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ON THE GROUP RINGS OF THE KLEIN'S FOUR GROUP

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ABSTRACT. Let K be a field of characteristic 0 and G a Klein's four group. We find the idempotent elements and units of the group ring KG by using the basic group table matrix of G.

1. Introduction

Every finite group G is isomorphic to a subgroup of the symmetric group of degree $|G|$. Given a finite group G, we can construct the group table of G from the symmetric group table.

Let $G = \{g_0 = 1, g_1, g_2, \cdots, g_{n-1}\}$ be a finite group with the fixed order $g_0 = 1, g_1, g_2, \cdots, g_{n-1}$ of elements. From the group table

	g_0	g_1	$\cdots$	g_j	$\cdots$	g_{n-1}
g_0				$\vdots$		
g_1				$\vdots$		
$\vdots$				$\vdots$		
g_i	$\cdots$	$\cdots$	$\cdots$	g_ig_j		
$\vdots$						
g_{n-1}						

we obtain the group table matrix

$$(g_ig_j).$$

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A basic group table matrix of G is a matrix with the diagonal entries 1 obtaining from the group table matrix $(g_i g_j)$ by elementary row operations interchanging two rows. The elements of the first column of the basic group table matrix are the inverses of elements $g_0 = 1, g_1, \cdots, g_{n-1}$ of G.

EXAMPLE 1. Let $G = < x \,|\, x^n = 1 >$ be a cyclic group with a fixed order $x^0 = 1, x, x^2, \cdots, x^{n-1}$ of elements. Then the basic group table matrix of G is

$$\begin{pmatrix} 1 & x & x^2 & x^3 & \dots & x^{n-1} \\ x^{n-1} & 1 & x & x^2 & \dots & x^{n-2} \\ x^{n-2} & x^{n-1} & 1 & x & \dots & x^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ x & x^2 & x^3 & x^4 & \dots & 1 \end{pmatrix}$$

The basic group table matrix of the Klein's four group $G = \{g_0 = 1, g_1, g_2, g_3\}$ with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements is the symmetric matrix

$$\begin{pmatrix} 1 & g_1 & g_2 & g_3 \\ g_1 & 1 & g_3 & g_2 \\ g_2 & g_3 & 1 & g_1 \\ g_3 & g_2 & g_1 & 1 \end{pmatrix}$$

which is identical with the group table matrix of G.

Let R be a ring with unity and $G = \{g_0 = 1, g_1, g_2, \cdots, g_{n-1}\}$ a finite group with the fixed order $g_0 = 1, g_1, g_2, \cdots, g_{n-1}$ of elements. From the element $\alpha = \sum_{i=0}^{n-1} r(g_i)g_i$ of the group ring RG, we obtain a following matrix M_α by putting $r(g_i)$ in the place of g_i in the basic group table matrix of G.

$$M_\alpha = \begin{pmatrix} r(1) & r(g_1) & r(g_2) & \dots & r(g_{n-1}) \\ r(g_1^{-1}) & r(1) & \cdot & \dots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r(g_{n-1}^{-1}) & \cdot & \cdot & \dots & r(1) \end{pmatrix}$$

EXAMPLE 2. Let R be a ring with unity and G be a Klein's four group with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements. Then for $\alpha = \sum_{i=0}^{3} r_i g_i \in RG$, we have

$$M_\alpha = \begin{pmatrix} r_0 & r_1 & r_2 & r_3 \\ r_1 & r_0 & r_3 & r_2 \\ r_2 & r_3 & r_0 & r_1 \\ r_3 & r_2 & r_1 & r_0 \end{pmatrix}$$

and for $\beta = \beta_0 + \beta_1 g_1 \in RG$, we have

$$M_\beta = \begin{pmatrix} \beta_0 & \beta_1 & 0 & 0 \\ \beta_1 & \beta_0 & 0 & 0 \\ 0 & 0 & \beta_0 & \beta_1 \\ 0 & 0 & \beta_1 & \beta_0 \end{pmatrix}$$

Let R be a ring with unity, G be a finite group and $M(R,G) = \{M_\alpha \mid \alpha \in RG\}$.

Then followings are trivial.

