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FIXED POINT THEOREMS FOR FUZZY MAPPINGS AND APPLICATIONS

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ABSTRACT. In this paper we obtain common fixed point theorems for sequences of fuzzy mappings on Menger probabilistic metric spaces, including common fixed point theorems for sequences of multi-valued mappings, which generalize and improve some results of Lee et al. [8] and Chang [2].

0. Introduction

In [14], Sehgal and Bharucha-Reid showed the existence of the fixed point for one-valued local contraction mappings on probabilistic metric spaces. Later the existences of fixed points for multi-valued mappings in probabilistic metric spaces were obtained by Chang, Hadzic and others [1, 3-6]. However, Pai and Veeramani's work [10] seems to be the first to establish a probabilistic analogue of Nadler's Banach contraction principle [9].

On the other hand, Heilpern [7] extended the Nadler's principle to the case of fuzzy mappings in 1981. His work seems to be the first to establish a fuzzy analogue of Nadler's principle. Most recently, Lee et al.[8] defined a contractive fuzzy mapping on a probabilistic metric space and presented some fixed point theorems for fuzzy mappings on probabilistic metric spaces.

In this paper we obtain a generalized common fixed point theorem for a sequence of fuzzy mappings on Menger probabilistic metric spaces. Also we show that a sequence of closed-valued mappings on Menger

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probabilistic metric spaces has a common fixed point as corollary. Our results in this paper generalize and improve Lee et al.'s [8] work, and Chang's results [2].

1. Preliminaries

Let (E, d) be a metric space. A fuzzy set A in E is a function from E into $[0,1]$. If $x \in E$, the function value $A(x)$ is called the grade of membership of x in A. The α-level set of A, denoted by $(A)_\alpha$, is defined by $(A)_\alpha = \{x | A(x) \geq \alpha\}$ if $\alpha \in (0,1]$. The collection of all fuzzy sets A in E such that each A_α is nonempty closed set in E is denoted by $W(E)$ and $C(E)$ denotes the collection of all nonempty closed subsets of E. For $A, B \in W(E)$, we denote by $A \subset B$, iff $A(x) \leq B(x)$ for each $x \in E$. If T is a mapping from E into $W(Y)$, where Y is a metric space, then T is called a fuzzy mapping. For each $x \in E$ we let $\{x\}$ be a fuzzy set with a membership function equals to a characteristic function of the set $\{x\}$. d_H denotes the Hausdorff metric induced by d as follows ;

$$d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{x \in B} \inf_{y \in A} d(x, y)\} \text{ for } A, B \in C(E)$$

and D the fuzzy-Hausdorff metric induced by d such that

$$D(A, B) = \sup_{\alpha \in (0,1]} d_H((A)_\alpha, (B)_\alpha) \quad \text{for} \quad A, B \in W(E).$$

For the sake of convenience, we recall some definitions, terms and notations in probabilistic metric spaces [1-3, 11 - 13].

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ the set of all positive integers.

Definition 1.1. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function, if it is nondecreasing and left-continuous with $\inf F(t) = 0$ and $\sup F(t) = 1$.

In what follows we always denote by D^+ the set of all distribution functions and by H the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 1, & \text{if} \quad t > 0. \end{cases}$$

DEFINITION 1.2. A probabilistic metric space (in short, a PM-space) is an ordered pair $(E, \Im)$, where E is a nonempty set and $\Im$ is a mapping from $E \times E$ into D^+. We denote the distribution function $\Im(x, y)$ by $F_{x,y}$ and $F_{x,y}(t)$ represents the value of $F_{x,y}$ at $t \in \mathbb{R}$ for each $x, y \in E$. The function $F_{x,y}$ is assumed to satisfy the following conditions :

(PM-1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;
(PM-2) $F_{x,y}(0) = 0$;
(PM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in \mathbb{R}$;
(PM-4) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$, then $F_{x,z}(t_1 + t_2) = 1$.

DEFINITION 1.3. A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm, if it satisfies the following conditions : for any $a, b, c, d \in [0, 1]$,

(T-1) $\Delta(a, 1) = a$;
(T-2) $\Delta(a, b) = \Delta(b, a)$;
(T-3) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a$ and $d \geq b$;
(T-4) $\Delta(\Delta(a, b,), c) = \Delta(a, \Delta(b, c))$.

DEFINITION 1.4. A Menger PM-space is a triplet $(E, \Im, \Delta)$, where $(E, \Im)$ is a PM-space and Δ is a t-norm satisfying the following triangle inequality

$$F_{x,z}(t_1 + t_2) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2)) \text{ for all } x, y, z \in E \text{ and } t_1, t_2 \geq 0.$$

Schweizer and Sklar [11] have proved that if $(E, \Im, \Delta)$ is a Menger PM-space with a continuous t-norm Δ, then $(E, \Im, \Delta)$ is a Hausdorff topological space in the topology τ induced by the family of neighborhoods :

$$\{U_p(\varepsilon, \lambda) : p \in E, \quad \varepsilon > 0, \quad \lambda > 0\},$$

where

$$U_p(\varepsilon, \lambda) = \{x \in E : F_{x,p}(\varepsilon) > 1 - \lambda\}.$$

DEFINITION 1.5. Let $(E, \Im, \Delta)$ be a Menger PM-space with a continuous t-norm Δ. Let $(x_n)_{n=1}^{\infty}$ be any sequence in E. $(x_n)_{n=1}^{\infty}$ is said to be τ-convergent to $x \in E$ (we write $x_n \xrightarrow{\tau} x$), if for any given $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.

