jacobi identity and the bracket defines a Lie algebra structure.

Observe that an associative algebra is a special case of left symmetric algebras since (3.3) is trivially satisfied. In fact, it is well observed that a bi-invariant affinely flat connection corresponds exactly to an associative algebra. (See [9], [15], [17].)

PROPOSITION 3.2. *Let ∇ be a left invariant affinely flat connection on G. Then ∇ is bi-invariant if and only if the associated algebra is associative.*

Proof. By 3.1, we have $[x, yz] = [x, y]z + y[x, z]$ for $x, y, z \in \mathfrak{g}$. Change the brackets in this equation into the difference of two products using (3.1), and the associativity follows. □

A simple example of bi-invariant flat connection can be obtained easily on $G = Gl(n, \mathbb{R})$ using 3.2, by defining $\nabla_X Y = XY$, the usual matrix multiplication, for $X, Y \in \mathfrak{gl}(n, \mathbb{R})$. This is the usual flat connection induced on $Gl(n, \mathbb{R})$ viewing as an open subset of $\mathbb{R}^{n^2}$.

Having an algebra structure on a vector space $\mathfrak{g}$ is the same as having a linear map $\lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, where $\lambda_x = \lambda(x)$ is the left multiplication, i.e., $\lambda_x(y) = xy$. The right multiplication ρ_x is defined as $\rho_x(y) = yx$. In terms of λ, the flatness condition (3.2) holds if and only if λ is a Lie algebra homomorphism. Hence having a left-symmetric algebra structure on $\mathfrak{g}$ which is compatible with the Lie structure of $\mathfrak{g}$ is the same as having a Lie algebra homomorphism $\lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ such that $[x, y] = \lambda_x(y) - \lambda_y(x)$ or equivalently $ad_x = \lambda_x - \rho_x$.

Now let's take a different approach to the subject and go back to the representation view point developed in the previous section and compare with our left-symmetric algebra structure on $\mathfrak{g}$. As we mentioned earlier, affinely flat structure is a special case of (A, X)-structure with $A = \mathrm{Aff}(n)$ and $X = \mathbb{E}^n$, the Euclidean space as an affine space with the standard affine structure. Hence a left invariant affinely flat structure on a Lie group G (which is a left-symmetric algebra structure on $\mathfrak{g}$ compatible with Lie structure of $\mathfrak{g}$) corresponds to a left invariant affine structure on G. By Theorem 2.4, we have a Lie group homomorphism $\phi : G \to \mathrm{Aff}(n)$ such that $d(ev_x)|_e$ is an isomorphism for some $x \in X$. Again assuming G is simply connected, this is equivalent to having a Lie algebra homomorphism $r : \mathfrak{g} \to \mathfrak{a} = \mathfrak{aff}(n)$ which is transversal to $\mathfrak{a}_x$ for some $x \in X$, where $\mathfrak{a}_x$ is the Lie algebra of the isotropy subgroup A_x of A. Choose $x \in X = \mathbb{E}^n$ as the origin so that the affine space X becomes a vector space $V(\cong \mathbb{R}^n)$ and identify $\mathfrak{a}$ with $\mathfrak{gl}(V) + V$, where $\mathfrak{gl}(V)$ corresponds to $\mathfrak{a}_x$, $x = o \in V$. If $(M, m), (N, n) \in \mathfrak{gl}(V) + V$, the Lie algebra structure is given by

$$\begin{aligned}[(M, m), (N, n)] &= (MN - NM, Mn - Nm) \\ &= ([M, N], Mn - Nm).\end{aligned}$$

This follows viewing $\mathrm{Aff}(n)$ as a subgroup $\{\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} | A \in Gl(n),\ a \in \mathbb{R}^n\}$ of $Gl(n+1)$, and its Lie algebra as $\{\begin{pmatrix} M & m \\ 0 & 0 \end{pmatrix} | M \in \mathfrak{gl}(n),\ m \in \mathbb{R}^n\}$. Now if we denote the Lie algebra representation $r : \mathfrak{g} \to \mathfrak{a}$ by

