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ON SPANNING 3-TREES IN INFINITE 3-CONNECTED PLANAR GRAPHS

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Abstract. In this paper the existence of spanning 3-trees in every 3-connected locally finite vertex-accumulation-point-free planar graph is verified, which is an extension of D. Barnette to infinite graphs and which improves the result of the author.

1. Introduction

A tree is a *k-tree* if its vertices have a degree at most k ($k = 2, 3, \cdots$). A *spanning k-tree* T of a connected graph G is a k-tree as a subgraph of G such that $V(T) = V(G)$. A *k-forest* and a *spanning k-forest* are defined similarly. It is clear that spanning 2-trees of G are identical to Hamiltonian paths. A necessary condition for a finite graph to contain a spanning k-tree ($k \geq 3$) was given by S. Win [9], which is an extension of Posa's theorem. In this paper it will be considered for the case $k = 3$.

Let H be a plane graph. If C is a cycle in H, then $\overline{C}$ denotes the subgraph of H consisting of the vertices and edges on C or in the interior of C. If $\overline{C} = C$, then C is called a *facial cycle*. On the other hand, if H is a 2-connected, we may denote by $\widehat{H}$ the outer cycle of H; i.e. the cycle of H incident to the unbounded face. A plane graph H is a *circuit graph*, if there exists a cycle C in a 3-connected plane graph such that $H = \overline{C}$.

The existence of spanning 3-trees in finite graphs was given by D. Barnette [1], who showed that every circuit graph contains a spanning 3-tree. This result in particular implies that every 3-connected planar graph has the same property. It may be noticed that the connectedness

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number is in fact minimal; i.e. not every 2-connected planar graph has the property; for instance, $K_{2,n}$ ($n \geq 6$) is 2-connected planar but does not contain a spanning 3-tree. To generalize Barnette's theorem to infinite graphs we can ask whether every infinite 3-connected planar graph has a spanning tree, but it is clearly false, because a graph must be countable to contain such a tree. Indeed, R. Halin [4] conjectured that this assertion is true if the graphs are countable, which is yet an open problem. For this problem, the positive answer is proved for "strong triangulations" by the author [5]. But this result is only the first step to solve the problem.

In this paper we shall establish the conjecture for all locally finite vertex-accumulation-point-free (abbreviated VAP-free) graphs under the corresponding conditions, which improves the result of [5].

Namely our main result is as follows:

THEOREM. *Every 3-connected locally finite VAP-free planar graph contains a spanning 3-tree.*

We present some equivalent forms for a circuit graph in Section 2, and a few extensions of Barnette's theorem to connected planar graphs whose blocks are circuit graphs in Section 3. In Section 4, according to the method of C. Thomassen [7], the structure of a locally finite VAP-free graph is investigated with respect to a sequence of infinite disjoint cycles in the given graph. We construct in Section 5 a spanning 3-forest in a bridge of type k ($k = 0, 1, 2$), defined below using the results of Section 3. Combining this results with those in Section 4, we prove the main theorem in Section 6.

We end this section with some terms of graphs: Our terminology is essentially the same as that of [2] or [8]. All graphs considered are finite or infinite simple graphs. Let G be a graph. For a vertex v and a subgraph H of G, we denote by $d_H(v)$ the cardinal number of vertices adjacent to v in H, and by $d_G(v, H)$ the minimal metric distance between v and $V(H)$ in G.

Clearly every finite connected graph H has the following decomposition: There exist 2-connected induced subgraphs $H_0, H_1, \cdots, H_m$ of H

such that

$$H = \bigcup_{j=0}^{m} H_j \quad \text{and} \quad \left[\bigcup_{i=0}^{j-1} H_i\right] \cap H_j = \{v_j\}, \ j = 1, \cdots, m,$$

In this decomposition, H_j is called a *block* and v_j an *articulation of* H. If a block contains at most one articulation, then it is called an *endblock of* H. H has a *linear decomposition* (or H *is linear* if it has at most 2 endblocks. The following terminology is similar to one in [6].

Let C and C' be disjoint cycles in a VAP-free planar graph G, such that C lies in the interior of C'. By a (C, C')*-ring* in G we mean a subgraph of G, which consists of not only C and C' but also the vertices and edges lying between C and C'. A *bridge* of a (C, C')-ring R is either isomorphic to K_2 joining C to C', or a connected component of $R - (C \cup C')$ together with all edges connecting this component with $C \cup C'$. If B is a bridge of R, then the elements of $V(B) \cap V(C)$ (or $V(B) \cap V(C')$) are called *the vertices of attachment on* C (or C', respectively). B is *of type* k, $(k = 0, 1, 2, \cdots)$, if the number of its vertices of attachment on C' is k. For a (C, C')-ring R, we may say that R is *normal* if it holds the following conditions:

[N1] C and C' are induced.
[N2] $|V(B) \cap V(C')| \leq 2$, for all bridges B of R.
[N3] If $V(B) \cap V(C') = \{z, z'\}$, $z \neq z'$, for a bridge B, then $zz' \in E(C')$.

Therefore, if G is 3-connected and R is normal, every bridge of R must be one of type 0, 1 and 2, by [N2].

2. Characterization of the circuit graphs

As defined in the previous section, a circuit graph is isomorphic to $\overline{C}$, if C is a cycle of a 3-connected plane graph. It can be easily verified that a plane graph H is a circuit graph if and only if H is "internally 3-connected", in the sense of [3]; i.e. $\widetilde{H} := H \cup (\widehat{H} \times K_1)$ is 3-connected, where $\widehat{H} \times K_1$ is a wheel with the cycle $\widehat{H}$ and the vertex of K_1.

We shall give another equivalent form for such a graph, which is for us very useful in the next section.

THEOREM 2.1. *A 2-connected plane graph H is a circuit graph if and only if H satisfies following property; if H is separated by $x, y \in V(H)$, then each component of $H - \{x, y\}$ contains a vertex of $\widehat{H}$.*

PROOF. Let $\widetilde{H} = H \cup (\widehat{H} \times K_1)$ be defined above, and let z be denoted the vertex of K_1. Then $\widetilde{H}$ can be embedded in the plane such that the vertex z lies in the exterior of $\widehat{H}$.