$(x_n)_{n=1}^{\infty} \subset E$ is called a τ-Cauchy sequence, if for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n, x_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \geq N$.

A Menger PM-space $(E, \Im, \Delta)$ is said to be τ-complete, if each τ-Cauchy sequence in E is τ-convergent to some point in E.

From [11] we have the following lemma needed.

LEMMA 1.1. *If $(x_n)_{n=1}^{\infty}$ converges to x in a probabilistic metric space, then $(F_{x,x_n})_{n=1}^{\infty}$ converges to $F_{x,x} = H$, i.e., for every $t \geq 0 (F_{x,x_n}(t))_{n=1}^{\infty}$ converges to $F_{x,x}(t)$, and conversely.*

DEFINTION 1.6 [2]. Let $(E, \Im, \Delta)$ be a Menger PM-space and $A, B \in C(E)$, $x \in E$, then the probabilistic distance from x to A, and the probabilistic distance from A to B is defined respectively as follows ;

$$(1.1)\quad \begin{aligned} F_{x,A}(t) &= \sup_{s<t} \sup_{y \in A} F_{x,y}(s), \\ F_{A,B}(t) &= \sup_{s<t} \Delta(\inf_{x \in A} \sup_{y \in B} F_{x,y}(s), \inf_{y \in B} \sup_{x \in A} F_{x,y}(s)), \quad t \in \mathbb{R}. \end{aligned}$$

We can easily obtain the following lemma by Definitions 1.3 and 1.6.

LEMMA 1.2. *(1) The probabilistic distances $F_{A,B}(t)$ and $F_{x,A}(t)$ are left-contniuous and nondecreasing.*

(2) For any $A, B \in C(E)$ and $x \in A$,

$$F_{x,B}(t) \geq F_{A,B}(t), \quad t \geq 0.$$

LEMMA 1.3 [2]. *Let $(E, \Im, \Delta)$ be a Menger PM-space and Δ a left-continuous t-norm. If $A \in C(E)$ and $x, y \in E$, then we have the following ;*

(1) *$F_{x,A}(t) = 1$ for all $t \geq 0$ if and only if $x \in A$,*

(2) *$F_{x,A}(t_1 + t_2) \geq \Delta(F_{x,y}(t_1), F_{y,A}(t_2))$ for all $t_1, t_2 \geq 0$.*

DEFINITION 1.7 [8]. Let $(E, \Im, \Delta)$ be a Menger PM-space, $A, B \in W(E)$ and $\{x\} \subset E$, then the fuzzy probabilistic distance from $\{x\}$ to A, and the fuzzy probabilistic distance from A to B are defined respectively as follows ;

$$\begin{aligned} F_{\{x\},A}(t) &= \inf_{\alpha\in(0,1]} F_{x,(A)_\alpha}(t) \\ F_{A,B}(t) &= \inf_{\alpha\in(0,1]} F_{(A)_\alpha,(B)_\alpha}(t), \quad t \geq 0. \end{aligned}$$

LEMMA 1.4. *For any $A, B \in W(E)$ and $\{x\} \subset A$, we have*

$$F_{\{x\},B}(t) \geq F_{A,B}(t), \quad t \geq 0.$$

PROOF. Since $\{x\} \subset A$, $x \in (A)_\alpha$ for each $\alpha \in (0,1]$. Thus by Lemma 1.2, we have

$$\begin{aligned} F_{x,(B)_\alpha}(t) &\geq F_{(A)_\alpha,(B)_\alpha}(t) \\ &\geq F_{A,B}(t) \text{ for each } \alpha \in (0,1] \text{ and } t \geq 0. \end{aligned}$$

Hence

$$\begin{aligned} F_{\{x\},B}(t) &= \inf_{\alpha\in(0,1]} F_{x,(B)_\alpha}(t) \\ &\geq F_{(A)_\alpha,(B)_\alpha}(t) \\ &\geq F_{A,B}(t), \quad t \geq 0. \end{aligned}$$

LEMMA 1.5 [13]. *Let $\phi : [0,+\infty) \to [0,+\infty)$ be a strictly increasing function such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = +\infty$. If we define a function $\psi : [0,+\infty) \to [0,+\infty)$ by*

$$\psi(t) = \begin{cases} 0, & t = 0, \\ \inf\{s > 0 : \phi(s) > t\} \quad , & t > 0, \end{cases}$$

then ψ is continuous and nondecreasing.

DEFINITION 1.8. We say that a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfies the condition (Φ) if it is a strictly increasing and left-continuous function such that $\phi(0) = 0$, $\lim_{t \to +\infty} \phi(t) = +\infty$ and $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$.

LEMMA 1.6 [3]. *Let $\phi : [0, +\infty) \to [0, +\infty)$ satisfy the condition (Φ) and let ψ be a function defined as Lemma 1.5. Then we have the following:*
(1) $\phi(t) \leq t$ *for all* $t \geq 0$,
(2) $\phi(\psi(t)) \leq t$ *and* $\psi(\phi(t)) = t$ *for all* $t \geq 0$,
(3) $\psi(t) \geq t$ *for all* $t \geq 0$,
(4) $\lim_{n \to \infty} \psi^n(t) = +\infty$ *for all* $t > 0$.

