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NONWANDERING POINTS OF A MAP ON THE CIRCLE

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§ 1. Introduction

In study of the dynamics of a map f from a topological space X to itself, a central role is played by the various recursive properties of the points of X. One such property is periodicity. A weaker property is that of being nonwandering. Intermediate recursive properties include almost periodicity and recurrence.

Let $C^0(X,X)$ denote the set of continuous maps from X into itself. And for any $f \in C^0(X,X)$, let $P(f), R(f), \Lambda(f), \Gamma(f)$ and $\Omega(f)$ denote the set of periodic points, recurrent points, ω-limit points, γ-limit points and nonwandering points of f, respectively.

In 1988, J.C.Xiong [4] proved the following sequence of the sets and inclusion relation hold;

$$P(f) \subset R(f) \subset \Gamma(f) \subset \overline{P(f)} \subset \Lambda(f) \subset \Omega(f)$$

for any continuous map f of the interval I. But the equalities need not hold.

For a continuous map f of the circle S^1, J.S.Bae, S.H.Cho and S.K.Yang [2] obtained the similar result;

$$P(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f).$$

Also, in this case, the equalities need not hold.

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On the other hand, in 1983, L. Block, E. Coven, I. Mulvey and Z. Nitecki [1] showed that for any continuous map f of the circle, if $P(f)$ is closed and non-empty, then $P(f) = \Omega(f)$, and hence

$$(P(f) =)R(f) = \cdots = \Omega(f).$$

In this paper, we will show that the above equalities hold unless $P(f)$ is non-empty. Consequently, we obtain the following result.

THEOREM A. *For any $f \in C^0(S^1, S^1)$, if $P(f)$ is empty, then*

$$R(f) = \Gamma(f) = \overline{R(f)} = \Lambda(f) = \Omega(f).$$

§ 2. Preliminaries and definitions

Let (X, d) be a metric space and $f \in C^0(X, X)$. And let $f^{n+1} = f \circ f^n$, for $n = 1, 2, 3, \cdots$

A point $x \in X$ is called a *recurrent point* of f if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$. We denote the set of recurrent points of f by $R(f)$.

A point $x \in X$ is called a *nonwandering point* of f if for every neighborhood U of x, there exists a positive integer m such that $f^m(U) \cap U \neq \phi$. We denote the set of nonwandering points of f by $\Omega(f)$.

A point $y \in X$ is called an *ω-limit point* of $x \in X$ if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. We denote the set of ω-limit points of x by $\omega(x)$. Define $\Lambda(f) = \bigcup_{x \in X} \omega(x)$.

A point $y \in X$ is called an *α-limit point* of $x \in X$ if there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{y_i\}$ of points such that $f^{n_i}(y_i) = x$ and $y_i \to y$. The symbol $\alpha(x)$ denotes the set of α-limit points of $x \in X$.

A point $y \in X$ is called a *γ-limit point* of x if $y \in \omega(x) \cap \alpha(x)$. The symbol $\gamma(x)$ denotes the set of γ-limit points of x and $\Gamma(f) = \bigcup_{x \in X} \gamma(x)$.

Let $x \in S^1$ and $f \in C^0(S^1, S^1)$ be given. Then we will use the symbol $\omega_+(x)$ (*resp.* $\omega_-(x)$) to denote the set of all points $y \in S^1$

such that there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$ and

$$y < \cdots < f^{n_i}(x) < \cdots < f^{n_2}(x) < f^{n_1}(x)$$
$$(\ \textit{resp.}\ \ f^{n_1}(x) < f^{n_2}(x) < \cdots < f^{n_i}(x) < \cdots < y\ ).$$

A set $E \subset X$ is said to be *invariant* under f if $f(E) \subset E$. It is clear that $P(f)$, $R(f)$, $\Lambda(f)$, $\Gamma(f)$ and $\Omega(f)$ are invariant under f.

Let R be the set of reals and Z be the set of integers. Formally, we will think of the circle S^1 as R/Z and use $\pi : R \to R/Z$ to denote the canonical projection. In fact, the map $\pi : R \to S^1$ is an example of a covering map, since it wraps R around S^1 without doubling back (i.e., without critical points). To study the dynamics of the circle map, it is helpful to using a *lifting* .

Let f be a continuous map on the circle. We say that a continuous map F from R to itself is a lifting of f if $f \circ \pi = \pi \circ F$.

We will use the following notations throughout this paper.

Let $a, b \in S^1$ with $a \neq b$, and let $A \in \pi^{-1}(a)$, $B \in \pi^{-1}(b)$ with $|A - B| < 1$ and $A < B$. Then we write $\pi\big((A, B)\big)$, $\pi\big([A, B]\big)$, $\pi\big([A, B)\big)$ and $\pi\big((A, B]\big)$ to denote the open, closed and half-open arcs from a counterclockwise to b, respectively, and we denote it by (a, b), $[a, b]$, $[a, b)$ and $(a, b]$.