First, suppose for a contradiction that there exist vertices $x, y \in V(H)$ and a component K of $H - \{x, y\}$ containing no vertices of $\widehat{H}$. Then all vertices of K lie in the interior of $\widehat{H}$, and consequently no vertex of K is adjacent to z. Therefore K must also be a component of $\widetilde{H} - \{x, y\}$, which contradicts the 3-connectedness of $\widetilde{H}$.

We now turn to the converse: Let x and y be distinct vertices of $\widetilde{H}$. If $x = z$ or $y = z$, then $\widetilde{H} - \{x, y\}$ is still connected since H is 2-connected, and thus $\widetilde{H}$ is 3-connected. Now we assume $x, y \in V(H) = V(\widetilde{H}) \setminus \{z\}$. Let $Q_1, \cdots, Q_r$, $r \geq 2$, denote the components of $H - \{x, y\}$. Then, for all $i \in \{1, \cdots, r\}$, Q_i contains a vertex, say u_i, of $\widehat{H}$ by the hypothesis. Since $u_i z \in E(\widetilde{H})$ it follows that $\widetilde{H} - \{x, y\}$ is still connected, and thus $\widetilde{H}$ is 3-connected, which is equivalent to the fact H is a circuit graph. $\square$

COROLLARY 2.2. *Let H be a circuit graph and let $x \in V(\widehat{H})$ be arbitrary. Then $H' := H - \{x\}$ is linear, and moreover every block of H' not isomorphic to K_2 is a circuit graph.*

PROOF. Consider first the case that H' is 2-connected. In this case we have only to show that H' is a circuit graph. For this, suppose to the contrary that it is not the case. Then, by Theorem 2.1, we have vertices $u, v \in V(H')$ and a component Q of $H' - \{u, v\}$ such that Q contains no vertices of $\widehat{H}'$. Since all vertices of Q must lie in the interior of $\widehat{H}'$ it follows that H is also separated by the vertices u, v, and therefore Q contains no vertex of $\widehat{H}$. Using again Theorem 2.1 H cannot be a circuit graph, which implies a contradiction to the hypothesis.

We now assume H' to be separated by a vertex, and suppose for a contradiction that H' is not linear; i.e. it has more than 3 endblocks. Then H' contains either an articulation which is included in at least 3 blocks, or a block in which at least 3 articulations are included. In any case we can find an endblock, say Q, which is contained entirely or except for an articulation, say v, in the interior of $\widehat{H}$. Then, since $\{v, x\}$ separates the graph H and $Q - v$ is a component of $H - \{v, x\}$ whose vertices lie in the interior of $\widehat{H}$, it follows that H is not a circuit graph by Theorem 2.1, contrary to the hypothesis. The fact that every block ($\neq K_2$) of H' is a circuit graph can be shown by analogous method for the case above. □

3. Some extensions of Barnette's theorem

In this section it will be given a generalization and a few variations of Barnette's theorem [1], which shows the existence of spanning 3-trees in a circuit graph. We will construct such trees (or forests) satisfying cirtain conditions not only in a circuit graph but also in a connected graph each of whose blocks is either a circuit graph or isomorphic to K_2. For convenience of descriptions we define $\widehat{Q} := Q$ if Q is a block isomorphic to K_2, while otherwise $\widehat{Q}$ is defined as the outer cycle of Q.

We begin with the theorem of Barnette; even if our theorem is a slightly stronger result than that of his, we can show it by almost similar method; i.e. by induction on the number of edges lying in the interior of the given circuit graph as he did. We will therefore omit to prove.

THEOREM 3.1 (D. BARNETTE [1]). *Let H be a circuit graph and let $u, v \in V(\widehat{H})$ (or $u, v, w \in V(\widehat{H})$, pairwise disjoint). Then there exists a spanning 3-tree T in H such that $d_T(u) = 1$ and $d_T(v) \leq 2$ (or $d_T(u) \leq 2$, $d_T(v) \leq 2$ and $d_T(w) \leq 2$).* □

COROLLARY 3.2. *Let H be a connected graph whose blocks ($\neq K_2$) are circuit graphs, and assume that H has a linear decomposition with the endblocks Q and Q'. Let u and u' denote the unique articulations of Q and Q', respectively. Then, for given $v \in V(\widehat{Q}) \setminus \{u\}$ and $v' \in V(\widehat{Q}') \setminus \{u'\}$, there exists a spanning 3-tree T in H such that $d_T(v) = 1$ and $d_T(v') \leq 2$.*

PROOF. It can be immediately obtained from Theorem 3.1. □

COROLLARY 3.3. *Let H, Q, Q', u, u', v and v' be the form described in Corollary 3.2. Then there exists a spanning 3-tree T in $H - v$ such that $d_T(v') \leq 2$.*

PROOF. If T' is a spanning 3-tree in H obtained from Corollary 3.2, then $T := T' - v'$ satisfies the conclusion of this corollary. □

COROLLARY 3.4. *Let H be a connected graph whose blocks ($\neq K_2$) are circuit graphs, and moreover H has exactly 3 endblocks Q_1, Q_2 and Q_3 with their articulations u_1, u_2 and u_3. Then, for given $v_i \in V(\widehat{Q}_i) \setminus \{u_i\}$ ($i = 1, 2, 3$), there exists a spanning 3-tree T in H such that $d_T(v_i) \leq 2$ for $i = 1, 2, 3$.*

PROOF. As is shown in the proof of Corollary 3.2, H belongs to one of 2 cases. i.e. H contains an articulation (say x) which is included in 3 blocks, or H contains a block (say Q) which includes 3 articulations (say x_1, x_2 and x_3).