2. Main results

Now we obtain our main result, a generalized common fixed point theorem for a sequence of fuzzy mappings on Menger PM-spaces.

THEOREM 2.1. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ, and $(T_i)_{i=1}^{\infty} : E \to W(E)$ a sequence of fuzzy mappings. Suppose that there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the condition (Φ) such that for any $i, j \in \mathbb{N}$, and any $x, y \in E$,*

$$F_{T_i x, T_j y}(\phi(t)) \geq \min\{F_{x,y}(t), F_{x,(T_i x)_1}(t), F_{y,(T_j y)_1}(t)\}, t \geq 0. \tag{2.1}$$

Suppose further that for every $x, y \in E$ and $\{u_x\} \subset T_i x$, there exists $\{v_y\} \subset T_j y$ such that

$$F_{u_x, v_y}(t) = F_{T_i x, T_j y}(t), \quad t \geq 0, \tag{2.2}$$

then there exists an $x_* \in E$ such that

$$\{x_*\} \subset T_i x_*, \quad i \in \mathbb{N}.$$

PROOF. For any given $x_0 \in E$ we take $x_1 \in E$ such that $\{x_1\} \subset T_1 x_0$, and by (2.2) take $x_2 \in E$ such that $\{x_2\} \subset T_2 x_1$ satisfying

$$F_{x_1, x_2}(t) = F_{T_1 x_0, T_2 x_1}(t), \quad t \geq 0.$$

Similarly we take $x_3 \in E$ such that $\{x_3\} \subset T_3 x_2$ satisfying

$$F_{x_2,x_3}(t) = F_{T_2 x_1, T_3 x_2}(t), \quad t \geq 0.$$

Continuing this process, we obtain a sequence $(x_n)_{n=1}^{\infty} \subset E$ satisfying

(i) $\{x_n\} \subset T_n x_{n-1}$ (i.e., $x_n \in (T_n x_{n-1})_1$) ;

(ii) $F_{x_n,x_{n+1}}(t) = F_{T_n x_{n-1}, T_{n+1} x_n}(t)$, $t \geq 0$.

On the other hand, by the hypothesis we obtain the following condition for each $n \in \mathbb{N}$.

(iii) $F_{T_n x_{n-1}, T_{n+1} x_n}(\phi(t))$
$\geq \min\{F_{x_{n-1},x_n}(t), F_{x_{n-1},(T_n x_{n-1})_1}(t), F_{x_n,(T_{n+1} x_n)_1}(t)\}, t \geq 0.$

Define a function $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \begin{cases} 0, & t = 0; \\ \inf\{s > 0 : \phi(s) > t\}, & t > 0, \end{cases}$$

then we have the following condition (iv) from the condition (iii) and Lemma 1.6,

(iv) $F_{T_n x_{n-1}, T_{n+1} x_n}(t) \geq F_{T_n x_{n-1}, T_{n+1} x_n}(\phi(\psi(t)))$
$\geq \min\{F_{x_{n-1},x_n}(\psi(t)), F_{x_{n-1},(T_n x_{n-1})_1}(\psi(t)), F_{x_n,(T_{n+1} x_n)_1}(\psi(t))\},$
$t \geq 0.$

Now we prove that $(x_n)_{n=1}^{\infty}$ is a τ-Cauchy sequence in $(E, \mathcal{F}, \Delta)$. In fact, since $x_n \in (T_n x_{n-1})_1$ for each $n \in \mathbb{N}$, by Lemma 1.4, from the conditions (ii) and (iv) we can obtain the following ;

$$\begin{aligned} & F_{x_n,x_{n+1}}(t) \\ =& F_{T_n x_{n-1}, T_{n+1} x_n}(t) \\ \geq& \min\{F_{x_{n-1},x_n}(\psi(t)), F_{x_{n-1},(T_n x_{n-1})_1}(\psi(t)), F_{x_n,(T_{n+1} x_n)_1}(\psi(t))\} \\ \geq& \min\{F_{x_{n-1},x_n}(\psi(t)), F_{\{x_{n-1}\},T_n x_{n-1}}(\psi(t)), F_{\{x_n\},T_{n+1} x_n}(\psi(t))\} \\ \geq& \min\{F_{x_{n-1},x_n}(\psi(t)), F_{T_{n-1} x_{n-2}, T_n x_{n-1}}(\psi(t)), F_{T_n x_{n-1}, T_{n+1} x_n}(\psi(t))\} \\ =& \min\{F_{x_{n-1},x_n}(\psi(t)), F_{x_{n-1},x_n}(\psi(t)), F_{x_n,x_{n+1}}(\psi(t))\} \quad t \geq 0. \end{aligned}$$

Since $F_{x_n,x_{n+1}}(t)$ is nondecreasing, by Lemma 1.6

$$F_{x_n,x_{n+1}}(t) \geq F_{x_{n-1},x_n}(\psi(t)), \quad t \geq 0.$$

Thus we obtain the following inequality

$$F_{x_n,x_{n+1}}(t) \geq F_{x_0,x_1}(\psi^n(t)), \quad t \geq 0,$$

for the sequence $(x_n)_{n=1}^{\infty}$.