$r = (h, t) : \mathfrak{g} \to \mathfrak{a} = \mathfrak{gl}(V) + V$, then the transversality condition exactly corresponds to that t is an isomorphism. Also note that $r = (h, t) : \mathfrak{g} \to \mathfrak{gl}(V) + V$ is a Lie algebra homomorphism if and only if h is a Lie algebra homomorphism and t is a 1-cocycle of $\mathfrak{g}$-module V with $\mathfrak{g}$ action on V given by h; i.e., $t : \mathfrak{g} \to V$ and $t([x, y]) = h(x)t(y) - h(y)t(x)$. Furthermore, if we identify $T_x\mathbb{E}^n$ with $\mathbb{R}^n$ as usual for arbitrary $x \in \mathbb{E}^n$, then we have

PROPOSITION 3.3. *For any $x \in \mathbb{E}^n = V$, $d(ev_x)|_e : T_eG = \mathfrak{g} \to T_x\mathbb{E}^n = V$ is given by $Y \mapsto h(Y)x + t(Y)$. In particular, $d(ev_o)|_e = t$.*

Proof. From the equation $ev_x = Ev_x \circ \phi$, we have $dev_x|_e = dEv_x|_o \circ d\phi = dEv_x|_o \circ r$ and the conclusion follows from the fact that $dEv_x|_o : (M, m) \mapsto Mx + m$. Indeed, take any $Y = (M, m) \in \mathfrak{a}$ and consider 1-parameter subgroup e^{tY} of A, then $dEv_x|_o(Y) = \frac{d}{dt}|_0 Ev_x(e^{tY}) = \frac{d}{dt}|_0 e^{tY}x = Mx + m$. In the calculation of the last equality, we identify Aff(n) as the subgroup of $Gl(n+1)$ mentioned above and note that

$$e^{tY} \cdot x = \left(I + t\begin{pmatrix} M & m \\ 0 & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} M & m \\ 0 & 0 \end{pmatrix}^2 + \cdots\right)\begin{pmatrix} x \\ 1 \end{pmatrix}. \quad \square$$

PROPOSITION 3.4. *Let $\phi : G \to A = Aff(n)$ be a Lie group homomorphism such that $d(ev_x)|_e$ is a linear isomorphism for some $x \in \mathbb{R}^n$, and let $d\phi = (h, t) : \mathfrak{g} \to \mathfrak{a} = \mathfrak{gl}(n) + \mathbb{R}^n$ be its differential. Then the pull back connection ∇ on G under the map $ev_x : G \to \mathbb{R}^n$ is given by*

$$d(ev_x)|_e(\nabla_X Y) = h(X)(h(Y)x + t(Y)), \ X, Y \in \mathfrak{g}.$$

In particular, if $x = 0$, then $\nabla_X Y = t^{-1}(h(X)t(Y))$.

Proof. We saw in the previous section that ev_x is a covering map onto its image and the pull back connection ∇ is left- invariant. Hence $\nabla_X Y \in \mathfrak{g} = T_eG$ for $X, Y \in \mathfrak{g}$ and is determined by $dev_x|_e(\nabla_X Y)$. Let $\tilde{X} = dev_x(X)$ be the local vector field on $\mathbb{R}^n$ defined near x which is ev_x-related to X. Then $dev_x|_e(\nabla_X Y) = (D_{\tilde{X}}\tilde{Y})(x)$, where D is the standard connection on $\mathbb{R}^n$. Consider 1-parameter subgroup e^{tX} and compute $\tilde{Y}$ along the curve $ev_x(e^{tX})$. Recall that $ev_x \circ l_g = \phi(g) \circ ev_x$.

Then

$$\begin{aligned}\tilde{Y} \text{ at } ev_x(e^{tX}) &= dev_x(Y \text{ at } e^{tX}) \\ &= dev_x(dl_{e^{tX}}|_e(Y)) \\ &= d(\phi(e^{tX}))(dev_x|_e(Y)) \\ &= L(e^{tX})(h(Y)x + t(Y))\end{aligned}$$

where $L(e^{tX})$ is the linear part of affine transformation $\phi(e^{tX})$, that is, $L(e^{tX}) = p \circ \phi(e^{tX})$, p is the canonical projection: $\mathrm{Aff}(n) \to Gl(n)$.

