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CONVEX DECOMPOSITIONS OF REAL PROJECTIVE SURFACES. III: FOR CLOSED OR NONORIENTABLE SURFACES

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The purpose of our research is to understand geometric and topological aspects of real projective structures on surfaces. A real projective surface is a differentiable surface with an atlas of charts to $\mathbf{R}P^2$ such that transition functions are restrictions of projective automorphisms of $\mathbf{R}P^2$. Since such an atlas lifts projective geometry on $\mathbf{R}P^2$ to the surface locally and consistently, one can study the global projective geometry of surfaces.

This paper is the final piece of the series of the papers [2] and [3]. With this paper, we get a satisfactory classification of all real projective structures on surfaces (see [4]). This final paper shows that a nonorientable real projective surface also has an admissible decomposition and a closed real projective surface decomposes into pieces that are convex real projective surfaces or π-Möbius bands.

Recall that the complement of a one-dimensional subspace in $\mathbf{R}P^2$ has a canonical affine structure of a complete affine plane. The complement is said to be an *affine patch*. A real projective surface has convex boundary if each point of the boundary has a neighborhood admitting a chart to a convex domain in an affine patch. Let S be an orientable or nonorientable real projective surface with convex or empty boundary. (We will often refer to [2] and [3] for definitions and results needed in this paper). Let $\tilde{S}$ denote the universal cover of S, and $\pi_1(S)$ the fundamental group of S, identified with the group of deck transformations. Given S, there is an immersion $\mathbf{dev} : \tilde{S} \to \mathbf{R}P^2$ and a homomorphism $h : \pi_1(S) \to \mathrm{PGL}(3, \mathbf{R})$ satisfying $\mathbf{dev} \circ \gamma = h(\gamma) \circ \mathbf{dev}$ for each deck

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transformation $\gamma \in \pi_1(S)$. The pair $(\mathbf{dev}, h)$ is said to be the development pair. The sphere $\mathbf{S}^2$ double covers $\mathbf{R}P^2$ and has an induced real projective structure. A projective map is a map preserving real projective structures. Consider $\mathbf{S}^2$ as the quotient of $\mathbf{R}^3 - \{O\}$ by the equivalence relation given by $x \sim y$ if $x = sy$ for $s \in \mathbf{R}^+$. An element of $\mathrm{GL}(3, \mathbf{R})$ induces a projective automorphism of $\mathbf{S}^2$, and every projective automorphism of $\mathbf{S}^2$ arises in this way. Hence the group $\mathrm{Aut}(\mathbf{S}^2)$ of projective automorphism is isomorphic to $\mathrm{GL}(3, \mathbf{R})$ quotient out by the subgroup $\{s\mathrm{I} | s \in \mathbf{R}^+\}$. Hence, $\mathrm{Aut}(\mathbf{S}^2)$ is isomorphic to the group $\mathrm{SL}_{\pm}(3, \mathbf{R})$ of linear maps of determinant ± 1. Given the development pair $(\mathbf{dev}, h)$ of S, we can lift $\mathbf{dev}$ to a projective immersion $\mathbf{dev}' : \tilde{S} \to \mathbf{S}^2$ and h to $h' : \pi_1(S) \to \mathrm{Aut}(\mathbf{S}^2)$ satisfying $\mathbf{dev}' \circ \gamma = h'(\gamma) \circ \mathbf{dev}$ for each deck transformation γ. $(\mathbf{dev}', h')$ is also said to be a *development pair*. We will say that $\mathbf{dev}'$ is a *developing map* and h' a *holonomy homomorphism* and drop the primes in the paper. (see 2, Section 1] for details.)

Since an open hemisphere in $\mathbf{S}^2$ identifies with a complement of a one dimensional subspace of $\mathbf{R}P^2$ under the double covering map, the open hemisphere has an affine structure of a complete affine plane, compatible with the projective structure in the sense that an arc in the hemisphere is affinely geodesic if and only if it is projectively geodesic up to parametrizations. $\mathbf{S}^2$ has a standard Riemannian metric of curvature 1 as $\mathbf{S}^2$ is realized as the standard sphere in $\mathbf{R}^3$. An arc in $\mathbf{S}^2$ is geodesic in this metric if and only if it is projectively geodesic up to parametrizations. Let us denote by $\mathbf{d}$ the distance metric on $\mathbf{S}^2$ induced by the Riemannian metric. A *convex segment* in $\mathbf{S}^2$ is a geodesic segment of $\mathbf{d}$-length $\leq \pi$. A *convex* subset of $\mathbf{S}^2$ is a subset such that given any two elements there exists a convex segment connecting them. Under this definition, $\mathbf{S}^2$ itself and any great circle is convex, and in an open hemisphere, the convexity in the affine sense agrees with our notion.

A surface S is *convex* if given two points in $\tilde{S}$, there exists a geodesic segment which $\mathbf{dev}$ maps homeomorphic to a convex segment. Given a convex compact surface S with $\chi(S) \leq 0$, we proved that $\mathbf{dev} : \tilde{S} \to \mathbf{S}^2$ is an imbedding onto a convex domain in a hemisphere of $\mathbf{S}^2$ (see Lemma 1.5 of [2]). Furthermore, if $\chi(S) < 0$, then $\mathbf{dev}(\tilde{S})$ is a convex subset of an open hemisphere bounded with respect to the affine coordinates, i.e., a simply convex subset of $\mathbf{S}^2$.

A Klein model of hyperbolic plane can be understood as the interior Ω of the unit disk in an affine patch in $\mathbf{R}P^2$ with the group $\mathrm{PSO}(2,1) \subset \mathrm{PGL}(3,\mathbf{R})$ acting on it as the group of isometries. Hence, each hyperbolic surface gives rise to the quotient surface Ω/Γ where Γ is a discrete torsion free subgroup of $\mathrm{PSO}(2,1)$. Since Γ consists of projective transformations, the quotient surface is a real projective surface. Ω/Γ is convex since its developing map is realized as a section $\Omega \longrightarrow \mathbf{S}^2$ of the double covering map. We can deform these surfaces to obtain other convex real projective surfaces (see Koszul [7] and Goldman [6]). In fact, the deformation space of convex real projective structures on a closed surface of genus g, $g > 1$, is determined by Goldman [6] to be topologically a cell of dimension $16g - 16$, and Goldman gives explicit parameters to describe any convex real projective surface of genus g up to isotopy.

Not all real projective surfaces with convex or empty boundary are convex. This was shown by Goldman [5] and Sullivan-Thurston [10] by grafting annuli with geodesic boundary to a convex real projective surface. However, we will show that orientable real projective surfaces decompose into convex pieces (see Theorem 2). Nonorientable surfaces decompose into convex pieces and π-Möbius bands.

We say that S is the *sum* of connected subsurfaces $S_1, \dots, S_n$ if S is the union of $S_1, \dots, S_n$ and $S_i \cap S_j$ is the union of imbedded closed geodesics disjoint from one another or the empty set whenever $i \neq j$, $i, j = 1, \dots, n$ (compare with Section 3.1 in [5]). If S is the sum of $S_1, \dots, S_n$, then we say that S *decomposes* into $S_1, \dots, S_n$ (*along closed geodesics*) and that $\{S_1, \dots, S_n\}$ is a *decomposition collection* of S. (See [3], [5], and [6].)

A *hyperbolic automorphism* of $\mathbf{S}^2$ is one induced by a linear map with three distinct positive eigenvalues. A *quasi-hyperbolic automorphism* of $\mathbf{S}^2$ is one induced by a linear map with two distinct positive eigenvalues and a nondiagnalizable matrix. Let D be a compact simply convex domain in $\mathbf{S}^2$ whose boundary is the disjoint union of a segment α and a compact smooth open arc β with two common endpoints p and q. The quotient surface of $D - \{p, q\}$ by a properly discontinuous and free action of $< \vartheta >$ where ϑ is a hyperbolic or quasi-hyperbolic projective automorphism of $\mathbf{S}^2$ has an induced real projective structure.

A real projective surface projectively homeomorphic to such a surface is called a *primitive trivial annulus*. It is homeomorphic to a compact annulus. One of its boundary components is not geodesic, and the other is geodesic. A *trivial annulus* in S is a primitive trivial annulus A imbedded in S such that a component of δS equals the nongeodesic component of δA. (We will give some examples in Section 1).

For a surface S of negative Euler characteristic, a simple closed curve α in S^o is freely homotopic to a multiple of a component curve of δS if and only if the union of α and a component β of δS forms the boundary of an annulus in S. (To see this use the model of $\tilde{S}$ in the hyperbolic plane.) Hence, α is freely homotopic to β or β^{-1} for a boundary component curve β but to no other multiple of β.

A *purely convex surface* is a convex compact surface A with negative Euler characteristic that does not include a compact annulus with geodesic boundary components freely homotopic to a component of δA or include a trivial annulus with respect to A. (See Appendix A of [3].) An elementary annulus is defined in Section 1. A *boundary* elementary annulus in S is an elementary annulus including a boundary component of S.

By Proposition 4.5 of [3] gives us:

PROPOSITION 1. *A convex compact orientable surface S of negative Euler characteristic is a sum of purely convex surface and boundary elementary annuli and trivial annuli. Furthermore, S is purely convex if and only if S does not include any boundary elementary annuli or trivial annuli.*

One can obtain a convex but not purely convex surface by attaching an elementary annulus to a purely convex surface (see Section 4.4 of [2]).

A *maximal purely convex surface* in S is a purely convex surface in S that is not included properly in a purely convex surface in S.

A *maximal annulus* in S is a compact annulus with geodesic boundary that is not included properly in a compact annulus or a compact Möbius band with geodesic boundary in S. A *maximal Möbius band* in S is a compact Möbius band with geodesic boundary that is not included properly in a compact Möbius band with geodesic boundary in S.