For $x, y \in [a, b]$ with $a \neq b$, let $X \in \pi^{-1}(x)$, $Y \in \pi^{-1}(y)$ with $X, Y \in [A, B]$, then we define for $x, y \in [a, b]$, $x > y$ if and only if $X > Y$. Let C be a subset of a closed arc $[a, b]$, then we define $\sup C = \pi\big(\sup(\pi^{-1}(C) \cap [A, B])\big)$ and $\inf C = \pi\big(\inf(\pi^{-1}(C) \cap [A, B])\big)$. In particular, for $a, b, c \in S^1$, $a < b < c$ means that b lies in the open arc (a, c), that is, $b \in (a, c)$.

Now we consider the notation of an *f-covering.* The important property of an f-covering lies in the fact that if J f^n-covers itself for some n, then f has a periodic point in J.

DEFINITION 2.1. Let X be I or S^1 and $f \in C^0(X, X)$. Let J and K be two closed intervals in X. We say that J f-covers K if there is a closed subinterval $L \subset J$ such that $f(L) = K$.

The following two lemmas appear in [3].

LEMMA 2.2. [3, Lemma 2] *Let X be S^1 or I and $f \in C^0(X,X)$. Let J and K be proper closed intervals in X such that J f-covers K. If L is a closed interval with $L \subset K$, then J f-covers L.*

LEMMA 2.3. [3, Lemma 3] *Let $f \in C^0(I,I)$. Suppose that J is a proper closed interval in X such that J f-covers J or $f(J) \subset J$. Then f has a fixed point in J.*

§ 3. Main Result

The following lemma appears in [2].

LEMMA 3.1. [2] *For any $f \in C^0(S^1,S^1)$, $x \in \Omega(f)$ if and only if $x \in \alpha(x)$.*

LEMMA 3.2. *Let $f \in C^0(S^1,S^1)$ and $I = [a,b]$ be an arc for some $a, b \in S^1$ with $a \neq b$, and let $I \cap P(f) = \phi$.*

(a) *Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x < f(x)$. Then*
  (1) *if $y \in I$, $x < y$ and $f(y) \notin [y,b]$, then $[x,y]$ f-covers $[f(x),b]$,*
  (2) *if $y \in I$, $x > y$ and $f(y) \notin [y,b]$, then $[y,x]$ f-covers $[f(x),b]$.*

(b) *Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x > f(x)$. Then*
  (1) *if $y \in I$, $x < y$ and $f(y) \notin [a,y]$, then $[x,y]$ f-covers $[a,f(x)]$,*
  (2) *if $y \in I$, $y < x$ and $f(y) \notin [a,y]$, then $[y,x]$ f-covers $[a,f(x)]$.*

Proof. We prove only part (a) because of the symmetry. Let $A, B \in R$ with $A < B$ such that $\pi((A,B)) = (a,b)$, and let $X \in (A,B) \cap \pi^{-1}(x)$. Then we can take a lifting F of f with $F(X) \in (A,B)$. By assumption, we know that $A < X < F(X) < B$.

(1) Let $Y \in (A,B) \cap \pi^{-1}(y)$. Then $F(Y) \notin (Y+N, B+N)$ for any integer N. If $y > x$, then $Y > X$, and hence $F(Y) > Y$ because also F has no periodic points in $[A,B]$. Since $F(Y) \notin (Y+N, B+N)$ for any integer N, $F(Y) > B > F(X)$. Hence $[X,Y]$ F-covers $[F(X),B]$, so that $[x,y]$ f-covers $[f(x),b]$.

(2) Let $Y \in (A,B) \cap \pi^{-1}(y)$. Then $F(Y) \notin (Y+N, B+N)$ for any integer N. If $y < x$, then $Y < X$, and hence $F(Y) > Y$. Therefore we have $F(Y) > B > F(X)$, so that $[Y,X]$ F-covers $[F(X),B]$, and hence $[y,x]$ f-covers $[f(x),b]$.

LEMMA 3.3. *Let* $f \in C^0(S^1, S^1)$ *and* $P(f) = \phi$. *Then*

$$\overline{R(f)} \subset \Gamma(f).$$

Proof. Without loss of generality, we assume that $x \in \overline{R(f)} \setminus R(f)$. Then there exists an open arc (a, b) in S^1 containing x such that $f^n(x) \notin (a, b)$ for any positive integer n, and hence we may assume that there exists a sequence $\{x_i\}$ of points with $x_i \in R(f)$ such that $a < x_1 < x_2 < \cdots < x_i < \cdots < x < b$ and $x_i \to x$. For each $i = 1, 2, \cdots$, there exist $y_i, z_i \in (x_{i-1}, x_{i+1})$ and n_i, m_i with $n_i < m_i$ such that