Consider the first case. In this case we can clearly decompose H into 3 induced subgraphs H_1, H_2 and H_3, such that

$$H = \bigcup_{i=1}^{3} H_i, \quad \bigcap_{i=1}^{3} H_i = \{x\} \quad \text{and} \quad v_i \in V(H_i), \text{ for } i = 1, 2, 3.$$

Then, for $i = 1, 2, 3$, we get a spanning 3-tree T_i in H_i with $d_{T_i}(x) = 1$ and $d_{T_i}(v_i) \leq 2$ by Corollary 3.2. By setting $T := \bigcup_{i=1}^{3} T_i$ we obtain a spanning 3-tree in H as desired.

We now investigate the another case. Similarly we decompose H into induced subgraphs H_1, H_2, H_3 and Q, such that

$$H = Q \cup \left\{ \bigcup_{i=1}^{3} H_i \right\}, \quad Q \cap H_i = \{x_i\} \quad \text{for } i = 1, 2, 3.$$

By Theorem 3.1, there exists a spanning 3-tree T' in Q with $d_{T'}(x_i) \leq 2$, for $i = 1, 2, 3$. For the subgraph H_i ($i = 1, 2, 3$) we use Corollary 3.2 to get a spanning 3-tree T_i with $d_{T_i}(x_i) = 1$. Then $T := T' \cup \left\{ \bigcup_{i=1}^{3} T_i \right\}$ is a desired spanning 3-tree in H. □

Now we can give the final result in this section, which is a generalization of Corollary 3.4.

THEOREM 3.5. *Let H be a connected graph whose blocks ($\neq K_2$) are circuit graphs. Assume $Q_1, \cdots, Q_n$ ($n \geq 3$) to be endblocks of H with their articulations $u_1, \cdots, u_n$. Further let $v_j \in V(\widehat{Q}_j) \setminus \{u_j\}$, $j \in \{1, \cdots, n\}$, and let $k, l, m \in \{1, \cdots, n\}$ be pairwise disjoint. Then there exists a 3-forest F in H such that*

(1) $V(F) = V(H)$;

(2) *F has exactly $n-2$ components;*

(3) *v_k, v_l and v_m lie on a common component, and each of the remaining components contains exactly one v_j ($j = \{1, \cdots, n\} \setminus \{k, l, m\}$); and*

(4) *$d_F(v_j) \leq 2$ for all $j \in \{1, \cdots, n\}$.*

PROOF. If $n = 3$, then, by Corollary 3.4, we can easily get a spanning 3-forest (in this case it is a 3-tree) F in H satisfying desired properties.

Now assume that $n \geq 4$, and without los of generality set $\{k, l, m\} = \{n-2, n-1, n\}$. We will decompose H into $n-2$ induced subgraphs H_0, $H_1, \cdots, H_{n-3}$ as follows:

First let H_0 denote the minimal connected subgraph of H which contains only v_k, v_l and v_m among $\{v_1, \cdots, v_n\}$ and has the property; if $E(Q) \cap E(H_0) \neq \emptyset$ for a block Q of H, then it must hold $Q \subseteq H_0$.

Next, we construct the induced subgraphs $H_1, \cdots, H_{n-3}$ iteratively: Set $G_0 := H_0$. For $j = 1, \cdots, n-3$, G_j denotes the subgraph induced by the component of $H - \left[\bigcup_{i=0}^{j-1} G_i\right]$ containing v_j and by its articulation (say x_j). If G_j is linear, then set $H_j := G_j$. Otherwise we denote H_j the linear connected subgraph of G_j which contains a path connecting x_j with v_j and has the property; if $|E(Q) \cap E(H_j)| \neq \emptyset$ for a block Q of G_j it must hold $Q \subseteq H_j$.

Then we have:

i) $H = \bigcup_{j=0}^{n-3} H_j$ and $\left[\bigcup_{i=0}^{j-1} H_i\right] \cap H_j = \{x_j\}$, $j = 1, \cdots, n-3$;

ii) H_0 has exactly 3 endblocks with $v_k, v_l, v_m \in V(H_0)$, and H_j has a linear decomposition with $v_j \in V(H_j)$, $j = 1, \cdots, n-3$.

For H_0, using Corollary 3.4, there is a spanning 3-tree T_0 in H_0 with $d_{T_0}(v_k) \leq 2$, $d_{T_0}(v_l) \leq 2$ and $d_{T_0}(v_m) \leq 2$. On the other hand, by Corollary 3.3, we get a spanning 3-tree T_j in $H_j - x_j$ with $d_{T_j}(v_j) \leq 2$, for $j = 1, \cdots, n-3$. Then $F := \bigcup_{j=0}^{n-3} T_j$ is obviously a 3-forest holding the properties (1)–(4), which completes our proof. □

4. Sequence of induced cycles

The aim of this section is to provide the necessary background for the proof of our main result. We shall present a sequence of infinite disjoint cycles in a 3-connected locally finite VAP-free planar graph satisfying some conditions, whose union together with their interior covers all vertices of the given graph.

For convenience' sake, we will abbreviate a 3-connected locally finite VAP-free graph to a *3LV-graph* in this and the next section.

LEMMA 4.1. *Let R be a normal $(C.C')$-ring of a 3LV-graph. Then:*

(1) *There exist at least 2 bridges of R of type 1 or 2.*
(2) *There is no edge of R joining two vertices of C (or C'), which is not an edge of C (or C').*
(3) $|V(B) \cap V(C)| \geq 1$, *for all bridge B of R.*

PROOF. By [N2] the assertion (1) and (3) are obvious, since G is 3-connected. The property (2) follows from the fact that the cycles C and C' are induced. □

LEMMA 4.2. *For an induced cycle C of a 3LV-graph G, there exists exactly one cycle C' such that the created (C, C')-ring is normal.*

PROOF. We first show the existence. Let V be the set of all vertices of facial cycles of G containing at least one vertex of C, and then let H denote the subgraph of G induced by V. Further we set the outer cycle of H by C'. Then it can be easily checked that C' is also an induced cycle and that C and C' are disjoint each other. Thus we have [N1]. We must now show that the (C, C')-ring satisfies the conditions [N2] and [N3].