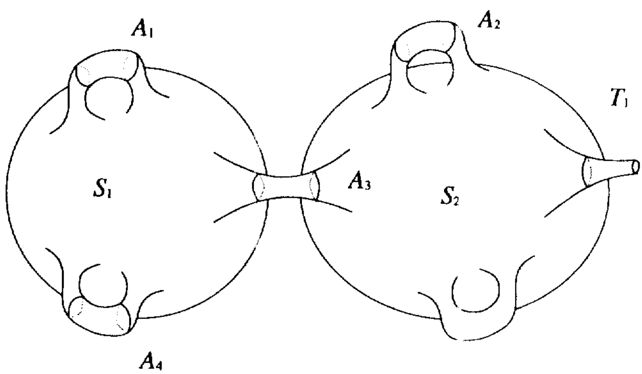

FIGURE 1. A picture of an admissible decomposition: purely convex surfaces S_1, S_2, a trivial annulus T_1, and maximal annuli A_i, $i = 1, 2, 3, 4$.

We say that the above three categories of subsurfaces of S, i.e., trivial annuli; maximal annuli or Möbius bands; and maximal purely convex surfaces in S are *admissible subsurfaces* in S. The three categories of surfaces are considered to have mutually distinct *types*. If S decomposes into admissible subsurfaces of S, then the decomposition collection is *admissible*.

We will prove the nonorientable case of the Admissible decomposition theorem of [3]:

THEOREM 1. *Let Σ be a nonorientable compact real projective surface with convex or empty boundary and negative Euler characteristic. Then Σ admits a unique admissible decomposition.*

For a surface with boundary, the admissible decomposition does not necessarily mean that we can further decompose it using Goldman's annulus decomposition theorem [3, Appendix B] to convex pieces since a maximal annulus whose fundamental group has a generator of quasi-hyperbolic holonomy does decompose into elementary annuli of type I, convex ones. One can easily construct a purely convex surface with a boundary component curve whose holonomy is quasi-hyperbolic as noted in Section 4.4 of [3]. An elementary annulus of type IIa is not convex (see Section 1). By attaching an elementary annulus of type IIa to a purely convex surface with a boundary component curve whose holonomy is quasi-hyperbolic, we can obtain admissible decomposition which does not further yield a decomposition into convex pieces. For a closed surface, we will show that elementary annuli of type II do not occur because of strange behavior of a closed geodesic curve whose holonomy is quasi-hyperbolic.

A *π-Möbius band* is a real projective Möbius band projectively homeomorphic to the quotient surface of a compact annulus that is a sum of two elementary annuli of type I by the action of the group generated by an order-two projective automorphism. We give a construction of π-Möbius bands at the end of the paper.

THEOREM 2. *Let Σ be a closed orientable or nonorientable real projective surface with negative Euler characteristic. Then Σ uniquely decomposes into purely convex subsurfaces, elementary annuli of type I* (which is convex), *and π-Möbius bands such that no two purely convex subsurfaces are adjacent.*

In particular, if Σ is orientable, then Σ decomposes into convex subsurfaces, since elementary annuli of type I are convex. This implies the Convex decomposition theorem, which we mentioned in [4]. These theorems answer the question of Thurston and Goldman raised around 1977 for closed or nonorientable surfaces.

In Section 1, we give examples of elementary annuli and trivial annuli. In Section 2, we prove that an admissible decomposition of a surface induces one on its covering and vice versa. This implies Theorem 1 easily. To prove this property, we discuss the properties of tight curves, show that purely convex surfaces, trivial annuli, and boundary elementary annuli are preserved under the action of the covering maps, whether pushing it to the quotient surface or pulling it back by taking the components of the inverse images under the covering map. Finally, we show that a decomposition into purely convex surfaces, trivial annuli, and maximal annuli or Möbius bands is admissible if and only if the adjacent surfaces are of different types. This will complete the proof of this property. In Section 3, we prove Theorem 2. Since any closed real projective surface has an admissible decomposition, we further decompose maximal annuli and Möbius bands into elementary annuli or π-Möbius bands by Goldman's annulus decomposition theorem. We show that elementary annuli of type II does not occur since the surface is closed. This follows since the boundary components of an elementary annulus of type II have quasi-hyperbolic holonomy. In Section 4, we classify π-Möbius bands.

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1. Examples

For convenience of the reader, we will repeat some examples given in [3]. Given a point x of $\mathbf{S}^2$ we let $-x$ denote its antipodal point. Let ϑ be a hyperbolic automorphism. It has three fixed points s, m, and w corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respectively where $\lambda_1 > \lambda_2 > \lambda_3 > 0$. Their antipodal points $-s, -m$, and $-w$ are also fixed points. The great circles including two pairs of antipodal fixed points are ϑ-invariant and each component arc of the circle removed with fixed points are ϑ-invariant lines. (A *line* in a real projective surface is an imbedded geodesic defined on an interval.) Let l_1 be the great circle containing $m, -m, w$, and $-w$; l_2 that containing $s, -s, w$, and $-w$; and l_3 that containing $s, -s, m$, and $-m$. $\mathbf{S}^2$ removed with these circles are simply convex triangles where ϑ acts on. s and $-s$ are attracting fixed points of the action of $< \vartheta >$, and the corresponding attracting basins are the open hemispheres that are components of $\mathbf{S}^2 - l_1$ containing s and $-s$ respectively. $w, -w$ are repelling fixed points and $m, -m$ fixed points of saddle type. The closure of any invariant open triangle has three fixed points from each of the pairs $\{s, -s\}$, $\{m, -m\}$, and $\{w, -w\}$.

For points y and z in $\mathbf{S}^1$ that are not antipodal, by $\overline{yz}$ we mean a unique minor geodesic segment connecting y and z. An oriented arc is *curved in one direction* if a convex neighborhood of each of its point has two components such that the one in the given direction is convex. An unoriented arc is *curved in one direction* if it is so when given an orientation. Given a point x in the open triangle, $\bigcup_{n\in\mathbf{Z}} \overline{\vartheta^n(x)\vartheta^{n+1}(x)}$ is an arc connecting the attracting fixed point and the repelling fixed point in the closure of the triangle. (See Figure 2.) The arc, say α, is curved in one-direction since one can see that the curve $\overline{x\vartheta(x)} \cup \overline{\vartheta(x)\vartheta^2(x)}$ is curved in one-direction, α is the union of the images of the curve under ϑ^n, and ϑ^n preserves the orientation of $\mathbf{S}^2$ and that of the curve.

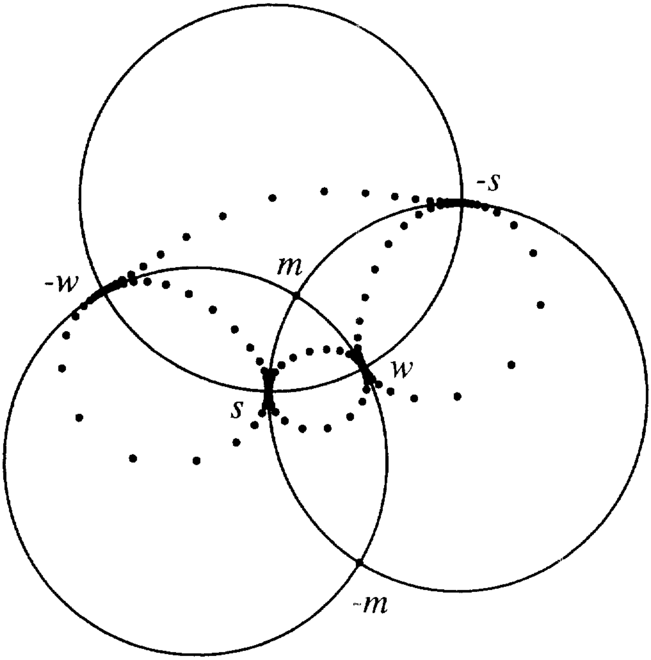

FIGURE 2. The action of a hyperbolic automorphism stereographically projected.

One way to understand Figure 2 is to realize ϑ as generated by a Lie algebra element corresponding to a vector field in $\mathbf{S}^2$. The vector field is zero at $s, -s, m, -m, w$, and $-w$, and the complement of the zero set is foliated by flow lines. The components of l_i removed with fixed points are flow lines. At s, each flow line in the open ϑ-invariant triangles is tangent to the lines $\overline{sm}$ or $\overline{s-m}$, at w to $\overline{wm}$ or $\overline{w-m}$, and m is a saddle type singularity. Similar statements hold at $-s, -w$, and $-m$. Thus, the orbit usually starts off tangent to a ϑ-invariant line.

Let φ be a quasi-hyperbolic automorphism. Then the linear map φ' corresponding to φ has two eigenvalues one of which corresponds to a two-dimensional subspace P where φ' is represented by the matrix

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

φ has four fixed points $s', -s', w'$, and $-w'$. Let l_1' be the great circle containing the four fixed points. Then exactly one of $\{s', -s'\}$ or $\{w', -w'\}$ is a subset of a φ-invariant great circle l_2' corresponding to P, the other is disjoint from l_2', and φ acts as an affine translation on l_2' removed with the fixed points of l_2'. (Since components l_2' removed with the fixed points are open 1-dimensional hemispheres, they have the natural affine structure of an affine 1-space.)

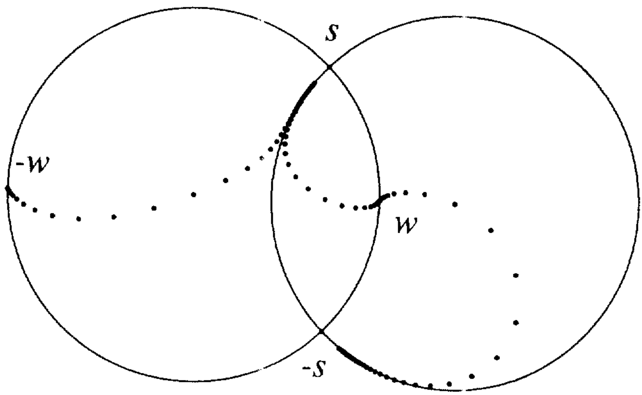

FIGURE 3 The action of a quasi-hyperbolic automorphism stereographically projected.