$$x_{i-1} < f^{n_i}(y_i) < y_i < x_{i+1} < x$$

and

$$x_{i-1} < z_i < f^{m_i}(z_i) < x_{i+1} < x.$$

By Lemma 3.2,

$$[y_i, x] \ \ f^{n_i}\text{-covers} \ \ [a, f^{n_i}(y_i)]$$

and

$$[z_i, x] \ \ f^{m_i}\text{-covers} \ \ [f^{m_i}(z_i), b].$$

Consequently,

$$(*) \qquad [x_{i-1}, x] \ \ f^{n_i}\text{-covers} \ \ [x_1, x_{i-1}] \quad \text{for each } i,$$

and

$$(**) \qquad [x_{i-1}, x] \ \ f^{m_i}\text{-covers} \ \ [x_{i+1}, x] \quad \text{for each } i.$$

Now, let $K_i = [x_i, x]$ for all positive integer i, Then K_{i-1} f^{m_i}-covers K_{i+1}. Hence we may choose a closed arc L_1 in K_1 such that $f^{m_1}(L_1) = K_3$. Also, we can take a closed arc L_2 in L_1 such that $f^{m_1+m_3}(L_2) = K_5$. Continuing this process, we may take a closed arc $L_i \subset K_1$ such that $L_1 \supset L_2 \supset \cdots$ and $f^{\sum_{i=1}^{k} m_{2i-1}}(L_i) = K_{2k+1}$ for each $k = 1, 2, \cdots$. Let $y \in \bigcap_{i=1}^{\infty} L_i$. Then $x \in \omega(y)$ and $y \in [x_1, x]$. Now, take N such that $x_{N-1} > y$. By $(*)$, for all $i \geq N$, there exists $y_i \in [x_{i-1}, x]$ such that $f^{n_i}(y_i) = y$. Since $x_i \to x$, we have $y_i \to x$, and hence $x \in \alpha(y)$. Thus $x \in \omega(y) \cap \alpha(y) \subset \Gamma(f)$.

The following lemma appears in [2].

LEMMA 3.4. [2] *Let $f \in C^0(S^1, S^1)$. Then we have*

$$\Gamma(f) \subset R(f) \cup \overline{P(f)}.$$

By using Lemma 3.3 and Lemma 3.4, we have the following proposition.

PROPOSITION 3.5. *Let $f \in C^0(S^1, S^1)$ and $P(f) = \phi$. Then we have $R(f) = \Gamma(f) = \overline{R(f)}$, and hence $R(f)$ is closed.*

Proof. Suppose that $P(f) = \phi$. Then by Lemma 3.3, we have $\overline{R(f)} \subset \Gamma(f)$, and by Lemma 3.4, we know that $\Gamma(f) \subset R(f)$. Therefore, we conclude $R(f) = \Gamma(f) = \overline{R(f)}$.

THEOREM A. *For any $f \in C^0(S^1, S^1)$, if $P(f)$ is empty, then*

$$R(f) = \Gamma(f) = \overline{R(f)} = \Lambda(f) = \Omega(f).$$

Proof. Let $x \in \Omega(f) \setminus R(f)$ and D be a connected component of $S^1 \setminus R(f)$ containing x. By Proposition 3.5, $R(f)$ is closed, and hence $D = (a, b)$ for some $a, b \in S^1$ with $a \neq b$. Then we know that $a, b \in R(f)$. Since $R(f)$ is invariant under f, $a \in \omega_-(a)$ and $b \in \omega_+(b)$. And since $x \in \Omega(f) \cap D$, there exists $k > 0$ such that $f^k(D) \cap D \neq \phi$. Therefore, there exists a point $y \in D$ with $f^k(y) \in D$. Without loss of generality, we may assume that $a < y < f^k(y) < b$. Then we know that $[y, b]$ f^k-covers $[f^k(y), f^k(b)]$ by Lemma 3.2, and $b \in (f^k(y), f^k(b))$ since $f^k(b) \in R(f) \subset [b, a]$ and $f^k(b) \neq b$. Since $b \in \omega_+(b)$, there exist positive integers m, n such that $b < f^{m+n}(b) < f^m(b) < f^k(b)$. Especially,

$$[y, b] \quad f^k\text{-covers} \quad [b, f^m(b)].$$

On the other hand, since $f^n(b) \notin (a, b) = D$, by Lemma 3.2,

$$[b, f^m(b)] \quad f^n\text{-covers} \quad [a, f^{m+n}(b)].$$

In particular,

$$[b, f^m(b)] \quad f^n\text{-covers} \quad [y, b]$$

By Lemma 2.2, $[y, b]$ f^{n+k}-covers itself, and hence f has a periodic point in $[a, b]$ by Lemma 2.3, which is a contradiction.

The proof of Theorem A is complete.

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