On the other hand, for any positive integers n, m and $t \geq 0$ we have

$$\begin{aligned}
&F_{x_n,x_{n+m}}(t)\\
\geq&\Delta\left\{F_{x_n,x_{n+1}}\left(\left(1-\frac{1}{k}\right)t\right), F_{x_{n+1},x_{n+m}}\left(\frac{t}{k}\right)\right\}\\
\geq&\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}, \Delta\left[F_{x_{n+1},x_{n+2}}\left\{\left(\frac{1}{k}-\frac{1}{k^2}\right)t\right\}, F_{x_{n+2},x_{n+m}}\left(\frac{1}{k^2}t\right)\right]\right]\\
\geq&\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}, \Delta\left[F_{x_0,x_1}\left\{\psi^{n+1}\left(\left(\frac{1}{k}-\frac{1}{k^2}\right)t\right)\right\}, F_{x_{n+2},x_{n+m}}\left(\frac{1}{k^2}t\right)\right]\right]\\
\geq&\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}, \Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\},\right.\right.\\
&\quad\left.\left.\Delta\left[F_{x_{n+2},x_{n+3}}\left\{\left(\frac{1}{k^2}-\frac{1}{k^3}\right)t\right\}, F_{x_{n+3},x_{n+m}}\left(\frac{1}{k^3}t\right)\right]\right]\right]\\
\geq&\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}, \Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\},\right.\right.\\
&\quad\left.\left.\Delta\left[F_{x_0,x_1}\left\{\psi^{n+2}\left(\left(\frac{1}{k^2}-\frac{1}{k^3}\right)t\right)\right\}, F_{x_{n+3},x_{n+m}}\left(\frac{1}{k^3}t\right)\right]\right]\right]\\
\geq&\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}, \Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\},\right.\right.\\
&\quad\left.\left.\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}, \Delta\left[F_{x_{n+3},x_{n+4}}\left\{\left(\frac{1}{k^3}-\frac{1}{k^4}\right)t\right\}, F_{x_{n+4},x_{n+m}}\left(\frac{1}{k^4}t\right)\right]\right]\right]\right]\\
\geq&\cdots\\
\geq&\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}, \Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\},\right.\right.\\
&\quad\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\},\right.\\
&\quad\Delta\left[\cdots, \Delta\left[F_{x_{n+m-2},x_{n+m-1}}\left\{\left(\frac{1}{k^{m-2}}-\frac{1}{k^{m-1}}\right)t\right\}, F_{x_{n+m-1},x_{n+m}}\left(\frac{1}{k^{m-1}}t\right)\right]\right]\cdots\Big]\\
\geq&\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}, \Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\},\right.\right.\\
&\quad\Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\},\right.\\
&\quad\Delta\left[\cdots, \Delta\left[F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}, F_{x_0,x_1}\left\{\psi^n\left(\left(1-\frac{1}{k}\right)t\right)\right\}\right]\right]\cdots\Big] \quad \text{for } k>1.
\end{aligned}$$

Letting $n \to \infty$, for any m we have

$$\lim_{n\to\infty} F_{x_n,x_{n+m}}(t) = 1, \quad t \geq 0.$$

This implies that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in E. Let $x_n \xrightarrow{\tau} x_* \in E$. Next we prove that $\{x_*\}$ is a common fixed point, i.e.,

$$\{x_*\} \subset \bigcap_{i=1}^{\infty} T_i x_*.$$

In fact, for $\{x_{n+1}\} \subset T_{n+1}x_n$ and $T_i x_*$ for each fixed $i \in \mathbb{N}$, we have

$$\begin{aligned}
&F_{x_{n+1},(T_i x_*)_1}(t)\\
\geq& F_{\{x_{n+1}\},T_i x_*}(t)\\
\geq& F_{T_{n+1}x_n,T_i x_*}(t)\\
\geq& \min\{F_{x_n,x_*}(\psi(t)), F_{x_n,(T_{n+1}x_n)_1}(\psi(t)), F_{x_*,(T_i x_*)_1}(\psi(t))\}\\
\geq& \min\{F_{x_n,x_*}(\psi(t)), F_{\{x_n\},T_{n+1}x_n}(\psi(t)), F_{x_*,(T_i x_*)_1}(\psi(t))\}\\
\geq& \min\{F_{x_n,x_*}(\psi(t)), F_{T_n x_{n-1},T_{n+1}x_n}(\psi(t)), F_{x_*,(T_i x_*)_1}(\psi(t))\}\\
=& \min\{F_{x_n,x_*}(\psi(t)), F_{x_n,x_{n+1}}(\psi(t)), F_{x_*,(T_i x_*)_1}(\psi(t))\}\\
\geq& \min\{F_{x_n,x_*}(\psi(t)), F_{x_0,x_1}(\psi^n(t)), F_{x_*,(T_i x_*)_1}(\psi(t))\}.
\end{aligned} \tag{2.3}$$

Letting $n \to \infty$ and taking limit inferior in (2.3) we have by Lemma 1.1 and Lemma 1.6

$$\varliminf_{n\to\infty} F_{x_{n+1},(T_i x_*)_1}(t) \geq F_{x_*,(T_i x_*)_1}(\psi(t)). \tag{2.4}$$