To show [N2] let B be a bridge of the (C, C')-ring, let $\{y_1, \cdots, y_m\}$ be the vertices of attachment of C', and suppose for the contradiction that $m \geq 3$. Since B is not isomorphic to K_2, it follows that

$$V(B) \setminus V(C \cup C') \neq \emptyset$$

Therefore we can find a y_1, y_m-path in $R - (V(C) \cup \{y_2, \cdots, y_{m-1}\})$, and hence y_k $(k = 2, \cdots m-1)$ cannot be contained in a facial cycle of G which contains a vertex of C, contrary to the construction of C' (or H).

To show [N3] let B be a bridge of R satisfying the hypothesis of [N3]. Since z and z' are vertices of attachment of C', they must lie on a commom facial cycle of G. If we suppose $zz' \notin E(C')$, then there exists a vertex $y \in V(C')$ which lies on the same facial cycle containing z and z'. By an argument similar to one above we again obtain a contradiction.

We now prove the uniqueness. Suppose that C_1 and C_2 are induced cycles satisfying this lemma. First we will show $|V(C_1) \cap V(C_2)| := m \geq 2$. For this, suppose that $m \leq 1$. Then it must hold either $C_1 \subseteq \overline{C}_2$ or $C_2 \subseteq \overline{C}_1$. Assume C_1 and C_2 are of the first case. If $m = 0$, i.e. $V(C_1) \cap V(C_2) = \emptyset$, then all vertices of C_1 must lie between C and C_2, and thus there exists only one bridge of type 1 or 2 of the (C, C_2)-ring since C_1 is a cycle (in particular, it is connected). Therefore we have a contradiction to Lemma 4.1 (2). On the other hand, if $V(C_1) \cap V(C_2) =: \{v\}$, the bridge containing all vertices of C_1 is of type 1 or 2, and moreover v must be the vertex of attachment or one of two vertices of attachment of the bridge on C_2. Then, from the planarity of G, the remaining bridges can at most have the vertex v as that of attachment on C_2, and thus G is separated by one or two vertices, which is impossible since G is 3-connected. The second case can be analogously verified.

By setting

$$\{z_1, \cdots z_m\} := V(C_1) \cap V(C_2), \quad m \geq 2$$

we will show that $z_i z_{i+1} \in E(G)$ for all $i \in \{1, \cdots, m\}$, where $z_{m+1} = z_1$. Then we get $C_1 = C_2$ as desired, since C_1 and C_2 are induced cycles.

For this, suppose that there exists a $k \in \{1, \cdots, m\}$ such that $z_k z_{k+1} \notin E(G)$. By setting P_j $(j = 1, 2)$ the path connecting z_k with z_{k+1} on C_j, we have $|V(P_j)| \geq 3$ and therefore $V(P_j - \{z_k, z_{k+1}\}) \neq \emptyset$. Moreover, we can see that either all vertices of $P_2 - \{z_k, z_{k+1}\}$ lie in the interior of C_1 or $P_1 - \{z_k, z_{k+1}\}$ in the interior of C_2. We will assume the first case. If there exists a path connecting $P_1 - \{z_k, z_{k+1}\}$ and $P_2 - \{z_k, z_{k+1}\}$ the bridge containing a vertex of $P_2 - \{z_k, z_{k+1}\}$ has more than 2 vertices of attachment of C_1, a contradiction to [N2]. Otherwise we have also a contradiction to [N3], since $P_2 - \{z_k, z_{k+1}\} \neq \emptyset$. □

With the aid of Lemma 4.2, we can show our main result in this section, which plays a crucial role in the remaining sections.

THEOREM 4.3. *Let G be a 3LV-graph, and let C_0 be an arbitrary facial cycle of G. Then there exists a sequence of induced cycles $\{C_0, C_1, C_2, \cdots\}$ satisfying following properties:*

(1) C_j lies in the interior of C_{j+1}, for all $j = 0, 1, 2, \cdots$.
(2) The (C_j, C_{j+1})-ring is normal, for all $j = 0, 1, 2, \cdots$.
(3) $G = \bigcup_{j=0}^{\infty} \overline{C}_j$.

Moreover, under the given facial cycle C_0, such a sequence of cycles is unique.

PROOF. From the 3-connectedness of G each facial cycle (also C_0) is induced. The existence and uniqueness of such a sequence of cycles satisfying the condition (1)–(2) can be immediately verified from Lemma 4.2. It remains only to be shown that $G = \bigcup_{j=0}^{\infty} \overline{C}_j$.

For this, let $x \in V(G)$ arbitrary. By the property (1) we have

$$x \in V(\overline{C}_{n_x}) \subseteq V\left(\bigcup_{j=0}^{\infty} \overline{C}_j\right), \quad \text{where } n_x = d_G(x, C_0).$$

Therefore we get $G \subseteq \bigcup_{j=0}^{\infty} \overline{C}_j$. Since it clearly holds $G \supseteq \bigcup_{j=0}^{\infty} \overline{C}_j$, we obtain the equality, which is as desired. □

REMARK. By considering of the property of 3LV-graph, it can be shown that the argument of Theorem 4.3 is not only necessary but also sufficient; i.e. an infinite locally finite 3-connected planar graph is a 3LV-graph if and only if there exists a sequence of induced cycles $\{C_0, C_1, C_2, \cdots\}$ satisfying the conditions [N1]–[N3] (compare with [7]).

5. Spanning 3-trees in a bridge

In this chapter, we investigate the structure of spanning 3-forests in a bridge of a normal (C, C')-ring R in a 3LV-graph G. Note that a spanning 3-forest F in a graph G is a subgraph of G containing no cycles such that $V(G) = V(F)$ and $d_F(x) \leq 3$ for all $x \in V(F)$. Let B be a bridge ($\neq K_2$) of R and let $\{x_1, \cdots, x_r = \overline{x}\}$ be the set of the vertices of attacment on C in the clockwise order. Then we may say that x_1 (or $\overline{x}$) is *the first* (or *the last*, respectively) *vertex of attachment of B on C*. We set further $H := B - (C \cup C')$. If $r = 1$, then B must be of type 2, and in this case H contains at most 2 endblocks.