$$\begin{aligned}D_{\tilde{X}}\tilde{Y} \text{ at } x &= \frac{d}{dt}\Big|_0(\tilde{Y} \text{ along } ev_x(e^{tX})) \\ &= \frac{d}{dt}\Big|_0(L(e^{tX})(h(Y)x + t(Y))) \\ &= \left(\frac{d}{dt}\Big|_0 L(e^{tX})\right)(h(Y)x + t(Y)).\end{aligned}$$

Now $\frac{d}{dt}\big|_0(L(e^{tX})) = \frac{d}{dt}\big|_0 p \circ \phi(e^{tX}) = dp \circ d\phi(\frac{d}{dt}\big|_0 e^{tX}) = dp \circ d\phi(X) = h(X)$. Note that $dp : \mathfrak{gl}(n) + \mathbb{R}^n \to \mathfrak{gl}(n)$ is the projection and hence $dp \circ d\phi = h$. Therefore

$$dev_x|_e(\nabla_X Y) = D_{\tilde{X}}\tilde{Y}(x) = h(X)(h(Y)x + t(Y)). \quad \square$$

Now if we identify $\mathbb{R}^n$ with $\mathfrak{g}$ by $t : \mathfrak{g} \to \mathbb{R}^n$, then $\nabla_X Y = \tilde{h}(X)Y$, where $\tilde{h} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by $\tilde{h}(X) = t^{-1} \circ h(X) \circ t$. Hence we recover back the left multiplication $\lambda = \tilde{h}$ of the associated left symmetric algebra structure on $\mathfrak{g}$.

Let's summarize as follows.

THEOREM 3.5. *Let G be a simply connected Lie group with its Lie algebra $\mathfrak{g}$. Then the followings are equivalent.*

(1) G admits a left invariant affinely flat connection.

(2) There is a representation $\phi : G \to \mathrm{Aff}(n)$ such that $d(ev_x)|_e$ is a linear isomorphism for some $x \in \mathbb{E}^n$. (Such a representation is called etale representation and we may phrase this as G acts on $\mathbb{E}^n$ locally simply transitively by affine transformations).

(3) *There is a Lie algebra representation* $r = (h, t) : \mathfrak{g} \to \mathfrak{aff}(n) = \mathfrak{gl}(n) + \mathbb{R}^n$ *such that* t *is a linear isomorphism.*

(4) *There is a Lie algebra representation* $h : \mathfrak{g} \to \mathfrak{gl}(n)$ *and a linear isomorphism* $t : \mathfrak{g} \to \mathbb{R}^n$ *such that*

$$t([x, y]) = h(x)t(y) - h(y)t(x).$$

(5) *There is a Lie algebra representation* $\lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *such that*

$$[x, y] = \lambda(x)y - \lambda(y)x.$$

(6) *There is a Lie algebra homomorphism* : $\mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$ *of the from* $a \mapsto (\lambda(a), a)$. *(Such a representation will be called canonical in this paper.)*

(7) *There is a left-symmetric algebra structure on* $\mathfrak{g}$ *which is compatible with the Lie structure of* $\mathfrak{g}$.

Note that a similar theorem was stated in [10] for the complete case. Now suppose that ∇ is a left invariant flat connection on G which is geodesically complete. Then the developing map $D : G \to \mathbb{E}^n$ is an affine equivalence and hence the affine structure is complete in the sense of the previous section. Conversely, it is obvious that the completeness in our sense implies the geodesic completeness. By 2.3, complete case corresponds to a representation ϕ where $d(ev_x)|_e$ is a linear isomorphism for *all* $x \in \mathbb{E}^n$, and G acts on $\mathbb{E}^n$ simply transitively. And on the Lie algebra level, this corresponds to a Lie algebra representation $r = (h, t)$ such that $Y \mapsto h(Y)x + t(Y)$ is an isomorphism for *all* $x \in \mathbb{R}^n$. Now if we identify $\mathfrak{g}$ with $\mathbb{R}^n$ by $t : \mathfrak{g} \to \mathbb{R}^n$, then this map $d(ev_x)|_e$ corresponds to the map : $\mathfrak{g} \to \mathfrak{g}$ given by $Y \mapsto \lambda(Y)(t^{-1}x) + Y$. Hence ∇ is complete if and only if $Y \mapsto Y + \nabla_Y X$ is an isomorphism for all $X \in \mathfrak{g}$, and this is to say that $1 + \rho_x$ is an isomorphism for all $x \in \mathfrak{g}$. (See [9], [17], [10].) This fact is a very interesting algebraic formulation of the condition for the geodesic completeness of ∇. It would be interesting if one can find a different proof of this condition for completeness using a more direct geometric reasonning.