(1) $M_{\alpha+\beta} = M_\alpha + M_\beta$

(2) $M_{\alpha\beta} = M_\alpha M_\beta$

(3) $M_{r\alpha} = rM_\alpha$.

Therefore $RG \cong M(R,G)$ by an algebra isomorphism.

The following fact is proved by Kaplansky and Zalerskii.

Let K be a field of characteristic 0 and G be a finite group. Then if $\alpha = \sum \alpha(g)g \in KG$ is a nontrivial idempotent, then $\alpha(1)$ is a rational number lying strictly between 0 and 1.

In this paper, we shall find the idempotent and unit elements in the group rings of the Klein's four group by using the matrix.

2. Main Theorems

Let K be a field of characteristic 0 and G be a Klein's four group with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements. For $\alpha = \sum_{i=0}^{3} r_i g_i \in KG$, we have

$$M_\alpha = \begin{pmatrix} r_0 & r_1 & r_2 & r_3 \\ r_1 & r_0 & r_3 & r_2 \\ r_2 & r_3 & r_0 & r_1 \\ r_3 & r_2 & r_1 & r_0 \end{pmatrix}$$

and

$$P^{-1}M_\alpha P = \text{ diag } \{r_0 + r_1 - r_2 - r_3 \quad r_0 - r_1 + r_2 - r_3 \\ r_0 - r_1 - r_2 + r_3 \quad r_0 + r_1 + r_2 + r_3\}$$

where

$$P = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence

$$M_\alpha^n = \frac{1}{4}\begin{pmatrix} a+b+c+d & a-b-c+d & -a+b-c+d & -a-b+c+d \\ a-b-c+d & a+b+c+d & -a-b+c+d & -a+b-c+d \\ -a+b-c+d & -a-b+c+d & a+b+c+d & a-b-c+d \\ -a-b+c+d & -a+b-c+d & a-b-c+d & a+b+c+d \end{pmatrix}$$

where

$$\begin{aligned} a &= (r_0 + r_1 - r_2 - r_3)^n \\ b &= (r_0 - r_1 + r_2 - r_3)^n \\ c &= (r_0 - r_1 - r_2 + r_3)^n \\ d &= (r_0 + r_1 + r_2 + r_3)^n. \end{aligned}$$

THEOREM 1. *Let K be a field of characteristic 0 and G be a Klein's four group with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements. If $\alpha = \sum_{i=0}^{3} r_i g_i$ is an idempotent element of KG, then*

(1) $r_0 = 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, 1$

(2) *KG has 14 nontrivial idempotent elements and KH has 2 nontrivial idempotent elements for every subgroup H of G of order 2.*

PROOF. Since $M_\alpha^2 = M_\alpha$, we have

$$\begin{aligned} r_0^2 + r_1^2 + r_2^2 + r_3^2 &= r_0 \\ (2r_0 - 1)r_1 + 2r_2 r_3 &= 0 \\ (2r_0 - 1)r_2 + 2r_1 r_3 &= 0 \\ (2r_0 - 1)r_3 + 2r_1 r_2 &= 0 \end{aligned}$$

and thus the values of r_0, r_1, r_2, r_3 are as follows:

r_0	r_1	r_2	r_3
0	0	0	0
1	0	0	0
$\frac{1}{2}$	0	0	$\pm\frac{1}{2}$
$\frac{1}{2}$	0	$\pm\frac{1}{2}$	0
$\frac{1}{2}$	$\pm\frac{1}{2}$	0	0
$\frac{1}{4}$	$\frac{1}{4}$	$\pm\frac{1}{4}$	$\pm\frac{1}{4}$
$\frac{1}{4}$	$-\frac{1}{4}$	$\pm\frac{1}{4}$	$\mp\frac{1}{4}$
$\frac{3}{4}$	$\frac{1}{4}$	$\pm\frac{1}{4}$	$\mp\frac{1}{4}$
$\frac{3}{4}$	$-\frac{1}{4}$	$\pm\frac{1}{4}$	$\pm\frac{1}{4}$

Since each proper subgroup H consist of 2 distinct elements of G, KH has 2 nontrivial idempotent elements by above values of r_0, r_1, r_2, r_3.