Assume without loss of generality that $s', -s' \in l_2'$. Each of the six components of $l_1' \cup l_2'$ removed with fixed points is a φ-invariant open line. Each of the four components of $\mathbf{S}^2 - l_1' - l_2'$ is a φ-invariant open lune. The closure of the open lune contains $s', -s'$ as vertices and exactly one of $w', -w'$ in the boundary. Suppose that the eigenvalue corresponding to w' is less than that to s'. Let B be the closure of one of the lunes above. Suppose that B contains w' in the boundary and the action of φ on the interior of $l_2' \cap B$ is a translation toward s'. Then for each point x in B^o, $\bigcup_{n\in\mathbf{Z}} \overline{\varphi^n(x)\varphi^{n+1}(x)}$ is an arc connecting

s' and w' curved in one-direction. This holds by a reason similar to the hyperbolic case. For any other choice of the lune B, the action on the interior of $l'_2 \cap B$, and the condition on the eigenvalues of s' and w', the corresponding statements hold (See Figure 3)

To understand Figure 3, we see φ as generated by a Lie algebra element corresponding to a vector field on $\mathbf{S}^2$. The zero set of the vector field is $\{s, -s, w, -w\}$. Components of l'_i removed with the fixed points are flow lines. At s, each flow line in an open φ-invariant lune is tangent to l'_2. At w, each such flow line is tangent to $\overline{-sw}$. Similar statements hold at $-s$ and $-w$. Hence, we can see the behavior of the orbits in Figure 3 (similar to the arcs of the yin-yang symbol).

Let T be the closure of a ϑ-invariant open triangle above. We may assume without loss of generality that s, w, and m are on the boundary of T. We obtain two elementary annuli:

$$T^o \cup \overline{ws}^o \cup \overline{ms}^o / < \vartheta >, T^o \cup \overline{ws}^o \cup \overline{wm}^o / < \vartheta > .$$

(The other choices of edges will give us non-Hausdorff quotient spaces.) The real projective surfaces projectively homeomorphic to the above annuli are said to be *elementary annuli of type I.* (Note ϑ may vary.)

Let B be the closure of one of the invariant lunes of φ above. We may assume without loss generality that w' is on the boundary of B and φ acts on the interior α of $l'_2 \cap B$ as a translation toward s'. Then there are two elementary annuli:

$$B^o \cup \overline{s'w'}^o \cup \overline{-s'w'}^o / < \varphi >, B^o \cup \overline{s'w'}^o \cup \alpha / < \varphi > .$$

(As above, the other choice of the edges will result in non-Hausdorff spaces.) The real projective surfaces projectively homeomorphic to the above annuli are said to be *elementary annuli of type IIa* and *IIb* respectively. (Note φ may vary.)

One can easily see that the above four are all the compact elementary annuli one can construct that include the open annuli obtained from T^o or B^o by the actions of $< \vartheta >$ or $< \varphi >$ respectively.

We can give examples of primitive trivial annuli. In T above, let α be the curve $\bigcup_{n \in \mathbf{Z}} \overline{\vartheta^n(x)\vartheta^{n+1}(x)}$. Then α and $\overline{ws}$ bound a convex ϑ-invariant open set K, and $K \cup \alpha \cup \overline{ws}^o / < \vartheta >$ is a primitive trivial annulus. We may choose α to be any ϑ-invariant curve in T^o connecting w and s curved in one direction and construct a primitive trivial annulus.

In B above, let α be the curve $\bigcup_{n\in\mathbf{Z}} \overline{\varphi^n(x)\varphi^{n+1}(x)}$. Then α and $\overline{w's'}$ bound a convex φ-invariant open set K, and $K \cup \alpha \cup \overline{w's'}^o / < \varphi >$ is a primitive trivial annulus. Again, we may choose α to be any ϑ-invariant curve in B^o connecting w and s curved in one direction.

Any trivial annulus T is projectively homeomorphic to one constructed as above: Let D be a compact simply convex domain in $\mathbf{S}^2$ whose boundary is the disjoint union of a segment β and a compact smooth open arc γ with two common endpoints p and q. Suppose that T is projectively homeomorphic to $D - \{p, q\} / < \vartheta >$ where ϑ is a hyperbolic automorphism.

Since $D - \{p, q\}$ covers a compact annulus, there is no fixed point of ϑ in $D - \{p, q\}$. If D meets a ϑ-invariant open line other than β^o, then since D is ϑ-invariant, D contains the endpoints of the open line. Since the endpoints are fixed points, they must be p and q. This means that the line is β^o. This is absurd. Thus $D - \overline{pq}$ meets no ϑ-invariant line and $D - \overline{pq}$ is included in one of the ϑ-invariant open triangle. Hence, D is bounded by a ϑ-invariant arc in the open triangle curved in one direction and an edge of the triangle, and T is projectively homeomorphic to one constructed above. If ϑ is quasi-hyperbolic, a similar argument will show the corresponding result.

We will need the following lemma.

LEMMA 1. *Let E be an elementary annulus. Then $\mathbf{dev}|\tilde{E}$ is an imbedding onto its image. Depending on whether E is an elementary annulus of type I, IIa, or IIb, $\mathbf{dev}(\tilde{E})$ equals one of the following sets respectively:*

$$\triangle^o \cup \overline{v_1v_2}^o \cup \overline{v_2v_3}^o, B^o \cup \alpha^o - \{x\}, B^o \cup \alpha^o \cup \beta,$$

where $\triangle$ is a triangle with vertices v_1, v_2, and v_3, B a lune, α a segment of $\mathbf{d}$-length π in the boundary of B, x a point of α^o, and β an open line of $\mathbf{d}$-length $< \pi$ also in the boundary of B disjoint from α but sharing an endpoint with α.

If E is a trivial annulus, then $\mathbf{dev}|\tilde{E}$ is an imbedding onto $D - \{p, q\}$ where D is a simply convex domain in $\mathbf{S}^2$ bounded by a geodesic and an arc sharing endpoints p and q.

Proof. Suppose that E is an elementary annulus of type I. E is projectively homeomorphic to a projective quotient surface of a convex domain Ω in a closed triangle in $\mathbf{S}^2$. Thus $\tilde{E}$ is projectively homeomorphic to Ω and E is convex. By the proof of Lemma 1.5 of [2], we obtain that $\mathbf{dev}|\tilde{E}$ is an imbedding onto its image. Since $\mathbf{dev}(\tilde{E})$ and Ω are projectively homeomorphic and any nonsingular projective map defined on an open domain of $\mathbf{S}^2$ extends to a global projective automorphism of $\mathbf{S}^2$, we obtain by the description of Ω in this section that

$$\mathbf{dev}(\tilde{E}) = \triangle^o \cup \overline{v_1 v_2}^o \cup \overline{v_2 v_3}^o,$$

where $\triangle$ is a triangle with vertices v_1, v_2, and v_3. The rest of the lemma follows similarly. □

2. Admissible Decomposition

Let S and S_f be compact projective surfaces with convex or empty boundary and negative Euler characteristic. Let $f : S_f \to S$ be a projective finite covering map. Suppose that S is the sum of $S_1, \dots, S_n$ and S_f the sum of $S_{f,1}, \dots, S_{f,m}$. We say that $\{S_{f,1}, \dots, S_{f,m}\}$ is a *decomposition collection pulled from* $\{S_1, \dots, S_n\}$ if $S_{f,i}$ for each i, $i = 1, \dots, m$, is a component of $f^{-1}(S_j)$ for some j, $j = 1, \dots, n$.

It is obvious that given each decomposition collection of S, there exists a unique decomposition collection of S_f pulled from it. Conversely, given a decomposition collection of S_f, there can be at most one decomposition of S from which the decomposition collection of S_f is pulled.

THEOREM 3. (1) *If a decomposition collection* $\{S_1, \dots, S_n\}$ *of* S *is admissible, then the decomposition collection of* S_f *pulled from it is admissible.*

(2) *If a decomposition collection* $\{S_{f,1}, \dots, S_{f,m}\}$ *of* S_f *is admissible, then there exists an admissible decomposition collection of* S *from which* $\{S_{f,1}, \dots, S_{f,m}\}$ *is pulled.*

Proof. It is sufficient to prove this theorem in case S_f is orientable: Assume that the above theorem is true if S_f is orientable. Now suppose that S_f is not orientable. Then there is an orientable double cover (S_f^d, p) of S_f.

Let us prove (1). Let $\mathcal{I}$ be an admissible decomposition collection of S. There exists a decomposition collection $\mathcal{J}$ on S_f pulled from $\mathcal{I}$ by f, and a decomposition collection $\mathcal{K}$ of S_f^d pulled from $\mathcal{J}$ by p. It is obvious that $\mathcal{K}$ is pulled from $\mathcal{I}$ by $f \circ p$. By (1) in the orientable case, $\mathcal{K}$ is admissible. (2) in the orientable case shows that there exists an admissible decomposition collection $\mathcal{J}'$ of S_f from which $\mathcal{K}$ is pulled. Clearly, $\mathcal{J} = \mathcal{J}'$, and (1) is proved.

Let us prove (2). Let $\mathcal{J}$ be an admissible decomposition collection of S_f. Then the decomposition collection $\mathcal{K}$ of S_f^d induced by p is also admissible by (1) in the orientable case. By (2) in the orientable case, there exists an admissible decomposition collection $\mathcal{I}$ of S from which $\mathcal{K}$ is pulled by $f \circ p$. Let $\mathcal{J}'$ be the decomposition of S_f pulled from $\mathcal{I}$. Then $\mathcal{K}$ is pulled from $\mathcal{J}'$. Hence, $\mathcal{J}' = \mathcal{J}$, and $\mathcal{J}$ is pulled from $\mathcal{I}$ by f. □

We are left with proving the theorem in the case when S_f is orientable.

Let M be an (orientable or nonorientable) real projective surface with convex or empty boundary. Recall that a geodesic in M always maps into M^o or else into δM as a covering of a component of δM up to parametrizations since δM is convex (see Lemma 3.4 of [2] or Section 2.1 of [3]).