4. Geometry of canonical representations

In general, suppose we have a group G acting on a space X so that the action induces a group homomorphism $\phi : G \to A$, where A is an automorphism group of X. Then from the definition of an action, we obtain the following two identities which are obvious but useful for our analysis.

$$ev_x \circ l_g = \phi(g) \circ ev_x \tag{4.1}$$

$$ev_{g\cdot x} = ev_x \circ r_g, \quad g \in G \text{ and } x \in X. \tag{4.2}$$

Here the $g \in G$ action on a point $x \in X$ is denoted by $g \cdot x$.

Suppose that a simply connected Lie group G has a left invariant flat affine structure so that we have a Lie group homomorphism $\phi : G \to A = \mathrm{Aff}(n)$ as described in the previous sections. Choose a point $x \in X = \mathbb{E}^n$ where $d(ev_x)|_e$ is an isomorphism as the origin. Then the affine space X becomes a vector space V and the affine transformation group A can be written as a semi-direct product $Gl(V) \ltimes V$ and its Lie algebra can be correspondingly written as a sum, $a = \mathfrak{gl}(V) + V$. Therefore the differential of ϕ has two components $d\phi = (h, t) : \mathfrak{g} \to \mathfrak{gl}(V) + V$ and the condition that $d(ev_x)|_e$ is an isomorphism is equivalent to that t is an isomorphism. Now if we identify V with $\mathfrak{g}$ by t, then we obtain a canonical representation, $r = (\lambda, id) : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) + \mathfrak{g}$. As we saw in the previous section, this λ is exactly the left multiplication determined by the left invariant connection ∇ pulled back by ev_x, i.e., $\lambda_x(y) = \nabla_x y$ for $x, y \in \mathfrak{g}$. Hence the algebraic formulas appeared in the representation or left symmetric algebra are expected to have differential geometric meanings and intrepretations, and vice versa.

Let us set and fix the notations for the canonical representation as follows.

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{r=(\lambda, id)} & \mathfrak{a} = \mathfrak{aff}(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g}) + \mathfrak{g} \\
\big\downarrow \exp & & \big\downarrow \exp \\
G & \xrightarrow{\phi=(L,q)} & A = \mathrm{Aff}(\mathfrak{g}) = Gl(\mathfrak{g}) \ltimes \mathfrak{g}
\end{array}$$

We will frequently use the notations $\lambda_a, L_g, \cdots$ for $\lambda(a), L(g), \cdots$, etc. in the computations below.

Using this set up, let's first calculate $ev_x : G \to \mathfrak{g}$ explicitly.

Let $g = \exp a$, $a \in \mathfrak{g}$ and $g \in G$. Then

$$\begin{aligned}\exp(\lambda_a, a) &= 1 + (\lambda_a, a) + \frac{1}{2!}(\lambda_a, a)^2 + \cdots \\ &= (1 + \lambda_a + \frac{1}{2!}\lambda_a^2 + \cdots, a + \frac{1}{2!}\lambda_a(a) + \frac{1}{3!}\lambda_a^2(a) + \cdots) \\ &= (e^{\lambda_a}, \text{“}e^a - 1\text{”})\end{aligned}$$

where “$e^a - 1$” $= a + \frac{1}{2!}a \cdot a + \frac{1}{3!}a \cdot (a \cdot a) + \cdots$. Recall that $(\lambda_a, a)^2, \cdots$ is calculated using $\begin{pmatrix} \lambda_a & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda_a & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_a^2 & \lambda_a(a) \\ 0 & 0 \end{pmatrix}, \cdots$, etc. Hence $ev_x(g) = g \cdot x = L_g(x) + q_g = e^{\lambda_a}(x) +$ “$e^a - 1$”, and record this as

$$(4.3) \qquad \begin{cases} g \cdot x = e^{\lambda_a}(x) + \text{“}e^a - 1\text{”}, \quad g = \exp a \\ L_g = e^{\lambda_a} \\ q_g = \text{“}e^a - 1\text{”} = a + \frac{1}{2!}a \cdot a + \frac{1}{3!}a \cdot (a \cdot a) + \cdots \end{cases}$$