On the other hand

$$F_{x_*,(T_i x_*)_1}(\psi(t)) \geq \Delta\{F_{x_*,x_{n+1}}(\delta), F_{x_{n+1},(T_i x_*)_1}(\psi(t)-\delta)\}, \quad \delta > 0.$$

Taking limit superior we have

$$F_{x_*,(T_i x_*)_1}(\psi(t)) \geq \varlimsup_{n\to\infty} F_{x_{n+1},(T_i x_*)_1}(\psi(t)-\delta), \quad \delta > 0.$$

By the arbitrariness of $\delta > 0$, we have

$$F_{x_*,(T_i x_*)_1}(\psi(t)) \geq \overline{\lim_{n\to\infty}} F_{x_{n+1},(T_i x_*)_1}(\psi(t)). \tag{2.5}$$

Combining (2.4) and (2.5) we have

$$\begin{aligned}\underline{\lim_{n\to\infty}} F_{x_{n+1},(T_i x_*)_1}(\psi(t)) &\geq \underline{\lim_{n\to\infty}} F_{x_{n+1},(T_i x_*)_1}(t)\\ &\geq F_{x_*,(T_i x_*)_1}(\psi(t))\\ &\geq \overline{\lim_{n\to\infty}} F_{x_{n+1},(T_i x_*)_1}(\psi(t))\\ &\geq \overline{\lim_{n\to\infty}} F_{x_{n+1},(T_i x_*)_1}(t).\end{aligned}$$

Therefore

$$\lim_{n\to\infty} F_{x_{n+1},(T_i x_*)_1}(\psi(t)) = F_{x_*,(T_i x_*)_1}(\psi(t)), \text{ and} \tag{2.6}$$

$$\lim_{n\to\infty} F_{x_{n+1},(T_i x_*)_1}(t) = F_{x_*,(T_i x_*)_1}(\psi(t)). \tag{2.7}$$

By the arbitrariness of t, from (2.6) we have

$$\lim_{n\to\infty} F_{x_{n+1},(T_i x_*)_1}(t) = F_{x_*,(T_i x_*)_1}(t), \quad t \geq 0.$$

Therefore from (2.7) we have

$$\begin{aligned}F_{x_*,(T_i x_*)_1}(t) &= F_{x_*,(T_i x_*)_1}(\psi(t))\\ &= F_{x_*,(T_i x_*)_1}(\psi^2(t))\\ &= \cdots\cdots\\ &= F_{x_*,(T_i x_*)_1}(\psi^m(t)).\end{aligned}$$

Letting $m \to \infty$, we have

$$F_{x_*,(T_i x_*)_1}(t) = 1, \quad t \geq 0.$$

This shows that $x_* \in (T_i x_*)_1$ by Lemma 1.3, i.e., $\{x_*\} \subset T_i x_*, i = 1, 2, \cdots$. This completes the proof.

THEOREM 2.2. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ satisfying $\Delta(t,t) \geq t, t \in [0,1]$. Let $(T_i)_{i=1}^{\infty} : E \to W(E)$ be a sequence of fuzzy mappings. Suppose that there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the condition (Φ) such that for any $i, j \in \mathbb{N}$ and any $x, y \in E$,*

(2.8)

$$\begin{aligned} &F_{T_i x, T_j y}(\phi(t)) \\ &\geq \min\left\{F_{x,y}(t), F_{x,(T_i x)_1}(t), F_{y,(T_j y)_1}(t), F_{x,(T_j y)_1}(2t), F_{y,(T_i x)_1}(2t)\right\}, \\ &\qquad t \geq 0. \end{aligned}$$

Suppose further that for every $x, y \in E$ and $\{u_x\} \subset T_i x$, there exists a $\{v_y\} \subset T_j y$ such that

$$F_{u_x, v_y}(t) = F_{T_i x, T_j y}(t), \quad t \geq 0.$$

Then there exists an $x_* \in E$ such that $\{x_*\} \subset \bigcap_{i=1}^{\infty} T_i x_*$.

PROOF. From the inequality (2.8) we have

$$\begin{aligned} &F_{T_i x, T_j y}(\phi(t)) \\ &\geq \min\left[F_{x,y}(t), F_{x,(T_i x)_1}(t), F_{y,(T_j y)_1}(t), \right. \\ &\qquad \left. \Delta\{F_{x,y}(t), F_{y,(T_j y)_1}(t)\}, \Delta\{F_{x,y}(t), F_{x,(T_i x)_1}(t)\}\right]. \end{aligned}$$

By $\Delta(t,t) \geq t, t \in [0,1]$, the condition (2.1) holds. Therefore $(T_i)_{i=1}^{\infty}$ has a common fixed point.

Letting $\phi(t) = kt$, $0 < k < 1$, in Theorem 2.1 and Theorem 2.2, we have the following two common fixed point theorems, Theorem 2.3 and Theorem 2.4.