We now assume that $r \geq 2$. Since H is still connected, there is a path in B connecting x_1 and $\overline{x}$, such that the intersection of the path and C is $\{x_1, \overline{x}\}$. We choose such a path P_B with

$$V(P_B) = \{x_1, z_1, \cdots, z_s, \overline{x}\},$$

where z_1 and z_s denote the first and the last vertices adjacent to x_1 and $\overline{x}$, respectively. Now we may denote Δ_B the set of all blocks of H containing at least one edge of P_B. Then we can easily see that the set Δ_B is unique for a given bridge B.

On the other hand, if the $x_1, \overline{x}$-path on C is denoted by P'_B, we have a cycle $J := P_B \cup P'_B$. The blocks of H lying entirely in the interior or exterior of J are said to be the *inner blocks* or *outer blocks* of H, respectively. The *inner endblocks* and *outer endblocks* of H are analogously defined. By the 3-connectedness of G, if B is of type 0 or 1, there exist no outer blocks, while there can be at most one such block if B is of type 2.

Let $Q_1, \cdots, Q_t$ be the inner endblocks of H with their articulations $u_1, \cdots, u_t$ in this order. Then, for each Q_i $(i = 1, \cdots, t)$, we can choose a system of pairwise disjoint vertices of attachment $\{\overline{x}_1, \cdots, \overline{x}_t\}$ on C such that $\overline{x}_i\,(i = 1, \cdots, t)$ is adjacent to a vertex of $Q_i - u_i$ and $\overline{x} \notin$

$\{\overline{x}_1, \cdots, \overline{x}_t\}$. For, let $\{x'_1, \cdots, x'_r = \overline{x}\}$ be the set of all vertices of attachment on C. Then the following are easy to see:

i) Every inner endblock is adjacent to at least 2 vertices of C;

ii) if $x'_{i_1}, \cdots, x'_{i_{k_i}}$, $(k_i \geq 2)$ denote the vertices of C adjacent to Q_i $(i = 1, \cdots, t)$, then

$$\underbrace{1 = 1_1 < 1_2 < \cdots < 1_{k_1}}_{\text{for } Q_1} \leq \underbrace{2_1 < 2_2 < \cdots < 2_{k_2}}_{\text{for } Q_2} \leq \cdots \leq \underbrace{t_1 < t_2 < \cdots < t_{k_t}}_{\text{for } Q_t}.$$

If we choose

$$\overline{x}_1 := x_{1_1}, \ \overline{x}_2 := x_{2_1}, \cdots, \overline{x}_t := x_{t_1},$$

then $\{\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_t\}$ satisfies the desired properties, since $k_i \geq 2$ and i). This set $\{\overline{x}_1, \cdots, \overline{x}_t\}$ may be called a *system of representatives of* $\{Q_1, \cdots, Q_t\}$.

We summerize these results in a lemma.

LEMMA 5.1. *Let B be a bridge of a normal (C, C')-ring in a 3LV-graph, and assume that $H := B - (C \cup C')$ contains at least one inner block. Then:*

(1) *If B is of type 0 or 1, then H contains no outer blocks. On the other hand, if B is of type 2, H can contain at most one such block.*

(2) *For the inner endblocks $Q_1, \cdots, Q_t$ of H, there exists a system of their representatives $\{\overline{x}_1, \cdots, \overline{x}_t\}$ on C.* □

In the next 3 theorems, a spanning 3-forest in a bridge of a normal (C, C')-ring of type k $(k = 0, 1, 2)$ will be constructed, such that neither of its components contains a vertex of C and that of C' simultaneously.

In order to simplify the following theorems or formulations, it will be assumed that every bridge contains at least one inner block. However we can without difficulty verify that the assertions are also true for the bridges having no inner blocks.

THEOREM 5.2. *Let B be a bridge of type 0 of a normal (C, C')-ring in a 3LV-graph, and let x_0 and $\overline{x}$ be the first and the last vertex of attachment of B on C, respectively. Then there exists a spanning 3-forest F in B such that*

(1) *every component of F contains exactly one vertex of attachment of B on C;* and

(2) $d_F(x_0) = d_F(\overline{x}) = 0$, *and* $d_F(x) \leq 1$ *for all* $x \in V(B) \cap V(C)$.

PROOF. Let $H := B - (C \cup C') = B - C$, and let Δ_B be the set of blocks of H defined at the beginnng of this section. Since B cannot have an outer block (by Lemma 5.1) every block is either inner or contained in Δ_B. Let $Q_1, \cdots, Q_t$ be the inner endblocks with the system of representatives $\{x_1, \cdots, x_t\}$. Further let v_i be a vertex of $Q_i - u_i$ adjacent to x_i and set $\widetilde{H} := H \cup \{z\overline{z}\}$, where u_i is the articulation of Q_i and $\overline{z}$ is a new vertex. By setting the first block of H as Q, we classify into two cases.

Case 1 : Q is not an endblock of H.

In this case, $Q_1, \cdots, Q_t$ and $z\overline{z}$ are the endblocks of $\widetilde{H}$ in all. If $t = 1$, we can choose a vertex ($\neq x_0$, $\overline{x}$), say $\tilde{x}$, of attachment of B, since $|V(B) \cap V(C)| \geq 3$. With a vertex of H adjacent to $\tilde{x}$ we get a spanning 3-tree T in $\widetilde{H} \cup \{\tilde{v}\tilde{x}\}$ by Corollary 3.2 or 3.4. Then, by setting

$$F := (T - \overline{z}) \cup [V(B) \cap V(C)],$$

we have a desired spanning 3-forest.

We now assume that $t \geq 2$. By Theorem 3.5 we have a 3-forest F' in $\widetilde{H}$ such that

i) $V(F') = V(\widetilde{H})$;
ii) F' contains exactly $(t-1)$ components $T_2', \cdots, T_t'$;
iii) $v_i \in T_i'$, $i = 2, \cdots, t-1$, and $v_1, v_t, \overline{z} \in T_t'$; and
iv) $d_{F'}(v_i) \leq 2$, $i = 2, \cdots, t$.