We know that ev_x is a covering map and a left translation l_g corresponds to g action on $\mathfrak{g}$ as given in (4.1). To compute locally the vector fields on $\mathfrak{g}$ which corresponds to left invariant vector fields on G, differentiate (4.1) to obtain $d(ev_x)\big|_g \cdot dl_g\big|_e = d(\phi(g))|_x \circ d(ev_x)|_e$.

Note that $d(ev_x)|_e = 1 + \rho_x$ by 3.3 (or we can compute this directly using explicit formula for ev_x given above) and since $\phi(g)$ is an affine transformation, its differential $d(\phi(g))$ is its linear part L_g. Hence we obtain

$$(4.4) \qquad d(ev_x)|_g \circ dl_g|_e = L_g \circ (1 + \rho_x), \quad x \in X = \mathfrak{g},\ g \in G.$$

Let $y \in \mathfrak{g}$ be a left invariant vector field on G. Then the corresponding vector field on $X = \mathfrak{g}$ under the map ev_x will be given by $L_g \circ (1 + \rho_x)(y)$ at the point $g \cdot x \in X = \mathfrak{g}$.

Similarly, to obtain the vector fields corresponding to the right invariant vector fields, differentiate (4.2) to have $d(ev_{g \cdot x})|_e = d(ev_x)|_g \circ dr_g|_e$. Then the right invariant vector field generated by $y \in T_eG \cong \mathfrak{g}$ descends on $X = \mathfrak{g}$ by ev_x to a vector field given by $(1 + \rho_{g \cdot x})(y)$ at the point $g \cdot x \in X = \mathfrak{g}$. In particular, the left (resp. right) invariant vector field generated by $y \in T_eG \cong \mathfrak{g}$ is locally (resp. globally) ev_o-related to the vector field given by $Y_{g \cdot o} = L_g(y)$ (resp. $\tilde{Y}_x = (1 + \rho_x)(y)$).

Also from this, we see that $1 + \rho_x$, $x = g \cdot o \in X = \mathfrak{g}$, can be seen as the differential of the local map of $X = \mathfrak{g}$ into itself which corresponds to the right translation r_g on a neighborhood of $e \in G$ through ev_o. More precisely, if we choose a neighborhood U of $e \in G$ so that $ev_o|_U$ is a diffeomorphism, and let $\bar{r}_g = ev_o \circ r_g \circ ev_o^{-1}$ on $ev_o(U) \subset \mathfrak{g}$, then we have

$$d\bar{r}_g|_o = 1 + \rho_{g \cdot o}. \tag{4.5}$$

We have a following interesting formula for a fixed point of G-action on $X = \mathfrak{g}$.

Proposition 4.1. *$x \in X = \mathfrak{g}$ is fixed by 1-parameter subgroup determined by $a \in \mathfrak{g}$ if and only if $(1 + \rho_x)a = 0$.*

Proof. Observe that

$$\begin{aligned}
\frac{d}{dt}(\exp(ta) \cdot x) &= \frac{d}{dt} ev_x(\exp(ta)) \\
&= d(ev_x)\big|_g\left(\frac{d}{dt}\exp(ta)\right) \\
&= d(ev_x)\big|_g(dl_g\big|_e(a)) \\
&= d(\phi(g))\big|_x d(ev_x)\big|_e(a) && \text{(differentiate (4.1))} \\
&= L_g(d(ev_x)\big|_e(a)) && \text{(recall that } \phi(g) = (L_g, q_g))
\end{aligned}$$

Hence $\exp(ta) \cdot x \equiv x$ if and only if $d(ev_x)|_e(a) = 0$, and note that $d(ev_x)|_e = 1 + \rho_x : T_eG = \mathfrak{g} \to \mathfrak{g}$. □

As an immediate corollary of 4.1, we obtain the following well known observation [9, 17].