THEOREM 2.3. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ, and $(T_i)_{i=1}^{\infty} : E \to W(E)$ a sequence of fuzzy mappings. Suppose that there exists a constant $k \in (0,1)$ such that for any $i, j \in \mathbb{N}$, and any $x, y \in E$,*

$$F_{T_i x, T_j y}(kt) \geq \min\{F_{x,y}(t), F_{x,(T_i x)_1}(t), F_{y,(T_j y)_1}(t)\}, t \geq 0.$$

Suppose further that for every $x, y \in E$ and $\{u_x\} \subset T_i x$, there exists $\{v_y\} \subset T_j y$ such that

$$F_{u_x, v_y}(t) = F_{T_i x, T_j y}(t), \quad t \geq 0.$$

Then there exists an $x_* \in E$ such that

$$\{x_*\} \subset T_i x_*, \quad i \in \mathbb{N}.$$

THEOREM 2.4. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ satisfying $\Delta(t,t) \geq t, t \in [0,1]$. Let $(T_i)_{i=1}^{\infty} : E \to W(E)$ be a sequence of fuzzy mappings. Suppose that there exists a constant $k \in (0,1)$ such that for any $i, j \in \mathbb{N}$ and any $x, y \in E$,*

$$\begin{aligned}
&F_{T_i x, T_j y}(kt) \\
&\geq \min\{F_{x,y}(t), F_{x,(T_i x)_1}(t), F_{y,(T_j y)_1}(t), F_{x,(T_j y)_1}(2t), F_{y,(T_i x)_1}(2t)\}. \\
&\quad t \geq 0.
\end{aligned}$$

Suppose further that for every $x, y \in E$ and $\{u_x\} \subset T_i x$, there exists $\{v_y\} \subset T_j y$ such that

$$F_{u_x, v_y}(t) = F_{T_i x, T_j y}(t), \quad t \geq 0.$$

Then there exists an $x_* \in E$ such that $\{x_*\} \subset \bigcap_{i=1}^{\infty} T_i x_*$.

Putting $i = j$ in Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4, we obtain the following fixed point theorems for fuzzy mappings as corollaries.

THEOREM 2.5. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ, and $T : E \to W(E)$ a fuzzy mapping. Suppose that there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the condition (Φ) such that for any $x, y \in E$,*

$$F_{Tx, Ty}(\phi(t)) \geq \min\{F_{x,y}(t), F_{x,(Tx)_1}(t), F_{y,(Ty)_1}(t)\}, t \geq 0.$$

Suppose further that for every $x, y \in E$ and $\{u_x\} \subset Tx$, there exists $\{v_y\} \subset Ty$ such that

$$F_{u_x, v_y}(t) = F_{Tx, Ty}(t), \quad t \geq 0.$$

Then there exists an $x_* \in E$ such that

$$\{x_*\} \subset Tx_*.$$

THEOREM 2.6. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left continuous t-norm Δ satisfying $\Delta(t,t) \geq t$, $t \in [0,1]$, and $T : E \to W(E)$ a fuzzy mapping. Suppose that there exists a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfying the condition (Φ) such that for any $x, y \in E$*

$$\begin{aligned} &F_{Tx,Ty}(\phi(t)) \\ &\geq \min\{F_{x,y}(t), F_{x,(Tx)_1}(t), F_{y,(Ty)_1}(t), F_{x,(Ty)_1}(2t), F_{y,(Tx)_1}(2t)\}, \\ &\qquad t \geq 0. \end{aligned}$$

Suppose further that for every $x, y \in E$ and $\{u_x\} \subset Tx$, there exists $\{v_y\} \subset Ty$ such that

$$F_{u_x,v_y}(t) = F_{Tx,Ty}(t), \quad t \geq 0.$$

Then there exists an $x_* \in E$ such that

$$\{x_*\} \subset Tx_*.$$

THEOREM 2.7. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ, and $T : E \to W(E)$ a fuzzy mapping. Suppose that there exists a constant $k \in (0,1)$ such that for any $x, y \in E$,*

$$F_{Tx,Ty}(kt) \geq \min\{F_{x,y}(t), F_{x,(Tx)_1}(t), F_{y,(Ty)_1}(t)\}, t \geq 0.$$

Suppose further that for every $x, y \in E$ and $\{u_x\} \subset Tx$, there exists $\{v_y\} \subset Ty$ such that

$$F_{u_x,v_y}(t) = F_{Tx,Ty}(t), \quad t \geq 0.$$

Then there exists an $x_* \in E$ such that

$$\{x_*\} \subset Tx_*.$$

THEOREM 2.8. *Let $(E, \mathcal{F}, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ satisfying $\Delta(t,t) \geq t, t \in [0,1]$, and $T : E \to W(E)$ a fuzzy mapping. Suppose that there exists a constant $k \in (0,1)$ such that for any $x, y \in E$*

$$F_{Tx,Ty}(kt) \geq \min\{F_{x,y}(t), F_{x,(Tx)_1}(t), F_{y,(Ty)_1}(t), F_{x,(Ty)_1}(2t), F_{y,(Tx)_1}(2t)\}, \quad t \geq 0.$$

Suppose further that for every $x, y \in E$ and $\{u_x\} \subset Tx$, there exists $\{v_y\} \subset Ty$ such that

$$F_{u_x,v_y}(t) = F_{Tx,Ty}(t), \quad t \geq 0$$

Then there exists an $x_* \in E$ such that

$$\{x_*\} \subset Tx_*.$$

From Theorem 2.5 we have the following fixed point theorem.