Then

$$F := [F' \cup \{x_i v_i \,|\, i = 2, \cdots, t\} \setminus \{z\overline{z}\}] \cup [V(B) \cap V(C)]$$

is a spanning 3-forest in B as desired.

Case 2 : Q is an endblock of H.

First, it may be noticed that $x_0 \neq x_1$, because $V(B) \cup V(C') = \emptyset$. Note in this case that $Q, Q_1, \cdots, Q_t$ and $z\overline{z}$ are endblocks of $\widetilde{H}$ in all. Thus, by Theorem 3.5, we obtain a 3-forest F' in $\widetilde{H}$ with the following properties:

i) $V(F') = V(\widetilde{H})$;

ii) F' has exactly t components;
iii) v_t and $\overline{z}$ lie on a common component, and each of the remaining components contains exactly one v_i $(i = 1, \cdots, t-1)$; and
iv) $d_{F'}(v_i) \leq 2$, $i = 2, \cdots, t$.

Then

$$F := [F' \cup \{x_i v_i \,|\, i = 1, \cdots, t\} \setminus \{z\overline{z}\}] \cup [V(B) \cap V(C)]$$

obviously satisfies the conclusion of this theorem. □

THEOREM 5.3. *Let B is a bridge of type 1 of a normal (C, C')-ring i n 3LV-graph, and let $\overline{x}$ be the last vertex of attachment of B on C. Further let $V(B) \cap V(C') = \{y\}$. Then there exists a spanning 3-forest F in B such that*

(1) *each component of F contains exactly one vertex of attachment of B on $C \cup C'$;*
(2) *$d_F(\overline{x}) = 0$, $d_F(x) \leq 1$ for all $x \in V(B) \cap V(C)$; and*
(3) *$d_F(y) = 0$.*

PROOF. Set $H := B - (C \cup C')$. Let Δ_B, $\{Q_1, \cdots, Q_t\}$, $\{x_1, \cdots, x_t\}$ and $\{v_1, \cdots, v_t\}$ be defined in the same way as in the proof of Theorem 5.2. Further we denote the first and the last block of Δ_B by Q and Q', respectively.

Case 1 : Q is not an endblock of H.

Choose a vertex v of H adjacent to y. If Q' is an endblock of H, then we set $\widetilde{H} = H \cup \{yv\}$. Otherwise, we add a new edge zz' to H, and in this case we set $\widetilde{H} = H \cup \{yv, z\overline{z}\}$. Then $Q_1, \cdots, Q_t$, yv and Q' are the endblocks in all, and hence by Theorem 3.4 we get a 3-forest F' in $\widetilde{H}$ such that

i) $V(F') = V(\widetilde{H})$;
ii) F' contains t components $T_1, \cdots, T_t$ with $y \in V(T_1)$ and $x_i \in V(T_i)$ for all $i = 1, \cdots, t$; and
iii) $d_{F'}(v_i) \leq 2$, $i = 1, \cdots, t$.

Then

$$F := [F' \cup \{x_i v_i \,|\, i = 1, \cdots, t\} \setminus \{yv\}] \cup [V(B) \cap V(C)]$$

is a spanning 3-forest in B as desired.

Case 2 : Q is an endblock of H.

In this case, we will only consider the case that Q' is an endblock of H. The another case can be analogously verified. Let u and u' are the unique articulation of Q and Q', respectively. If y is adjacent to $Q - u$ we set $\widetilde{H} = H \cup \{yv\}$, where v is an arbitrary vertex of $Q - u$. Then, by a similar argument above, we also get a desired 3-forest in B.

We now assume y is not adjacent to $Q - u$. By putting x_0 to the first vertex of attachment of B on C, we first have $x_0 \neq x_1$ from the 3-connectedness of the given graph. Therefore there exists a vertex $v_0 \in V(Q) \setminus \{u\}$ such that $x_0 v_0 \in E(B)$. Now let a vertex adjacent to y be denoted by v, and put $\widetilde{H} = H \cup \{yv, x_0 v_0\}$. Then the entire endblocks of $\widetilde{H}$ are $x_0 v_0$, $Q_1, \cdots, Q_t$, Q' and yv, and so we can find a 3-forest F' in $\widetilde{H}$ such that

i) $V(F') = V(\widetilde{H})$;

ii) F' contains $t+1$ components $T_0, T_1, \cdots. T_t$ with $x_0, y \in V(T_0)$ and $v_i \in V(T_i)$ for all $i = 1, \cdots, t$; and

iii) $d_{F'}(v_i) \leq 2$, $i = 0, 1, \cdots, t$.

Then by setting

$$F := [F' \cup \{x_i v_i \,|\, i = 0, 1, \cdots, t\} \setminus \{yv\}] \cup [V(B) \cap V(C)]$$

our proof is complete. □

From Lemma 5.1 we know that every bridge of type 2 can contain at most one outer endblock, and moreover the union of all outer blocks has a linear decomposition. To continue proving the next theorems for such bridges we will give a lemma without proof, which can be immediately verified by considering the connectedness number of the graph.

LEMMA 5.4. *Let B be a bridge of type 2 of a (C, C')-ring in a 3LV-graph with its outer endblock $\widetilde{Q}$, and set $\{y_1, y_2\} = V(B) \cap V(C')$. Further let $\widetilde{Q} = \widetilde{Q}_1, \cdots. \widetilde{Q}_r$ be the outer blocks of $H := B - (C \cup C')$ with the articulations $u_1, \cdots, u_{r-1}$ in this order. Finally let $u_r \in V(\widetilde{Q}_r)$ be the articulation contained in Δ_B. Then:*

(1) *$\widetilde{Q}_1 - u_1$ is adjacent to y_1 and y_2.*

(2) *If $\widetilde{Q}_r - u_r$ is 2-connected, then there exists a vertex $v \in V(\widetilde{Q}_r) \setminus \{u_{r-1}, u_r\}$ which is adjacent to either y_1 or y_2. Otherwise, there exist $v \in V(S) \setminus \{w\}$ and $v' \in V(S') \setminus \{w'\}$ such that $y_1 v, y_2 v' \in E(G)$, where S and S' are the endblocks of $\widetilde{Q}_r - u_r$ with the articulations w and w', respectively.* □