A *geodesic complex* K in M is a compact subset with the following property: for each point p of K, the surface M includes an open neighborhood $\mathcal{U}$ of p such that

$$\mathcal{U} \cap K = \bigcup_{i=1}^{n} l_i$$

holds where each $l_i, i = 1, \ldots, n$, is a maximal line in $\mathcal{U}$ passing through p. Recall that an image of a closed geodesic is a geodesic complex. (See [3].)

Let K be a geodesic complex. A *regular point* of K is a point of K with a neighborhood in K that is a line, a *vertex* of K is a point of K that is not regular, and a *regular arc* of K is a component arc of K removed the set of vertices. Regular arcs of K are imbedded geodesics.

LEMMA 2. *Let K be a geodesic complex in M^o, and p a vertex of K. Then there exists a closed geodesic into K passing through p in the direction of each regular arc ending at p.*

Proof. See Section 3.6 in [3]. □

Recall that a geodesic in $\tilde{M}$ always imbeds to a one-dimensional manifold by Section 4.8 of [2] since $\chi(M) \leq 0$. A *convex* line in $\tilde{M}$ is an imbedded line l such that $\mathbf{dev}|l$ is an imbedding onto a convex line in $\mathbf{S}^2$.

A closed geodesic $\alpha : \mathbf{S}^1 \to M$, where $\mathbf{S}^1$ is a circle, is called a *tight curve* in M if its lift to $\tilde{M}$ is a geodesic imbedding onto a convex open line (see Section 2 of [3]). For example, a closed geodesic in a convex real projective surface of negative Euler characteristic is a tight curve (see Lemma 1.5 of [2]). The boundary components of elementary annuli are tight curves by Lemma 1; so is the geodesic boundary component of a trivial annulus. Basic properties of tight curves are described in Proposition 2.2 of [3]. They are very similar to those of closed geodesics in hyperbolic surfaces.

LEMMA 3. *Let α be a tight curve and β a closed geodesic with same image as α. Then β is a tight curve. Moreover, if $g : M \to M'$ is a covering map, and $g \circ \gamma$ for a closed curve γ is a tight curve, then so is γ.*

Proof. Since a geodesic returning to its starting point tangent to its initial vector must be an infinite cyclic cover of a closed curve up to parametrizations, α equals $\beta' \circ c$ up to parametrizations where c is a finite covering map $\mathbf{S}^1 \to \mathbf{S}^1$ and β' is a tight curve injective except at finitely many points. Since β must equal $\beta' \circ c'$ for a finite covering map $\mathbf{S}^1 \to \mathbf{S}^1$ by same reason, β is a tight curve also. The last part follows from the first. □

Let M' be a real projective surface with a projective covering map $g : M' \to M$. Let $\alpha \subset M'^o$ be an imbedded tight curve. Then $g(\alpha)$ is a geodesic complex in M^o. Moreover, $g^{-1}(g(\alpha))$ is a geodesic complex in M'^o. Suppose that $g|\alpha$ is not a covering map onto its image. Then some points of α are vertices of the complex $g^{-1}(g(\alpha))$.

LEMMA 4. *Under above assumptions, there exists a tight curve in M^o intersecting essentially with α in homotopy.*

Proof. By Lemma 2 there exists a closed geodesic β in $g^{-1}(g(\alpha))$ passing p transversally with respect to α at p. Since $g \circ \beta$ is also a closed curve mapping into $g(\alpha)$, Lemma 3 shows that $g \circ \beta$ is a tight curve. Hence, β is a tight curve.

We claim that β intersects α essentially. We obtain a finite cover M'' of M' with covering map $p : M'' \to M'$ so that β lifts to a simple tight curve β'' (see Proposition 2.2 of [3], a result following from Scott [8] and [9]). M'' includes an imbedded tight curve α'' corresponding to α under p that intersects with β''.

If β intersects trivially with α in homotopy, then there is a homotopy $\{\beta_t\}$, $0 \leq t \leq 1$, so that $\beta_0 = \beta$ and β_1 does not meet α. The lift $\{\beta''_t\}$ of $\{\beta_t\}$ to M'' with $\beta''_0 = \beta''$ has the property that the lift β''_1 does not meet α''. That is, β'' intersects α'' trivially in homotopy. By Proposition 2.2 (4) of [3], α'' equals the image curve of β''. Hence, α equals the image curve of β. This is a contradiction. □

We will now begin to show Proposition 2, which says that the covering maps preserve the types of surfaces.

LEMMA 5. *Let M be a convex surface. Then a covering of M is convex, and so is a surface covered by M.*

Proof. We defined a convex surface to be a projective surface whose universal cover is convex. (See also Section 1.5 of [2].)

A *boundary elementary annulus* in S is an elementary annulus (see [3]) including a component of δS.

PROPOSITION 2. *Let A and A_f be subsurfaces of S and S_f respectively such that $f|A_f$ is a finite covering map onto A. Then the following statements hold :*

(1) *A_f is a trivial annulus in S_f if and only if A is a trivial annulus in S.*
(2) *A_f is a boundary elementary annulus in S_f if and only if A is a boundary elementary annulus in S.*
(3) *A_f is a purely convex surface if and only if A is a purely convex surface.*

Proof. (1) If A_f is a trivial annulus in S_f, then by Lemma 2.5 of [3], one of its boundary components is the unique imbedded tight curve in A_f, and the other component is a component of δS_f. Hence, A is homeomorphic to an annulus, and one of its boundary components is a component of δS. The other boundary component c is an imbedded tight curve since it is covered by the boundary component of A_f, which is a tight curve. c must be the unique imbedded tight curve since A_f would have more than one imbedded tight curves by taking inverse images under f otherwise. Hence, A is a trivial annulus by Lemma 2.5 of [3].

Conversely, if A is a trivial annulus in S, then one of its boundary component c is a unique imbedded tight curve and the other component is a component of δS by Lemma 2.5 of [3]. Hence, A_f is homeomorphic to an annulus, and one of its boundary component c_f is an imbedded tight curve and the other component is a component of δS_f. Let γ be a tight curve to A. Then γ is freely homotopic to a finite covering of c since $\pi_1(A) = \mathbf{Z}$. By Proposition 2.2 of [3], γ maps into a tight curve, which must be c since A is a trivial annulus. Since all tight curves in A map into c, it follows that given any imbedded tight curve γ' in A_f, $f(\gamma') = c$, and, hence, c_f is the unique imbedded tight curve in A_f. Hence, A_f is a trivial annulus in S_f by Lemma 2.5 of [3].

(2) This is proved similarly to (1).

(3) Suppose that A_f is a purely convex subsurface of S_f. Then A is convex by Lemma 5 and has negative Euler characteristic. If A is not purely convex, then A includes E, a trivial annulus or an annulus with geodesic boundary, whose components are freely homotopic to a boundary component of A. By (1), a component of the inverse image of E in A_f is a trivial annulus or an annulus with geodesic boundary, whose components are freely homotopic to a component of δA_f. This is a contradiction, and A is purely convex.

Suppose that A is purely convex. Then A_f is convex by Lemma 5 and has negative Euler characteristic. If A_f is not purely convex, then A_f includes E, a trivial annulus or a boundary elementary annulus by Proposition 1.

A component c of δE is an imbedded tight curve in S^o_f. The inverse image $f^{-1}(f(c))$ is a geodesic complex in S^o_f (see above). Since c is freely homotopic to a component of δS_f, closed curves intersect trivially with c in homotopy. By Lemma 4, there is no vertex of $f^{-1}(f(c))$ in c, and $f|c$ is a covering map onto an imbedded tight curve c_d in S^o.

Since $f(c)$ is an imbedded tight curve, $f^{-1}(f(c))$ consists of components that are disjoint imbedded tight curves. By Lemma 2.5 of [3], E^o includes no component of $f^{-1}(f(c))$ since E is a trivial annulus or an elementary annulus. Hence, int$E$ is disjoint from $f^{-1}(f(c))$ and is a component of $S_f - f^{-1}(f(c))$.

Let d be the other boundary component of E. Then $f|d$ is a covering map onto a boundary component d_d of S. Let S_d be the component of $S - c_d$ including d_d. Clearly, $f|\text{int}E$ is a covering map onto S_d, and S_d is homeomorphic to an annulus. If c_d is not separating, then since S is the union of S_d and c_d, S is homeomorphic to a Möbius band, which is a contradiction. By Lemma 6, $S_d \cup c_d$ is a compact subsurface of S with a boundary component c_d, and $f|E$ is a covering map onto $S_d \cup c_d$. Hence, $S_d \cup c_d$ is an annulus.

By (1) and (2), $S_d \cup c_d$ is a trivial annulus or a boundary elementary annulus with respect to A. This contradicts the assumption that A is purely convex. Hence, A_f is purely convex.

LEMMA 6. *Let I be the union of disjoint simple closed curves in S^o. Suppose that there is no closed curve intersecting I at a point transversally and intersecting I at no other point. Let T be the closure of a component of $S - I$, and T_f that of a component of $S_f - f^{-1}(I)$. Then*

(1) *T is a surface whose boundary is the union of components of I and δS intersecting T.*
(2) *T_f is a surface whose boundary is the union of components of $f^{-1}(I)$ and δS_f intersecting T_f.*
(3) *If a point of intT_f maps into int$T$, then $f|T_f$ is a covering map onto T.*

Proof. Straightforward.

Now we show that decomposition of surfaces are admissible if and only if the types of adjacent surfaces are different. Let M be a real projective surface with convex or empty boundary.