THEOREM 2. *Let K be a field of characteristic 0 and G a Klein's four group with the fixed order $g_0 = 1, g_1, g_2, g_3$ of elements. Then*

(1) *If $\alpha = \sum_{i=0}^{3} r_i g_i \in KG$ is a unit such that $\alpha^n = 1$, then α satisfy*

$$\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{1}{4}\left\{ \rho_1 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + \rho_2 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \rho_3 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \rho_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

where ρ_1, ρ_2, ρ_3, and ρ_4 are n-th roots of unity in K.

(2) *The number of units α such that $\alpha^2 = 1$ is 16.*

(3) *KG has no nonzero nilpotent elements.*

PROOF. (1) From $M_\alpha^n = I$, we have

$$\begin{aligned} a+b+c+d&=4\\ a-b-c+d&=0\\ -a+b-c+d&=0\\ -a-b+c+d&=0 \end{aligned}$$

and thus $a=b=c=d=1$. Therefore

$$\begin{aligned} r_0 &= \frac{1}{4}(\rho_1+\rho_2+\rho_3+\rho_4)\\ r_1 &= \frac{1}{4}(\rho_1-\rho_2-\rho_3+\rho_4)\\ r_2 &= \frac{1}{4}(-\rho_1+\rho_2-\rho_3+\rho_4)\\ r_3 &= \frac{1}{4}(-\rho_1-\rho_2+\rho_3+\rho_4) \end{aligned}$$

where ρ_1, ρ_2, ρ_3, and ρ_4 are n-th roots of unity in K.

(2) From (1), the values of r_0, r_1, r_2, r_3 are as follows:

r_0	r_1	r_2	r_3
0	± 1	0	0
0	0	± 1	0
0	0	0	± 1
$\frac{1}{2}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\mp\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$
$-\frac{1}{2}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\pm\frac{1}{2}$	$\mp\frac{1}{2}$
± 1	0	0	0 .

(3) Let $\alpha = \sum_{i=0}^{3} r_i g_i \in KG$ be a nilpotent element. Then $M_\alpha^n = 0$ for some n. Hence we have

$$\begin{aligned} a+b+c+d&=0\\ a-b-c+d&=0\\ -a+b-c+d&=0\\ -a-b+c+d&=0 \end{aligned}$$

and thus $a=b=c=d=0$. Therefore $r_0 = r_1 = r_2 = r_3 = 0$.

$\alpha = \sum r(g)g \in KG$ is called an 1-unit if α is a unit and $\sum r(g) = 1$. For the units of KG, we have the following theorem.

THEOREM 3. *Let K be a field of characteristic 0 and G be a Klein's four group with elements $g_0 = 1, g_1, g_2, g_3$. Then*

(1) *$\alpha = \sum_{i=0}^{3} r_i g_i \in KG$ is a unit if and only if $r_0 + r_1 \neq \pm(r_2 + r_3)$ and $r_0 - r_1 \neq \pm(r_2 - r_3)$.*
(2) *$\alpha = \sum_{i=0}^{3} r_i g_i \in KG$ is a 1-unit if and only if $r_0+r_1+r_2+r_3 = 1$ and $r_i + r_j \neq \frac{1}{2}$ $(i \neq j)$.*

PROOF. (1) α is a unit if and only if

$$\begin{aligned} 0 \neq |M_\alpha| &= |P^{-1}M_\alpha P| \\ &= (r_0 + r_1 - r_2 - r_3)(r_0 - r_1 + r_2 - r_3) \\ &\qquad (r_0 - r_1 - r_2 + r_3)(r_0 + r_1 + r_2 + r_3). \end{aligned}$$

Therefore α is a unit if and only if $r_0 + r_1 \neq \pm(r_2 + r_3)$ and $r_0 - r_1 \neq \pm(r_2 - r_3)$.

(2) α is a 1-unit if and only if $r_0+r_1+r_2+r_3 = 1$, $r_0+r_1 \neq r_2+r_3$ and $r_0-r_1 = \pm(r_2-r_3)$. Hence α is a 1-unit if and only if $r_0+r_1+r_2+r_3 = 1$ and $r_i + r_j \neq \frac{1}{2}$ $(i \neq j)$.

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