THEOREM 5.5. *Let B be a bridge of type 2 of a (C, C')-ring in a 3LV-graph, and let $\overline{x}$ be the last vertex of attachment of B on C. Further let $\{y_1, y_2\} = V(B) \cap V(C')$. Then there exists a spanning 3-forest F in B such that*

(1) *y_1 and y_2 lie on a common component of F, and in addition each of its remaining components contains exactly one vertex of attachment of B on C;*
(2) *$d_F(\overline{x}) = 0$, and $d_F(x) \leq 1$ for all $x \in V(B) \cap V(C)$; and*
(3) *$d_F(y_1) = d_F(y_2) = 1$.*

PROOF. First, consider the case that $H := B - (C \cup C')$ has at least one outer block. Let $\{\widetilde{Q}_1, \cdots, \widetilde{Q}_r\}$ and $\{u_1, \cdots, u_r\}$ as in the theorem above be given. By setting

$$H_2 = \left[\bigcup_{i=1}^{r} \widetilde{Q}_i \right] - u_r \quad \text{and} \quad H_2 = H - H_1$$

we define B_i $(i = 1, 2)$ to the subgraphs of H induced by H_i and by the vertices of attachment of B_i on C and C'. Then we see that B_1 and B_2 are connected subgraphs of B with

$$V(B_1) \cup V(B_2) = V(B) \quad \text{and} \quad V(B_1) \cap V(B_2) = \emptyset.$$

For B_1, we can find a 3-forest F_1 such that

i) $V(F_1) = V(B_1)$; and
ii) $d_{F_1}(\overline{x}) = 0$, and $d_{F_1}(x) \leq 1$ for all $x \in V(B) \cap V(C)$.

We will now contruct a 3-forest F_2 in B_2. If it is done, then by setting $F := F_1 \cup F_2$ our proof of this theorem is complete.

If H contains no outer block, then we simply set $F_2 = \{y_1, y_2\}$. Otherwise put $L = \widetilde{Q}_r - u_r$.

Case 1 : L is 2-connected.

According to Lemma 5.4 we can choose a vertex $v \in V(L) \setminus \{u_{r-1}\}$ with $y_1 v \in E(B)$ or $y_2 v \in E(B)$. Without los of generality we assume $y_1 v \in E(B)$. By making the same use of the lemma, we can again find a vertex $\overline{v} \in V(\widetilde{Q}_1) \setminus \{u_1\}$ with $y_2 \overline{v} \in E(B)$. Then, following Corollary 3.2, there exists a 3-tree T in H_2 with $d_T(v) \leq 2$ and $d_T(\overline{v}) \leq 2$. By setting $F_2 = T \cup \{y_1 v, y_2 \overline{v}\}$ we get a 3-forest in B_2as desired.

Case 2 : L is separated by a vertex.

Let v and $\overline{v}$ be defined in Lemma 5.4. Since H_2 contains at most 3 endblocks Corollary 3.4 can be used in this case, and thus we get a spanning 3-tree T in H_2 such that $d_T(v) \leq 2$ and $d_T(\overline{v}) \leq 2$. Then $F_2 := T \cup \{y_1 v, y_2 \overline{v}\}$ is a desired spanning 3-forest in B_2 with $d_{F_2}(v_i) = 1$ $(i = 1, 2)$. □

In order to "graft" a constructed spanning tree in a (C, C')-ring on that in the next ring, we have to choose a bridge connecting C and C'. It is clear that such a bridge must be of type 1 or 2, and moreover always exists because G is connected. In the next theorems we will construct a spanning 3-forest in such a bridge, one component of which connects a vertex of C with that of C'. It may be noticed that we will only consider the case that the bridge contains at least 2 vertices of attachment on C. For, we can otherwise proceed analogously as we will do in the proof of Theorem 5.7.

THEOREM 5.6. *Let B, $\overline{x}$ and y as in Theorem 5.3 be given. Then there exists a spanning 3-forest F in B such that*

(1) *each component of F contains exactly one vertex of attachment of B on C;*
(2) $d_F(\overline{x}) = 0$, $d_F(x) \leq 1$ *for all* $x \in V(B) \cap V(C)$; *and*
(3) $d_F(y) = 1$.

PROOF. If we denote a spanning 3-forest in B constructed in Theorem 5.3 by $\widetilde{F}$ and if we let v be the vertex chosen in the theorem, then $F := \widetilde{F} \cup \{yv\}$ obviously satisfies the conclusion of this theorem. □

THEOREM 5.7. *Let B, $\overline{x}$, y_1 and y_2 as in Theorem 5.5 be given. Further assume $|V(B) \cap V(C)| \geq 2$. Then there exists a spannng 3-forest F in B such that*

(1) *y_1 is contained in a component of F, and in addition this component has a vertex of attachment on C no more;*
(2) *each of the remaining components contains exactly one vertex of attachment of B on C;*
(3) *$d_F(\overline{x}) = 0$, $d_F(x) \leq 1$ for all $x \in V(B) \cap V(C)$;* and
(4) *$d_F(y_1) \leq 1$, $d_F(y_2) = 1$.*

PROOF. Let H, Δ_B, $\{Q_1, \cdots, Q_t\}$, $\{v_1, \cdots, v_t\}$ and $\{x_1, \cdots, x_t\}$ be defined in the proof of Theorem 5.2. We will only give the proof for the case that H has 2 endblocks Q, Q' in Δ_B and an outer endblock, say $\widetilde{Q}$ (The remaining cases can be analogously verified). In this case the entire endblocks are exactly $Q_1, \cdots, Q_t$, Q, Q' and $\widetilde{Q}$. If there is a vertex $v \in V(\widetilde{Q})$ with $vy_1 \in E(B)$, then, by Theorem 3.5, we get a 3-forest F' satisfying the following properties:

i) $V(F') = V(H)$;
ii) F' has exactly $t+1$ components $T_1, \cdots, T_t, T_{t+1}$, such that $v_1, \overline{v} \in T_1$, $v \in T_{t+1}$ and $v_i \in T_i$ $(i = 1, \cdots, t)$; and
iii) $d_{F'}(v) \leq 2$, $d_{F'}(\overline{v}) \leq 2$ and $d_{F'}(v_i) \leq 2$ $(i = 1, \cdots, t)$.