PROPOSITION 3. *Suppose that two admissible subsurfaces in M (orientable or nonorientable) are adjacent. Then their types are different from each other.*

Proof. As before, it follows from the following Lemma 7, a nonorientable version of Lemma 5.4 of [3]. □

LEMMA 7. *Let $M_1, \ldots, M_n$ be purely convex subsurfaces of M. If $\{M_1, \ldots, M_n\}$ is a decomposition collection of a connected subsurface M', then M' is a purely convex subsurface of M.*

Proof. We may assume that M is not orientable by Lemma 5.4 of [3]. Let M^d be the orientable double cover of M with double covering map p. Then let M'' be a component of $p^{-1}(M')$, where $p|M''$ is a covering map onto M'. Now, M'' admits a pulled decomposition into subsurfaces $M''_1, \ldots, M''_m$. Since M''_i for each i covers M_j for some j, M''_i is purely convex for every i, $i = 1, \ldots, m$ by Proposition 2. By Lemma 5.4 of [3], M'' is purely convex. By Proposition 2, M' is purely convex. □

COROLLARY 1. *Suppose that M admits an admissible decomposition. If an imbedded tight curve α is a boundary component of a surface in the admissible decomposition collection, and α is not freely homotopic to a boundary component of M, then α is a common boundary component of a maximal annulus or Möbius band A and a purely convex surface B.*

Proof. A curve in a trivial annulus is always freely homotopic to a curve in a boundary component of M. □

A converse of Proposition 3 is proved.

PROPOSITION 4. *Let $\{S'_1, \ldots, S'_m\}$ be a decomposition of S into trivial annuli, annuli or Möbius bands with geodesic boundary, and purely convex surfaces. Suppose that the types of adjacent surfaces are distinct. Then it is an admissible decomposition.*

Proof. We only need to show that the annuli or Möbius bands and purely convex surfaces are maximal. First, we assume that S is orientable. Suppose that S'_i be an annulus with geodesic boundary and that A is a compact annulus with geodesic boundary including S'_i properly. Then there exists an adjacent surface S'_j of S'_i, which could be a trivial annulus or a purely convex surface, whose boundary component is included in A^o. This is a contradiction by Corollary 4.6 of [3]. Hence, S'_i is maximal. If S'_i is a purely convex surface, then a similar argument shows that S'_i is maximal.

Suppose that S is not orientable and S_i' an annulus with geodesic boundary. If S_i' is not maximal, then S_i' is included properly in a compact annulus or Möbius band A with geodesic boundary. Let S'' be the double cover of S and $\{S_1'', \dots, S_n''\}$ for some n be the induced decomposition of S''. Then there exists S_j'' covering S_i' and a component A', a compact annulus with geodesic boundary, of $p^{-1}(A)$ including S_j'' properly. This leads to contradiction as in the above paragraph by Corollary 4.6 of [3]. When S_i' is a Möbius band or purely convex surface, a similar reasoning shows that S_i' are maximal. □

We give the proof of Theorem 3.

(1) Let each $S_{f,i}$ be a component of $f^{-1}(S_j)$ for some j. Hence, by Proposition 2, each $S_{f,i}$ is a trivial annulus, an annulus with geodesic boundary, or a purely convex surface. Since the types of adjacent surfaces in $\{S_1, \dots, S_n\}$ are different, the types of adjacent surfaces in $\{S_{f,1}, \dots, S_{f,m}\}$ are different. By Proposition 4, the admissibility is proved.

(2) We have an admissible decomposition collection $\{S_{f,1}, \dots, S_{f,m}\}$ of S_f. There is the collection of imbedded tight curves that are components of $S_{f,i} \cap S_{f,j}$ for $i, j = 1, \dots, m$, $i \neq j$. Let us denote this collection by $\mathcal{I}_f$. This collection is composed of imbedded tight curves in S_f^o disjoint from one another. Let

$$I_f = \bigcup_{\alpha \in \mathcal{I}_f} \alpha.$$

Naturally, $\mathrm{int} S_{f,i}$ for each i is a component of $S_f - I_f$ and that the closure of each component of $S_f - I_f$ is $S_{f,j}$ for some j.

We prove (2) by the following three steps:

(i) We claim that for each tight curve α in $\mathcal{I}_f$, the map $f|\alpha$ is a covering map onto an imbedded tight curve in S^o. It is clear that $f(\alpha) \subset S^o$. Suppose that $f|\alpha$ is not a covering map onto its image. By Lemma 4, α intersects essentially with a tight curve β, and hence, α is not freely homotopic to a component of δS_f. By Corollary 1, α is a component of δA for a maximal annulus A in the decomposition collection of S_f. Since each boundary component of A is not freely homotopic to a component of δS_f, all boundary components of A are boundary components of maximal purely convex surfaces in the decomposition collection of S_f. By Proposition 4.5 (8) and Lemma 5.5 of [3],

A includes a π-annulus B. However, Lemma 3.8 of [3] contradicts the previous claim that α intersects essentially with a tight curve β. Hence $f|\alpha$ is a covering map onto an imbedded tight curve in S^o. Similarly, Lemma 3.8 of [3] shows that given two distinct elements α and β of $\mathcal{I}_f$, the imbedded tight curves $f(\alpha)$ and $f(\beta)$ in S^o are either identical or disjoint.

(ii) Let $\mathcal{I}$ be the collection of mutually disjoint imbedded tight curves in S^o that are images of elements of $\mathcal{I}_f$ under f. Let

$$I = \bigcup_{\alpha \in \mathcal{I}} \alpha.$$

Clearly, $I_f \subset I_f^u = f^{-1}(I)$. We let $\mathcal{I}_f^u$ denote the collection of imbedded tight curves that are the components of I_f^u. (Each component is not null-homotopic.)

We claim that S includes no closed curve intersecting I at a point transversally and intersecting I at no other point. Suppose not. Let α be an element of $\mathcal{I}$ such that an imbedded closed curve β in S^o intersects α at a point transversally and intersects I at no other point. A component S^α of $S - I$ includes $\beta - \alpha$. Let α_f be an element of $\mathcal{I}_f$ mapping to α, and $S_{f,i}$ and $S_{f,j}$ the subsurfaces in the admissible decomposition collection of S_f with a common boundary component α_f. Then every finite-covering curve γ of α intersects with β essentially in homotopy: We can obtain a neighborhood U of $\alpha \cup \beta$ homeomorphic to a punctured torus or a Klein bottle. If U is homeomorphic to a punctured torus, then under the orientable double covering S' of S, it follows that γ and α lift to a punctured torus in S' corresponding to U under the covering map. But in S', the closed curves respectively homotopic to the lifts of γ and β must always meet by the oriented intersection theory. By the homotopy lifting property, γ and β intersect essentially. If U is homeomorphic to a Klein bottle, then under the same covering U is covered by an orientable surface U' in S', and γ, α, and β^2, a double covering curve of β, lift to U'. Then a similar argument shows that γ and β^2 intersect essentially. Thus, no conjugate of the homotopy class of γ equals a multiple of a homotopy class of any component curve of δS. Hence, no conjugate of the homotopy class of the curve corresponding to $p|\alpha_f$ equals a multiple of a homotopy class of any component curve of δS. Since $p_* : \pi_1(S_f) \to \pi_1(S)$ is injective, α_f is not freely homotopic to a component of δS_f.

By Corollary 1, we may assume without loss of generality that $S_{f,i}$ is a purely convex surface and $S_{f,j}$ a maximal annulus. Let C_i be the component of $\mathrm{int} S_{f,i} - I_f^u$ intersecting a one-sided neighborhood of α, and C_j that of $\mathrm{int} S_{f,j} - I_f^u$. Then C_i is an open surface of negative Euler characteristic. Otherwise, we would have two freely homotopic tight curves in the purely convex surface $S_{f,i}$, which is a contradiction by Lemma 4.5 of [3]. However, C_j is an annulus. Since C_i and C_j are components of $S_f - I_f^u$, $f|C_i$ and $f|C_j$ are covering maps of $\mathrm{int} S^\alpha$. This is a contradiction by the Euler characteristic consideration.

(iii) Lemma 6 and (ii) imply that the closures of components of $S - I$ form a decomposition collection. Let us denote it by $\{S_1'', \dots, S_n''\}$. Similarly, the closures of components of $S_f - I_f^u$ form a decomposition collection of purely convex surfaces, compact annuli or Möbius bands with geodesic boundary, or trivial annuli. Moreover, given each component R of $S_f - I_f^u$ and its closure $\mathrm{Cl}(R)$ of R in S_f, $f|\mathrm{Cl}(R)$ is a covering map onto $\mathrm{Cl}(R')$ for a component R' of $S - I$.

Proposition 2 implies that each S_i'' is either a trivial annulus, an annulus or Möbius band with geodesic boundary, or a purely convex subsurface. We can form a new decomposition collection $\{S_1', \dots, S_m'\}$ by taking unions of adjacent surfaces whenever they have the same type until every pair of adjacent surfaces have different types. By Lemma 7, the union of adjacent surfaces of one type is still a surface of that type. So, each S_i' is either a trivial annulus, an annulus or Möbius band with geodesic boundary, or a purely convex surface, and adjacent surfaces in $\{S_1', \dots, S_m'\}$ are of different types. By Proposition 4, $\{S_1', \dots, S_m'\}$ is admissible.

The pulled decomposition collection of S^f of $\{S_1', \dots, S_m'\}$ is admissible by (1), and hence is identical with $\{S_{f,1}, \dots, S_{f,m}\}$ by the uniqueness part of the Admissible decomposition theorem of [3]. Hence, this completes the proof of (2).

We will now prove Theorem 1 : That is, we show that when Σ is not orientable, Σ has a unique admissible decomposition collection.

We first prove existence. Let Σ^d be the orientable double cover of Σ. The cover Σ^d has an admissible decomposition collection $\{\Sigma_1^d, \dots, \Sigma_n^d\}$ by the Admissible decomposition theorem in [3]. By Theorem 3 (2), Σ

has an admissible decomposition collection $\{\Sigma_1, \ldots, \Sigma_m\}$.

We now prove uniqueness. Let $\{\Sigma'_1, \ldots, \Sigma'_l\}$ be an admissible decomposition collection of Σ. Then by Theorem 3 (1), the decomposition collection $\{\Sigma''_1, \ldots, \Sigma''_o\}$ of Σ^d pulled from it is also admissible and, hence, is identical with $\{\Sigma^d_1, \ldots, \Sigma^d_n\}$ by the uniqueness in orientable case in [3]. Thus $\{\Sigma'_1, \ldots, \Sigma'_l\}$ is identical with $\{\Sigma_1, \ldots, \Sigma_m\}$.