Corollary 4.2. *$-u \in X = \mathfrak{g}$ is a fixed point of the canonical G-action on $\mathfrak{g}$ if and only if $1 - \rho_u = 0$, i.e., u is the right identity of the left-symmetric algebra structure on $\mathfrak{g}$.*

Note that in general, the proof of 4.1 shows that a G action on X has a fixed point $x \in X$ if and only if $d(ev_x)|_e = 0$. In affine case, if we choose x as our origin, then G acts as a linear transformation and $\phi : G \to \mathrm{Gl}(V)$, where V is the vector space obtained from the affine space by choosing x as the origin. Such an affine structure is called radiant [6] and this happens exactly when the 1-cocyle t becomes a coboundary. In terms of the canonical representation, t corresponds to the identity on $\mathfrak{g}$ and being a coboundary means that this equals to ρ_x for some $x \in \mathfrak{g}$, which is exactly the content of 4.2.

The proof of 4.1 shows that the infinitesimal action of $a \in \mathfrak{g}$ in affine case is given by

$$\frac{d}{dt}\Big|_0(\exp(ta)\cdot x) = dev_x\Big|_e(a) = (1+\rho_x)(a) = a\cdot x + a.$$

This also can be seen using (4.1), by differentiating the formula $\exp(ta)\cdot x = e^{t\lambda_a}(x) +$ "$e^{ta} - 1$" at $t = 0$. From this we also see that the infinitesimal linear holonomy action of $a \in \mathfrak{g}$ is given by

$$(4.6) \qquad \begin{cases} \frac{d}{dt}L_{\exp(ta)}(x) = \frac{d}{dt}e^{t\lambda_a}(x) = e^{t\lambda_a}(a\cdot x) \\ \frac{d}{dt}\Big|_0 L_{\exp(ta)}(x) = a\cdot x \end{cases}$$

In fact, we can interprete $a \cdot x$ directly. Since $a\cdot x = \nabla_a x$ and $ev_o : G \to \Omega = ev_o(G) \subset X = \mathfrak{g}$ is an affine equivalence, $a \cdot x$ corresponds locally to $D_A X$ on Ω, where A and X are local vector fields on Ω corresponding to left invariant vector fields a and x on G. Of course, $D_A X$ is a derivative of X ($X_{g\cdot o} = L_g(x)$) in the direction of A.

From the proof of 4.1 and (4.6), we can easily calculate the higher derivatives of the curve given as an orbit of x under the action of 1-parameter subgroup generated by $a \in \mathfrak{g}$.

$$\begin{aligned}
\frac{d}{dt}\exp(ta)\cdot x &= L_{\exp(ta)}((1+\rho_x)(a)) = e^{t\lambda_a}(1+\rho_x)(a) \\
\frac{d^2}{dt^2}\exp(ta)\cdot x &= \frac{d}{dt}e^{t\lambda_a}((1+\rho_x)(a)) = e^{t\lambda_a}(a\cdot(1+\rho_x)(a)) \\
&= e^{t\lambda_a}(\lambda_a(1+\rho_x)(a)) \\
&\vdots \\
\frac{d^n}{dt^n}\exp(ta)\cdot x &= e^{t\lambda_a}(\lambda_a^{n-1}(1+\rho_x)(a))
\end{aligned}$$

and

$$\frac{d^n}{dt^n}\Big|_0 \exp(ta) \cdot x = \lambda_a^{n-1}(1 + \rho_x)(a) = a(a(\cdots(a + ax)\cdots)).$$

In particular, if $x = 0$, we can interprete a, $a \cdot a, a(a \cdot a), \cdots$ as the first derivative (=tangent vector), the second derivative (=acceleration vector), the third derivative,$\cdots$ of the image curve under ev_o of of 1-parameter subgroup generated by $a \in \mathfrak{g}$ at the origin. This also follows from (4.3) directly.