THEOREM 2.9. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ, and $T : E \to W(E)$ a fuzzy mapping. Suppose that there is a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the condition (Φ) such that for any $x, y \in E$,*

$$F_{Tx,Ty}(\psi(t)) \geq F_{x,y}(t), \quad t \geq 0.$$

Suppose further that for every $x, y \in E$ and $\{u_x\} \subset Tx$, there exists $\{v_y\} \subset Ty$ such that

$$F_{u_x,v_y}(t) = F_{Tx,Ty}(t), \quad t \geq 0$$

Then there exists an $x_* \in E$ such that

$$\{x_*\} \subset Tx_*.$$

Now we obtain the following common fixed point theorem for a sequence of nonempty closed-valued mappings $(f_i)_{i=1}^{\infty} : E \to C(E)$ in PM-spaces. This theorem generalizes Corollary 2.5 in [8].

COROLLARY 2.10. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ and $(f_i)_{i=1}^{\infty} : E \to C(E)$ a sequence of nonempty closed-valued mappings. Suppose that there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the condition (Φ) such that for any $i, j \in \mathbb{N}$ and any $x, y \in E$,*

$$F_{f_i x, f_j y}(\phi(t)) \geq \min\{F_{x,y}(t), F_{x,f_i x}(t), F_{y,f_j y}(t)\}.$$

Suppose further that for any $x, y \in E$ and $u_x \in f_i x$, there exists $v_y \in f_j y$ such that for every $t \geq 0$

$$F_{u_x,v_y}(t) = F_{f_i x, f_j y}(t).$$

Then there exists an $x_* \in E$ such that

$$x_* \in f_i x_*, \quad i \in \mathbb{N}.$$

PROOF. Define $T_i : E \to W(E)$ by $T_i x = \chi_{f_i x}$, then $(T_i x)_1 = f_i x, i \in \mathbb{N}$. Thus we have

$$\begin{aligned} F_{T_i x, T_i y}(\phi(t)) &= F_{\chi_{f_i x}, \chi_{f_j y}}(\phi(t)) \\ &= \inf_{\alpha \in (0,1]} F_{(\chi_{f_i x})_\alpha, (\chi_{f_j y})_\alpha}(\phi(t)) \\ &= \inf_{\alpha \in (0,1]} F_{f_i x, f_j y}(\phi(t)) \\ &= F_{f_i x, f_j y}(\phi(t)), \quad t \geq 0. \end{aligned}$$

Hence

$$\begin{aligned} \text{(i)} \quad F_{T_i x, T_j y}(\phi(t)) &= F_{f_i x, f_j y}(\phi(t)) \\ &\geq \min\{F_{x,y}(t), F_{x,f_i x}(t), F_{y,f_j y}(t)\}, \\ &= \min\{F_{x,y}(t), F_{x,(T_i x)_1}(t), F_{y,(T_j y)_1}(t)\}. \\ \text{(ii)} \quad F_{u_x,v_y}(t) &= F_{f_i x, f_j y}(t) = F_{T_i x T_j y}(t), \quad t \geq 0. \\ \text{(iii)} \quad \{u_x\} \subset \chi_{f_i x} &= T_i x \quad \text{and } \{v_y\} \subset \chi_{f_j y} = T_j y. \end{aligned}$$

From Theorem 2.1 there exists an $x_* \in E$ such that $\{x_*\} \subset T_i x_* = \chi_{f_i x_*}$, that is,

$$x_* \in f_i x_*, \quad i \in \mathbb{N}.$$

Putting $\phi(t) = kt$, $0 < k < 1$ in Corollary 2.10, we have the following result which is also a direct consequence of Theorem 2.3 to the case of sequences of nonempty closed-valued mappings $(f_i)_{i=1}^{\infty} : E \to C(E)$. This theorem generalizes and improves Theorem 2.1 in [2] and Corollary 2.5 in [8].

COROLLARY 2.11. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ and $(f_i)_{i=1}^{\infty} : E \to C(E)$ a sequence of nonempty closed-valued mappings. Suppose that there exists a constant $k \in (0,1)$ such that for any $x, y \in E$*

$$F_{f_i x, f_j y}(kt) \geq \min\{F_{x,y}(t), F_{x,f_i x}(t), F_{y,f_j y}(t)\}.$$

Suppose further that for any $x, y \in E$ and $u_x \in f_i x$, there exists $v_y \in f_j y$ such that for every $t \geq 0$

$$F_{u_x, v_y}(t) = F_{f_i x, f_j y}(t).$$

Then $(f_i)_{i=1}^{\infty}$ has a common fixed point.

Putting $i = j$ in Corollary 2.10, we have the following theorem which generalizes Corollary 2.6 in [8].

COROLLARY 2.12. *Let $(E, \Im, \Delta)$ be a τ-complete Menger PM-space with a left-continuous t-norm Δ and $f : E \to C(E)$ a nonempty closed-valued mapping. Suppose that there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the condition (Φ) such that for any $x, y \in E$*

$$F_{fx,fy}(\phi(t)) \geq \min\{F_{x,y}(t), F_{x,fx}(t), F_{y,fy}(t)\}.$$

Suppose further that for any $x, y \in E$ and $u_x \in fx$, there exists $v_y \in fy$ such that for every $t \geq 0$

$$F_{u_x, v_y}(t) = F_{fx,fy}(t),$$

then f has a fixed point.