It is interesting to note that we can prove the existence of admissible decomposition not using Theorem 3. The cover Σ^d has an admissible decomposition collection $\{\Sigma^d_1, \ldots, \Sigma^d_n\}$. Since S^d is a regular covering, there is an order-two deck transformation θ acting on S^d. Since the admissible decomposition is unique, θ preserves the decomposition collection. Hence, there exists a decomposition collection $\{\Sigma_1, \ldots, \Sigma_m\}$ of S from which $\{\Sigma^d_1, \ldots, \Sigma^d_n\}$ is pulled. As before by Propositions 2 and 4, it is admissible. The uniqueness follows since the induced decompositions of Σ^d from any admissible decomposition of Σ is admissible. (To prove this, we have to follow the proof of Theorem 3 (1).)

3. Convex Decomposition

A closed curve or an annulus is said to be *hyperbolic* (resp. *quasi-hyperbolic*) if the holonomy of the generator of the fundamental group is hyperbolic (resp. quasi-hyperbolic). A Möbius band is said to be *hyperbolic* (resp. *quasi-hyperbolic*) if the holonomy of the square of a generator is hyperbolic (resp. quasi-hyperbolic).

We will now prove Theorem 2. Since $\delta\Sigma = \emptyset$, the surface Σ is the sum of maximal purely convex surfaces and maximal annuli or Möbius bands. Let us denote by $\mathcal{I}$ the admissible decomposition collection. By Lemma 4.3 in [3], every closed curve in a purely convex surface is hyperbolic or quasi-hyperbolic. Every boundary component curve α of a maximal annulus or Möbius band in $\mathcal{I}$ is hyperbolic or quasi-hyperbolic since α equals a boundary component curve in a purely convex surface in $\mathcal{I}$.

By following Proposition 5, each maximal annulus or Möbius band in $\mathcal{I}$ is hyperbolic since Σ is a closed surface. By Goldman's annulus decomposition theorem in [3], every hyperbolic maximal annulus is the unique sum of elementary annuli of type I.

We claim that a hyperbolic maximal Möbius band is a sum of elementary annuli of type I and one π-Möbius band: Let M be a maximal Möbius band, and A the orientable double cover of M. Then by Goldman's annulus decomposition theorem in [3], A is the sum of elementary annuli of type I. The orientation-reversing covering transformation ϑ of order two acts on A. By the uniqueness of the decomposition, ϑ preserves the decomposition. Because of strong and weak boundary components (see Section 1 and Lemma 3.1 of [3]), an elementary annulus of type I does not admit an orientation-reversing order-two projective self-homeomorphism. Thus, there are even number of elementary annuli of type I such that $< \vartheta >$ acts on the union of the pair of elementary annuli in the middle. Hence, M is the sum of elementary annuli of type I and a π-Möbius band.

The above two paragraphs imply the the existence of the decomposition of S into purely convex surfaces, elementary annuli of type I, and π-Möbius bands. By construction, no two purely convex surfaces in the decomposition collection are adjacent.

Let us call the above decomposition collection $\mathcal{J}$. Let $\mathcal{J}'$ be any decomposition collection of S into purely convex surfaces, elementary annuli of type I, and π-Möbius bands such that no two purely convex surfaces are adjacent. We take unions of adjacent elementary annuli of type I and π-Möbius bands until none of the elementary annuli of type I or π-Möbius bands is left. This gives a new decomposition collection $\mathcal{I}'$ of S into purely convex surfaces and annuli or Möbius bands with geodesic boundary. By Proposition 4, $\mathcal{I}'$ is admissible. Hence, $\mathcal{I}' = \mathcal{I}$. Since the decomposition of an annulus or Möbius band with geodesic boundary into elementary annuli and π-Möbius bands is unique by the annulus decomposition theorem in [3], $\mathcal{J}' = \mathcal{J}$. The uniqueness is proved. □

Recall the definition of elementary annuli of type IIa and IIb in Section 1.

PROPOSITION 5. *Let S be an orientable compact real projective surface with convex or empty boundary and negative Euler characteristic. If A is a quasi-hyperbolic maximal annulus in S, then δS is not empty, and A includes a component of δS. Moreover, A decomposes into elementary annuli of type IIa.*

If S is not orientable, then there is no quasi-hyperbolic maximal

Möbius band in the admissible decomposition collection of S. This follows from taking the orientable double covering of S, the above claim, and the fact that the boundary component of a maximal Möbius band lies in S^o.

To prove our proposition, we need a slightly lengthy discussion introducing an assignment of 1 *or* -1 *for quasi-hyperbolic imbedded tight curves.* This will be based on the observation that two sides of a quasi-hyperbolic imbedded tight curve are not projectively equivalent.

Let α be an arbitrary quasi-hyperbolic imbedded tight curve in S. Then we have either $\alpha \subset S^o$ or $\alpha \subset \delta S$. Let U be a neighborhood of α. We assume that if $\alpha \subset S^o$, then U is homeomorphic to an open annulus and that if $\alpha \subset \delta S$, then U is homeomorphic to a compact disk removed a single interior point. Thus $U - \alpha$ has one or two components. We call the relative closure in U of a component of $U - \alpha$ a *one-sided neighborhood* of α. A one-sided neighborhood is homeomorphic to a compact disk removed a single interior point. Suppose that α is oriented. Depending on which side of α the one-sided neighborhood lies, we call it a *left-sided* or *right-sided* neighborhood of α.

Let α and β be quasi-hyperbolic imbedded tight curves respectively in real projective surfaces S_1 and S_2. Two respective one-sided neighborhoods U and V of α and β are called *projectively equivalent* if there are a one-sided neighborhood U' of α in U and a one-sided neighborhood V' of β in V such that U' and V' are projectively homeomorphic. This is an equivalence relation.

Let ϑ be an arbitrary quasi-hyperbolic projective automorphism of $\mathbf{S}^2$, and $\mathbf{S}^1$ the great circle of $\mathbf{S}^2$ on parts of which ϑ acts as an affine translation (see Section 1). Let w and $-w$ be the fixed points of ϑ not on $\mathbf{S}^1$, and let s and $-s$ be the fixed points on $\mathbf{S}^1$. (w and s do not necessarily indicate fixed points of largest and smallest eigenvalues.) Let H_1 be the open hemisphere of $\mathbf{S}^2 - \mathbf{S}^1$ including $\overline{ws}^o$. Let α_1 and α_2 be the two segments sharing endpoints s and $-s$ such that $\mathrm{bd} H_1 = \alpha_1 \cup \alpha_2$. Let B_1 be the open lune bounded by $\overline{ws}$, $\overline{w{-s}}$, and α_1, and B_2 the open lune bounded by $\overline{ws}$, $\overline{w{-s}}$, and α_2.

LEMMA 8. *Exactly one of $B_1 \cup \overline{ws}^o$ and $B_2 \cup \overline{ws}^o$ includes a ϑ-invariant simply convex one-sided neighborhood of $\overline{ws}^o$. Furthermore, every one-sided neighborhood of $\overline{ws}^o$ in the set includes a ϑ-invariant simply convex one-sided neighborhood of $\overline{ws}^o$.*

Proof. By replacing ϑ by ϑ^{-1} and relabeling if necessary, we may assume without loss of generality that the eigenvalue corresponding to s is greater than that of w and the action of ϑ on α_1^o is an affine translation toward s. (See Figure 3.) Then the action of ϑ on α_2^o is an affine translation toward $-s$.

Let β be $\bigcup_{n\in\mathbf{Z}} \overline{\vartheta^n(x)\vartheta^{n+1}(x)}$ in B_1 for $x \in B_1$. This is curved in one direction. Then from Figure 4, it is easy to see that β and $\overline{ws}$ bound an open simply convex domain D in B_1, and $\overline{ws}^o \cup D$ is a ϑ-invariant simply convex one-sided neighborhood of $\overline{ws}^o$. (The reason that D is convex is that the boundary of $\mathrm{Cl}(D)$ can be easily shown to be curved in one direction only. D is simply convex since $\mathrm{Cl}(D)$ is a subset of a lune $\mathrm{Cl}(B_1)$ and does not contain both of the vertices of $\mathrm{Cl}(B_1)$.)

Moreover, if there is a ϑ-invariant convex open domain D' in B_2 such that $\overline{ws}^o \cup D'$ is a one-sided neighborhood, then $\vartheta^n(\overline{zt})$ for a point z of $\overline{ws}^o$ and t of D' converges to α_2 as $n \to \infty$. Since α_2 includes a pair of antipodal points and the closure of D' includes α_2, it follows that D' is not simply convex. Hence, $B_2 \cup \overline{ws}$ includes no simply convex one-sided neighborhood.

Since the choice of x in B_1 is arbitrary, the final statement of the lemma holds. □

We let B^+ to be B_i, $i = 1, 2$, such that $B_i \cup \overline{ws}^o$ includes a ϑ-invariant simply convex one-sided neighborhood of $\overline{ws}^o$ (see Figure 4). Let B^- be B_i, $i = 1, 2$, such that $B_i \cup \overline{ws}^o$ includes no ϑ-invariant simply convex one-sided neighborhood of $\overline{ws}^o$.

Let E be the open annulus given by $(H_1 - \overline{w-s})/ < \vartheta >$, and α_ϑ the imbedded tight curve in E corresponding to $\overline{ws}^o$. Then E is the union of two one-sided neighborhoods $E_{\vartheta,+}$ and $E_{\vartheta,-}$ of α_ϑ that are the respective images of $B^+ \cup \overline{ws}^o$ and $B^- \cup \overline{ws}^o$ under the quotient map.

By Lemma 8, $E_{\vartheta,+}$ includes a simply convex one-sided neighborhood of α_ϑ and $E_{\vartheta,-}$ no simply convex one-sided neighborhood of α_ϑ. (Moreover, every one-sided neighborhood of α_ϑ in $E_{\vartheta,+}$ includes a simply convex one-sided neighborhood of α_ϑ.) Therefore, $E_{\vartheta,+}$ and $E_{\vartheta,-}$ are not projectively equivalent one-sided neighborhoods of α_ϑ.