Now let us examine the geometric meaning of the equation $a \cdot x = 0$. From (4.6), we see immediately that this holds if and only if $L_{\exp(ta)}(x) = x$ for all $t \in \mathbb{R}$. This means that the local vector field X on Ω which is ev_o-related to the left invariant vector field $x \in \mathfrak{g}$ is invariant (i.e., parallel) along the integral curve of A, the local vector field ev_o-related to $a \in \mathfrak{g}$. In fact, this is exactly what $\nabla_a x = 0$ or $D_A X = 0$ means. The following proposition is obvious from this observation.

PROPOSITION 4.3. *For each $x \in \mathfrak{g}$, the following statements are equivalent. (Recall G is connected.)*

(1) $\rho_x = 0$.

(2) *The left invariant vector field x is parallel.*

(3) *The vector field X on Ω ev_o-related to x is a constant field. (Hence globally well-defined.)*

(4) $L_g(x) = x$ *for all $g \in G$.*

On the other hand, for $\lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, we have the following.

PROPOSITION 4.4. (i) *$T = \ker\lambda$ is a two sided ideal of the left-symmetric algebra $\mathfrak{g}$.*

(ii) $\lambda_x = 0$ *if and only if $\phi(g)$, $g = \exp x$, is a translation.*

Proof. (i) If $\lambda_x = 0$, $x \cdot a = 0$ for all $a \in \mathfrak{g}$, and $a \cdot x \in T$ since $(a \cdot x) \cdot y = [a, x] \cdot y = a(xy) - x(ay) = 0$ for all $y \in \mathfrak{g}$. (ii) follows from the fact that $L_{\exp x} = e^{\lambda_x} = 1$. □

We have a following interesting equation for ρ_x.

THEOREM 4.5. *For any $x \in \mathfrak{g}$ and any $g \in G$, we have*

$$\rho_{L_g(x)} = L_g \rho_x L_g^{-1}(1 + \rho_{g \cdot o}).$$

To prove this theorem, we start with the following lemma which also shows an interesting (but almost obvious) interpretation of ρ_x. Recall $\rho_x = \nabla x$.

LEMMA 4.6. *Let U be a small neighborhood of e in G. For a left invariant vector field $x \in \mathfrak{g}$, let (x, g) be the local vector field on $ev_o(gU)$ given by $ev_{o*}(x|_{gU})$. If we view (x, g) as a local map : $ev_o(gU) \subset \mathfrak{g} \to \mathfrak{g}$ as usual, then $\rho_x = D(x, e)|_o$, the differential of a map (x, e) at o, and $D(x, g)|_{g \cdot o} = L_g \rho_x L_g^{-1}$.*

Proof. $\rho_x(y) = \nabla_y x = D_y(x, e)$ (note that $d(ev_o)|_e : T_eG = \mathfrak{g} \to \mathfrak{g}$ is the identity map), where D is the usual flat connection on a vector space $\mathfrak{g}$, and $D_y(x, e)$ is the directional derivative in the direction of y and hence equals $D(x, e)(y)$.

Since l_g and $\phi(g)$ are ev_o-related, from the very definition of (x, g) we have $L_g \circ (x, e) = (x, g) \circ \phi(g)$, again viewing (x, e) and (x, g) as maps: $\mathfrak{g} \to \mathfrak{g}$. Taking differentials on both sides at o, we get

$$L_g \circ D(x, e)\big|_o = D(x, g)\big|_{g \cdot o} \circ d(\phi(g))\big|_o = D(x, g)\big|_{g \cdot o} \circ L_g. \qquad \square$$

As a consequence of this lemma, we see that

$$\text{(4.7)} \qquad \operatorname{tr}(\rho_x) = \operatorname{div}(x, g)\big|_{g \cdot o} \text{ for all } g \in G,$$

where $\operatorname{tr}(\rho_x)$ is the trace of ρ_x and $\operatorname{div}(x, g)$ is the divergence of the vector field (x, g) which is defined as the trace of the differential $D(x, g)$. Hence the divergence of the left invariant vector field x is well-defined and equals $\operatorname{tr}(\rho_x)$.