3. Applications

In this section we study the existence of fixed points for fuzzy mappings in a metric space (E, d) using the results in section 2, and then we obtain similar result for multi-valued mappings as corollary.

THEOREM 3.1. *Let (E, d) be a complete metric space and $(T_i)_{i=1}^{\infty}$ a sequence of fuzzy mappings from (E, d) to $(W(E), D)$. Suppose that there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the condition (Φ) such that for any $i, j \in \mathbb{N}$ and $x, y \in E$*

$$D(T_i x, T_j y) \leq \frac{\phi(t)}{t} \max\{d(x, y), d(x, (T_i x)_1), d(y, (T_j y)_1)\}, \quad t > 0.$$

Suppose further that for any $x, y \in E$ and $u_x \in T_i x$, there exists $v_y \in T_j y$ such that for every $t > 0$

$$F_{u_x, v_y}(t) = F_{T_i x, T_j y}(t).$$

Then there exists an $x_* \in E$ such that

$$\{x_*\} \subset \bigcap_{i=1}^{\infty} T_i x_*.$$

PROOF. First we define $\Im : E \times E \to D^+$ by

$$F_{x,y}(t) = H(t - d(x, y)), \quad x, y \in E. \tag{3.2}$$

Then the space $(E, \Im, \min)$ with a t-norm $\Delta = \min$ is a τ-complete Menger PM-space and the topology induced by the metric d coincides with the topology τ. And it is easily proved that

$$F_{x,K}(t) = H(t - d(x, K)), \quad x \in E, \quad K \in C(E)$$

and

$$F_{K,C}(t) = H(t - d_H(K, C)), \quad K, \quad C \in C(E).$$

Then for any $x, y \in E$, and $i, j \in \mathbb{N}$ we have

$$\begin{aligned}
F_{T_i x, T_j y}(\phi(t)) &= \inf_{\alpha\in(0,1]} F_{(T_i x)_\alpha,(T_j y)_\alpha}(\phi(t)) \\
&= \inf_{\alpha\in(0,1]} H(\phi(t) - d_H((T_i x)_\alpha, (T_j y)_\alpha)) \\
&= H(\phi(t) - \sup_{\alpha\in(0,1]} d_H((T_i x)_\alpha, (T_j y)_\alpha)) \\
&= H(\phi(t) - D(T_i x, T_j y)) \\
&\geq H[\phi(t) - \frac{\phi(t)}{t}\max\{d(x,y), d(x,(T_i x)_1), d(y,(T_j y)_1)\}] \\
&= H[t - \max\{d(x,y), d(x,(T_i x)_1), d(y,(T_j y)_1)\}] \\
&= \min\{F_{x,y}(t), F_{x,(T_i x)_1}(t), F_{y,(T_j y)_1}(t)\}, \quad t > 0.
\end{aligned}$$

Thus Theorem 3.1 follows from Theorem 2.1 immediately.

COROLLARY 3.2. *Let (E, d) be a complete metric space and $(f_i)_{i=1}^{\infty} : (E, d) \to (C(E), d_H)$ a sequence of multi-valued mappings. Suppose that there exists a function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the condition (Φ) such that for any $i, j \in \mathbb{N}$ and any $x, y \in E$,*

$$d_H(f_i x, f_j y) \leq \frac{\phi(t)}{t}\max\{d(x,y), d(x, f_i x), d(y, f_j y)\}, t > 0.$$

Suppose further that for $u_x \in f_i x$, there exists $v_y \in f_j y$ such that for every $t > 0$,

$$F_{u_x, v_y}(t) = F_{f_i x, f_j y}(t).$$

Then there exists an $x_* \in E$ such that

$$x_* \in \bigcap_{i=1}^{\infty} T_i x_*.$$

PROOF. Define $T_i : (E, d) \to (W(E), D)$ by $T_i x = \chi_{f_i x}$ for all $i \in \mathbb{N}$.

Then for any $x, y \in E$,

$$
\begin{aligned}
D(T_i x, T_j y) &= D(\chi_{f_i x}, \chi_{f_j y}) \\
&= \sup_{\alpha \in (0,1]} d_H((\chi_{f_i x})_\alpha, (\chi_{f_j y})_\alpha) \\
&= d_H(f_i x, f_j y) \\
&\leq \frac{\phi(t)}{t} \max\{d(x,y), d(x, f_i x), d(y, f_j y)\} \\
&= \frac{\phi(t)}{t} \max\{d(x,y), d(x, (T_i x)_1), d(y, (T_j y)_1)\}, \quad t > 0
\end{aligned}
$$

Therefore by Theorem 3.1 there exists a point $x_* \in E$ such that

$$\{x_*\} \subset T_i x_* = \chi_{f_i x_*} \quad \text{for all } i \in \mathbb{N},$$

i.e., $x_* \in f_i x_*$ for all $i \in \mathbb{N}$.

Remark. (i) Theorem 3.1 and Corollary 3.2 generalizes Theorem 3.1 and Corollary 3.2 in [8] respectively.

(ii) Putting $\phi(t) = kt$, $0 < k < 1$, in Theorem 3.1 and Corollary 3.2, we also can obtain other common fixed point theorems.

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