A *short* tight curve is a tight curve such that **dev** composed with its lift to the universal cover is an imbedding onto a convex line in $\mathbf{S}^2$ that is simply convex. For example, by Lemma 1, the boundary components of elementary annuli of type I are short, and so are those of elementary annuli of type IIa. However, only one boundary component of an elementary annulus of type IIb is short; the other one corresponds to a line connecting a pair of antipodal points. Since the developing map of a convex compact surface of negative Euler characteristic is an imbedding onto a simply convex subset of $\mathbf{S}^2$, it follows that any tight curve in such a surface is short.

Let α be an arbitrary imbedded short tight curve in S whose holonomy is conjugate to ϑ. We claim that an open extended surface S' of S includes an open neighborhood of α projectively homeomorphic to an open neighborhood of α_ϑ in E_ϑ: Let α' be the component of $p^{-1}(\alpha)$ in $\tilde{S}$. Then a deck transformation γ corresponding to α acts on α' where $h(\gamma)$ is quasi-hyperbolic. Recall that we can always change the development pair $(\mathbf{dev}, h)$ of the given real projective surface S as follows:

$$\mathbf{dev}' = \beta \circ \mathbf{dev}, h'(\cdot) = \beta \circ h(\cdot) \circ \beta^{-1}$$

for any $\beta \in \mathrm{Aut}(\mathbf{S}^2)$ (see [2] and [10]). By changing the development pair if necessary, we may assume that $h(\gamma) = \vartheta$. Then since α' is an imbedded line, $\mathbf{dev}|\alpha'$ is an imbedding onto a ϑ-invariant simply convex line in $\mathbf{S}^2$. Thus, it is one of the four ϑ-invariant simply convex lines: $\overline{ws}^o, \overline{w-s}^o, \overline{-ws}^o$, or $\overline{-w-s}^o$. By changing $\mathbf{dev}$ by automorphisms commuting with ϑ if necessary, we may assume without loss of generality that it is $\overline{ws}^o$. Therefore, $\mathbf{dev}$ immerses a γ-invariant open neighborhood of α' to a ϑ-invariant open neighborhood of $\overline{ws}^o$. Hence, $\mathbf{dev}$ induces a projective immersion f from a neighborhood of α to that of α_ϑ in E_ϑ. Obviously, f restricts to an imbedding from a neighborhood of α to that of α_ϑ.

The above shows that each one-sided neighborhood in S of a quasi-hyperbolic tight curve α is projectively equivalent to $E_{\vartheta,+}$ or $E_{\vartheta,-}$. Moreover, a one-sided neighborhood of α is projectively equivalent to $E_{\vartheta,+}$ if and only if a one-sided neighborhood of α on the other side is projectively equivalent to $E_{\vartheta,-}$. We now introduce an assignment of integers 1 or -1 to α given an orientation. Suppose that the right-sided neighborhood of α is projectively equivalent to $E_{\vartheta,+}$ or that the left-sided neighborhood of α is projectively equivalent to $E_{\vartheta,-}$. Then we let $\mathrm{sign}(\alpha) = 1$. Suppose that the right-sided neighborhood of α is projectively equivalent to $E_{\vartheta,-}$ or that the left-sided neighborhood of

α is projectively equivalent to $E_{\vartheta,+}$. Then we let $\mathrm{sign}(\alpha) = -1$. (See Figure 4).

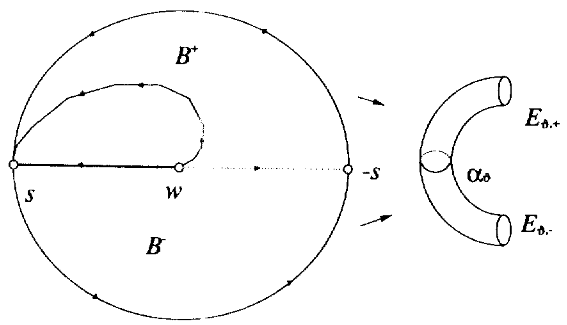

FIGURE 4. $E_{\vartheta,+}$ and $E_{\vartheta,-}$

Let us now list properties of this assignment and introduce another assignment:

(i) Suppose that α is a boundary component of a purely convex surface or trivial annulus P in S. Let α be given a boundary orientation.

Suppose that P is a purely convex surface. Let $\tilde{\alpha}$ be a component of $p^{-1}(\alpha)$ with a deck transformation γ acting on it corresponding to α. Let P' be a component of $p^{-1}(P)$ including $\tilde{\alpha}$ in the boundary. Then since P' is a universal cover of P, and P is purely convex, it follows that $\mathbf{dev}|P'$ is an imbedding onto a simply convex domain D in $\mathbf{S}^2$ (see the proof of Lemma 1.5 of [2]). Let α' be the image of $\tilde{\alpha}$. Then $h(\gamma)$ acts on α' and $h(\gamma)$ is quasi-hyperbolic.

Choose a small segment I in D meeting α' transversally. Let p be the endpoint of I in D^o and q one on α'. The endpoints of α', say w and s, are fixed points. Since α' is simply convex, we have $w \neq -s$. Let $\mathbf{S}^1$ be the great circle where $h(\gamma)$ is represented as a nondiagonal matrix and assume without loss of generality that $s, -s \in \mathbf{S}^1$ and $w, -w \notin \mathbf{S}^1$ (see Section 1). If D intersects $\mathbf{S}^1 - \{s, -s\}$, then since γ acts on P', $h(\gamma)$ acts on D hence, and D is convex, it follows that D includes a component of $\mathbf{S}^1 - \{s, -s\}$. This is a contradiction since D is simply convex. Thus, D is a subset of a closed lune B bounded by a segment β in $\mathbf{S}^1$ with endpoints s and $-s$, and the segment $\overline{ws} \cup \overline{w-s}$.

By changing γ to γ^{-1} if necessary, we may assume without loss of generality that the eigenvalue corresponding to s is greater than that of w. $h(\gamma)$ acts as an affine translation on β^o. If the action of $h(\gamma)$ on β^o is a translation toward $-s$, then $h(\gamma)^n(I)$ converges to β as $n \to \infty$. Since the closure $\mathrm{Cl}(D)$ of D is simply convex, this is a contradiction. Hence, the action of $h(\gamma)$ on β^o is an affine translation toward s.

The proof of Lemma 1.5 of [2] shows that $\mathbf{dev}|P'$ induces a projective homeomorphism $f : P \to D/\Gamma$ where Γ is the image under h of the group of deck transformations acting on P'. Let $q : D \to D/\Gamma$ denote the quotient map and A a one-sided neighborhood of α in P. Let A' be the component of $q^{-1}(f(A))$ including $\overline{ws}^o$. Then we may choose I short enough so that the convex quadrilateral E bounded by I, $h(\gamma)(I)$, $\overline{ph(\gamma)(p)}$, and $\overline{qh(\gamma)(q)}$ is included in A'. Then $\bigcup_{n\in\mathbf{Z}} h(\gamma)^n(E)$ is a simply convex neighborhood of $\overline{ws}^o$ in A' as in the proof of Lemma 8 and corresponds to a simply convex one-sided neighborhood of α in A (up to removing a boundary component).

If P is a trivial annulus, then P removed with a boundary component is a simply convex one-sided neighborhood of α. Hence, $\mathrm{sign}(\alpha) = -1$.

(ii) Given an elementary annulus E of type IIa, let α and β be components of δE, which are not oriented. We let $\mathrm{sign}_E(\alpha)$ be the sign of α given the boundary orientation from E and $\mathrm{sign}_E(\beta)$ that of β given the boundary orientation. (That is, an interior one-sided neighborhood of α includes a simply convex one-sided neighborhood if and only if $\mathrm{sign}_E(\alpha) = -1$.) Then from the model of elementary annuli of type IIa in Section 1, we see that if α has a simply convex one-sided neighborhood, then β does not and vice-versa: Again this depends on the direction of affine translations on the interior of $l'_2 \cap B$ and the eigenvalues associated with fixed points w' and s'. (Use the notation of Section 1.) To illustrate, suppose that the eigenvalue associated with s' is greater than that of w'. If the action of φ in the interior of $l'_2 \cap B$ is toward s, then $\overline{ws}^o$ has a simply convex one-sided neighborhood in $B \cup \overline{ws}^o$. If not, then $\overline{w-s}$ does. Therefore, we have $\mathrm{sign}_E(\alpha) = -\mathrm{sign}_E(\beta)$.

(iii) Let E be an elementary annulus of type IIb. Suppose that α is the short tight-curve component of δE. Section 1 shows that the short tight-curve component corresponds to $\overline{w's'}^o$ which has a φ-invariant simply convex one-sided neighborhood by the proof of Lemma 8. Thus, the interior of any one-sided neighborhood of α includes a convex one-sided neighborhood. Thus, we let $\mathrm{sign}_E(\alpha) = -1$. (Since the other boundary component of E is not short, we do not assign any value on it.)

(iv) Let E and E_1 respectively be two elementary annuli of type IIa and IIb adjacent to each other. If β is a common component of δE and δE_1, then β is a short tight curve and $\mathrm{sign}_E(\beta) = -\mathrm{sign}_{E_1}(\beta)$. If E and E_1 both are of type IIa, then the same statement holds again.

We now prove Proposition 5 using the above four properties (i), (ii), (iii), and (iv): Let A be the quasi-hyperbolic maximal annulus in S. Then A is the sum of elementary annuli E_i, $i = 1, \ldots, n$, of type IIa or IIb by the annulus decomposition theorem of Appendix B of [3]. We may assume without loss of generality that $E_i \cap E_j$ is empty if $|i - j| > 1$ and that $E_i \cap E_{i+1}$ is the common component of δE_i and δE_{i+1} for $i = 1, \ldots, n-1$. Let α_i, $i = 1, \ldots, n-1$, denote the common component of δE_i and δE_{i+1}. Let α_0 denote the component of δE_1 not among α_i, $i = 1, \ldots, n-1$, and α_n the component of δE_n not among the same collection. The admissible decomposition theorem implies that A is adjacent to a purely convex surface. We may assume without loss of generality that α_0 is the common boundary component of A and the purely convex surface.