Then

$$F := F' \cup \left[\bigcup_{i=1}^{t} x_i v_i\right] \cup \{y_1 v, y_2 \overline{v}\} \cup [V(E) \cap V(C)]$$

obviously holds the assertions of this theorem.

On the other hand, if there exists no such a vertex, we let

$$F := F' \cup \left[\bigcup_{i=1}^{t} x_i v_i\right] \cup \{x_0 v, y_2 \overline{v}\} \cup [V(B) \cap V(C)] \cup \{y_1\},$$

which is a spanning 3-forest in B as desired, where F' is the 3-forest constructed above. □

6. Proof of the main theorem

Given a normal (C, C')-ring R in a 3-connected locally finite VAP-free planar graph G, assume that a spanning 3-tree in $\overline{C}$, denoted by T_C, is already constructed with the property: $d_{T_C}(x) \leq 2$ for all $x \in V(C)$. For the proof of the main result, we will extend it to a new 3-tree in $\overline{C}'$ holding the analogous property. First, for every bridge B of type 0, we find a spanning 3-forest F_B in B satisfying the properties (1) and (2) in Theorem 5.2. We now discuss the remaining bridges of R.

If all bridges of type 1 and 2 are isomorphic to K_2, then we first choose such a bridge B_0 ($= xy$, with $x \in V(C)$ and $y \in V(C')$) arbitrary. In succession let y' be the vertex of C' adjacent to y in the clockwise order, and set $P_y = C' - \{yy'\}$. Then finally set $T_{C'} := T_C \cup \left[\bigcup_{\substack{B\,:\,\text{bridge} \\ \text{of type } 0}} F_B \right] \cup \{xy\} \cup P_y$.

Now assume otherwise; i.e. there exists a bridge which is not isomorphic to K_2. Choose such a bridge and denote it by B_0. Then, using the result of Theorem 5.6 or 5.7, there exists a spanning 3-forest F_{B_0} if B_0 is of type 1 or 2, respectively. For the remaining bridges B we use Theorem 5.3 or 5.5, and so we also get a spanning 3-forest F_B in B with the corresponding properties.

We will now "graft" the constructed 3-forests on the tree T_C. For this, let $V(C') = \{y_1, \cdots, y_m\}$ with $\{y_1\} = V(C') \cap V(B_0)$ or $\{y_m, y_1\} = V(C') \cap V(B_0)$, if B_0 is of type 1 or 2, respectively. Further set, for $j = 1, \cdots, m$,

$$P_j = \begin{cases} \emptyset, & \text{if } y_j,\ y_{j+1} \in V(B) \text{ for a bridge } B \text{ of } R. \\ \{y_j y_{j+1}\}, & \text{otherwise.} \end{cases}$$

Then finally set,

$$T_{C'} = \begin{cases} \left[T_C \cup \left(\bigcup_{\substack{\text{B : bridge} \\ \text{of } R}} F_B \right) \cup \left(\bigcup_{j=1}^{m} P_j \right) \right] - \{y_m y_1\}, \\ \qquad \text{if } B_0 \text{ is of type 1.} \\ T_C \cup \left(\bigcup_{\substack{B\text{:bridge} \\ \text{of } R}} F_B \right) \cup \left(\bigcup_{j=1}^{m} P_j \right), \\ \qquad \text{if } B_0 \text{ is of type 2.} \end{cases}$$

In any case we can show the following

CLAIM. *The constructed $T_{C'}$ is again a spanning 3-tree in $\overline{C}'$ with $d_{T_{c'}}(y) \leq 2$ for all $y \in V(C')$.*

PROOF. If every bridge of type 1 or 2 is isomorphic to K_2, then the claim is obvious, and so we can assume that there is a bridge ($\neq K_2$) of type 1 or 2. First we have $d_{T_{C'}}(x) \leq 3$ for all $x \in V(C)$, since $d_{T_C}(x) \leq 2$ and $d_{F_B}(x) \leq 1$ for every bridge B of R. The fact that $T_{C'}$ is connected and contains no cycles follows from the constructions in the theorems above. It can be easy to see that $d_{T_{C'}}(y_i) \leq 2$ for all $i \in \{1, \cdots, m\}$, and thus our proof is complete. □

We are now prepared to prove the main theorem of this paper:

THEOREM. *Every 3-connected locally finite VAP-free planar graph contains a spanning 3-tree.*

PROOF. Let G be such a graph, and let C_0 be an arbitrary facial cycle of G. By Theorem 4.3 there exists a sequence of disjoint cycles $\{C_0, C_1, C_2, \cdots\}$ such that the (C_j, C_{j+1})-ring ($j = 0, 1, 2, \cdots$) is normal and $G = \bigcup_{j=0}^{\infty} \overline{C}_j$.

Obviously we can find a spanning 3-tree T_0 in $C_0 = \overline{C}_0$ with $d_{T_0}(x) \leq 2$ for all $x \in V(C_0)$. Now assume that, for $j \geq 1$, a spanning 3-tree T_j in $\overline{C}_j$ with $d_{T_j}(x) \leq 2$ for all $x \in V(C_j)$ is already constructed. Then, by the Claim above and the fact that (C_j, C_{j+1})-ring is normal, we again get a spanning 3-tree T_{j+1} in $\overline{C}_{j+1}$ with the corresponding property. Therefore we have a sequence of 3-trees $\{T_0, T_1, T_2, \cdots\}$ in G with

$$T_j \subset T_{j+1} \quad \text{and} \quad V(T_j) = V(\overline{C}_j) \quad \text{for all } j = 0, 1, 2, \cdots.$$

By setting $T = \bigcup_{j=0}^{\infty} T_j$, we get a desired spanning 3-tree in G since $G = \bigcup_{j=0}^{\infty} \overline{C}_j$, and thus the proof is complete.

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