Proof of 4.5. Let $y \in T_e G = \mathfrak{g}$. Then

$$\begin{aligned}
&\rho_{L_g(x)}(y) \\
&= D(L_g(x), e)\big|_o(y) \\
&= \frac{d}{dt}\Big|_o L_{e^{ty}}(L_g(x)) \qquad (\frac{d}{dt}\Big|_o e^{ty}\cdot o = y \text{ and recall } (v,e)_{h\cdot o} = L_h(v)) \\
&= \frac{d}{dt}\Big|_o L_{e^{ty}\cdot g}(x) \\
&= D(x,g)\big|_{g\cdot o}(ev_{o*}r_{g*}y). \qquad (\frac{d}{dt}\Big|_o e^{ty}\cdot g\cdot o = ev_{o*}r_{g*}y \\
&\qquad\qquad\qquad\qquad \text{and } (x,g)_{h\cdot o} = L_h(x))
\end{aligned}$$

Now if we let $z = ev_{o*}r_{g*}y$, then $z = \bar{r}_{g*}ev_{o*}y = (1+\rho_{g\cdot o})(y)$ by (4.5) and $ev_{o*}|_e = 1$. Hence

$$\begin{aligned}
L_g\rho_x L_g^{-1}(z) &= D(x,g)\big|_{g\cdot o}(z) \qquad \text{(Lemma 4.6)} \\
&= \rho_{L_g(x)}(y) \\
&= \rho_{L_g(x)}(1+\rho_{g\cdot o})^{-1}(z).
\end{aligned}$$

□

COROLLARY 4.7. *For all $x, y \in \mathfrak{g}$ and $g \in G$, we have*

(i) $1 + \rho_{g\cdot x} = L_g(1+\rho_x)L_g^{-1}(1+\rho_{g\cdot o})$
(ii) $L_g(x\cdot y) = L_g(x)\cdot L_g(y) - L_g(L_g^{-1}(L_g(x)\cdot(g\cdot o))\cdot y)$
(iii) $L_g(x\cdot y) = ((1+\rho_{g\cdot o})^{-1}L_g(x))\cdot L_g(y)$.

The proof of this corollary is immediate from the theorem and will be omitted. (ii) shows explicitly the deficiency for L_g, the linear holonomy, to be a left-symmetric algebra homomorphism and the corresponding equation for Lie algebra level is that

$$\lambda_a(x\cdot y) = (\lambda_a x)\cdot y + x\cdot(\lambda_a y) - (\rho_a x)\cdot y,$$

which follows from the left symmetry of the algebra and shows the deficiency for λ_a to be a derivation.

In (i), if we let $x = h\cdot o$, then (i) can be written as

$$(1+\rho_{gh\cdot o}) = L_g(1+\rho_{h\cdot o})L_g^{-1}(1+\rho_{g\cdot o}), \tag{4.8}$$

and reveals a cocycle property of $(1+\rho_{g\cdot o})^{-1}$. In fact, (4.8) can be proved in a more motivating way. The identity $r_h \circ l_g = l_g \circ r_h$ on G descends on $\Omega \subset \mathfrak{g}$ locally via ev_o to the equation $\bar{r}_h \circ \phi(g) = \phi(g) \circ \bar{r}_h$. Taking differential on both sides and use (4.5). But of course, the statement of 4.6 is more general since x can be any point in $\mathfrak{g}$, not only in Ω.

As we saw in the previous discussions, the function $1+\rho_x\ (= dev_x|_e) : \mathfrak{g} \to \mathfrak{g}$ plays an important role in understanding the geometry of left-symmetric algebra. The polynomial function $p(x) = \det(1+\rho_x) : \mathfrak{g} \to \mathbb{R}$ is called the *characteristic polynomial* in [9].

The characteristic polynomial has a following interesting property well known for complex affine case [9]. And the real case follows immediately from Corollary 4.7 and (4.8). Let $\mathbb{R}_+$ be the multiplicative group of positive real numbers.

COROLLARY 4.8. *Let* $p(x) = \det(1+\rho_x) : \mathfrak{g} \to \mathbb{R}$. *Then* $p(g\cdot x) = \Delta(g)p(x)$, $g \in G$, *where* $\Delta : G \to \mathbb{R}_+$ *is a group homomorphism given by* $\Delta(g) = \det(1+\rho_{g\cdot o})$.

The characteristic polynomial is very useful for studying the developing maps or evaluation maps. This together with some applications of Corollary 4.8 will be discussed in the forthcoming paper.

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