By (i), we have $\mathrm{sign}_{E_1}(\alpha_0) = 1$. By (iii), the only short boundary component of every elementary annulus of type IIb is given -1; thus, E_1 is an elementary annulus of type IIa, α_1 is short, and $\mathrm{sign}_{E_1}(\alpha_1) = -1$ by (ii). An induction using (i), (ii), (iii), and (iv) implies that each α_i is short for $i = 0, \ldots, n$, each E_i is an elementary annulus of type IIa, and that we have

$$\begin{array}{ll}
\mathrm{sign}_{E_1}(\alpha_0) = 1, & \mathrm{sign}_{E_1}(\alpha_1) = -1, \\
\mathrm{sign}_{E_2}(\alpha_1) = 1, & \mathrm{sign}_{E_2}(\alpha_2) = -1, \\
\vdots & \vdots \\
\mathrm{sign}_{E_n}(\alpha_{n-1}) = 1, & \mathrm{sign}_{E_n}(\alpha_n) = -1.
\end{array}$$

By (i), α_n is not a boundary component of an adjacent purely convex surface or trivial annulus. Hence, Theorem 1 implies that α_n is a component of δS and $\delta S \neq \emptyset$. □

4. π-Möbius bands

We give a construction of π-Möbius bands. Let ϑ be an arbitrary hyperbolic projective automorphism of $\mathbf{S}^2$. Let $s, m, w, -s, -m$, and $-w$ denote the fixed points of ϑ as in Section 1 (but without the assumption on the eigenvalues). Then there exists a lune B bounded by two invariant segments α and β ending at points m and $-m$. α contains a fixed point s and β the fixed point w. (B is just the union of two ϑ-invariant triangles). We assume that s is the attracting or repelling fixed point of the action of $< \vartheta >$. There exists a unique orientation-reversing projective automorphism φ where we have $\varphi^2 = \vartheta, \varphi(m) = -m$ and s and w are fixed points (to see this diagonalize ϑ). Consider the set $B^* = B^o \cup (\alpha^o - \{s\})$. Then $< \vartheta >$ and $< \varphi >$ act properly discontinuously and freely on B^*. By the definition of elementary annuli in Section 1, $B^*/ < \vartheta >$ is a compact annulus which is a sum of two elementary annuli of type I, and $B^*/ < \varphi >$ is a compact Möbius band. There exists a projective double covering map $f : B^*/ < \vartheta > \rightarrow B^*/ < \varphi >$. Since the covering is a regular two-fold-covering, there exists an order-two deck transformation θ on $B^*/ < \vartheta >$ induced by φ so that the quotient of $B^*/ < \vartheta >$ by the action of $< \theta >$ is projectively homeomorphic to $B^*/ < \varphi >$. Hence, we constructed a π-Möbius band $B^*/ < \varphi >$.

We will now show that any π-Möbius band is projectively homeomorphic to one constructed as above. Let A be the annulus that double covers a π-Möbius band M where A is the sum of two elementary annuli E_1 and E_2 of type I. Then the universal cover $\tilde{M}$ of M is the union of two subsurfaces $\tilde{E}_1$ and $\tilde{E}_2$ meeting each other on a line α and they cover E_1 and E_2 respectively. $\tilde{E}_1$ has two boundary components α, a subset of $\tilde{M}^o$, and a line α_1; $\tilde{E}_2$ two boundary components α and a line α_2, such that α_1 and α_2 are all of the boundary components of $\tilde{M}$.

Let φ be the deck transformation of $\tilde{M}$ corresponding to the generator of $\pi_1(M)$. Then ϑ, $\vartheta = \varphi^2$, is the deck transformation corresponding to the generator of $\pi_1(A)$. Let us choose a development pair $(\mathbf{dev}, h)$ of $\tilde{M}$. Here $\mathbf{dev} : \tilde{M} \rightarrow \mathbf{S}$ is a real projective map and h satisfies $h(\theta) \circ \mathbf{dev} = \mathbf{dev} \circ \theta$ for each deck transformation θ of $\tilde{M}$ where $h(\theta) \in \mathrm{Aut}(\mathbf{S}^2)$.

Since E_1 is an elementary annulus of type I, Lemma 1 shows that

$$\mathbf{dev}(\tilde{E}_1) = \triangle_1^o \cup \overline{v_1v_2}^o \cup \overline{v_2v_3}^o, \tag{1}$$

where $\triangle_1$ is a triangle with vertices v_1, v_2, and v_3. Clearly, $\triangle_1$ is $h(\vartheta)$-invariant. Similarly, $\mathbf{dev}|\tilde{E}_2$ is an imbedding onto a subset of another $h(\vartheta)$-invariant triangle $\triangle_2$.

It is clear that $\triangle_1$ and $\triangle_2$ meet at $\mathrm{Cl}(\mathbf{dev}(\alpha))$, and their union is an $h(\vartheta)$-invariant lune B. We call s and w the endpoints of $\mathrm{Cl}(\mathbf{dev}(\alpha))$, and call m and $-m$ the vertices of the lune B so that $m \in \triangle_1$ and $-m \in \triangle_2$. By equation 1, $\mathbf{dev}(\tilde{E}_1)$ equals $\triangle_1^o \cup \overline{ms}^o \cup \overline{ws}^o$ or $\triangle_1^o \cup \overline{mw}^o \cup \overline{ws}^o$. A similar statement holds for $\mathbf{dev}(\tilde{E}_2)$. Hence, $\mathbf{dev}(\tilde{M})$ equals one of the four sets: (i) $B^o \cup \overline{mw}^o \cup \overline{-mw}^o$, (ii) $B^o \cup \overline{ms}^o \cup \overline{-ms}^o$, (iii) $B^o \cup \overline{mw}^o \cup \overline{-ms}^o$, or (iv) $B^o \cup \overline{ms}^o \cup \overline{-mw}^o$.

Above paragraphs show that $\mathbf{dev}|E_1^o \cup E_2^o$ is injective, and $\mathbf{dev}|\alpha$ is an injective map into the complement of $\mathbf{dev}(E_1^o \cup E_2^o)$. Since $\tilde{M}^o$ equals $E_1^o \cup E_2^o \cup \alpha$, $\mathbf{dev}|\tilde{M}^o$ is an injective immersion onto B^o, and hence is a projective homeomorphism. Since B^o is convex, $\tilde{M}^o$ is convex. Recall the projective completion $\check{M}$ of M from Section 1 of [2]. By Section 1.4 of [2], for the closure $\mathrm{Cl}(\tilde{M}^o)$ of $\tilde{M}^o$ in $\check{M}$, $\mathbf{dev}|\mathrm{Cl}(\tilde{M}^o)$ is an imbedding onto B since $\tilde{M}^o$ is convex. Since $\mathrm{Cl}(\tilde{M}^o)$ equals $\check{M}$ and hence includes $\tilde{M}$, it follows that $\mathbf{dev}|\tilde{M}$ is a projective homeomorphism onto $\mathbf{dev}(\tilde{M})$.

Moreover, $< h(\varphi) >$ and $< h(\vartheta) >$ act on $\mathbf{dev}(\tilde{M})$ with the following commutative diagram holding:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\ \varphi\ } & \tilde{M} \\ \downarrow \mathbf{dev} & & \downarrow \mathbf{dev} \\ \mathbf{dev}(\tilde{M}) & \xrightarrow{\ h(\varphi)\ } & \mathbf{dev}(\tilde{M}), \end{array}$$

where the same diagram with φ replaced by ϑ also holds. Since the actions of $< \varphi >$ and $< \vartheta >$ are properly discontinuous and free on $\tilde{M}$, the actions of $< h(\varphi) >$ and $< h(\vartheta) >$ on $\mathbf{dev}(\tilde{M})$ are properly discontinuous and free.

Since the deck transformation φ commutes with ϑ, the action of $< \varphi >$ descends to an order-two projective action on A. Since the decomposition of A into elementary annuli are unique by the annulus decomposition theorem (see [3, Appendix B]), the action preserves the simple closed curve $E_1 \cap E_2$. Hence φ preserves α, and $h(\varphi)$ preserves $\overline{ws}$. Since $h(\varphi)$ does not fix a point of $\overline{ws}^o$, w and s are fixed points of $h(\varphi)$.

Since $h(\varphi)$ is orientation reversing, $h(\varphi)(m) = -m$ and $h(\varphi)(-m) = m$. In order that $< h(\varphi) >$ acts on $\mathbf{dev}(\tilde{M})$, it follows that $\mathbf{dev}(\tilde{M})$ can only be of the form (i) or (ii). Suppose that

$$\mathbf{dev}(\tilde{M}) = B^o \cup \overline{mw}^o \cup \overline{-mw}^o = B^{\&}.(\mathrm{i})$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & B^{\&} \\ \downarrow p & & \downarrow q \\ M & \xrightarrow{\mathbf{dev}} & B^{\&}/ < h(\varphi) >, \end{array}$$

where p is the covering map, q the quotient map, and $\mathbf{dev}'$ the induced map. It follows that $\mathbf{dev}'$ is a projective homeomorphism. Since $< \vartheta >$, where $\vartheta = \varphi^2$, acts on $B^{\&}$ properly discontinuously, w is an attracting or repelling fixed point of the action of $< \vartheta >$ on $\mathbf{S}^2$: If w is a fixed point of saddle type, then the action of $< \vartheta >$ on $B^{\&}$ is not properly discontinuous. It follows that $B^{\&}/ < h(\varphi) >$ is a π-Möbius band constructed as in the beginning of this section. If $B^{\&}$ is of the form (ii), we can show similarly that M is projectively homeomorphic to a π-Möbius band constructed as above.

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