Comm. Korean Math. Soc. 11 (1996), No. 1, pp. 117–129

NONLINAER OPERATOR EQUATIONS WITH PROBABILISTIC CONTRACTORS IN PROBABILISTIC NORMED SPACES

S. S. Chang, Y. J. Cho, J. S. Jung and F. Wang

Abstract. In this paper, some existence theorems of solutions for nonlinear operator equations with probabilistic contractor in probabilistic normed spaces are given.

1. Introduction and preliminaries

Recently, the authors in [3] introduced the concept of probabilistic contractor in probabilistic normed spaces (briefly, PN-spaces) and showed the existence and uniqueness problems of solutions for set-valued and single-valued nonlinear operator equations in PN-spaces.

The purpose of this paper is to show the existence and uniqueness problems of solutions for set-valued and single-valued nonlinear operator equations with probabilistic contractor in PN-spaces. The results presented in this paper extend and improve the corresponding results of [1]-[7].

Throughout this paper, let $R = (-\infty, +\infty)$ and $R^+ = [0, +\infty)$. We denote $\mathcal{D}$ and H by the set of all distribution functions and the special distribution function defined by

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0, \end{cases}$$

respectively.

Received June 16, 1995. Revised December 28, 1995.

1991 AMS Subject Classification: 47H10, 54H25.

Key words and phrases: Menger probabilistic normed space, t-norm of h-type, probabilistic contractor.

A function $\triangle : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

(T-1) $\triangle(a,1) = a$ and $\triangle(0,0) = 0$,

(T-2) $\triangle(a,b) \le \triangle(c,d)$ for $a \le c$ and $b \le d$,

(T-3) $\triangle(a,b) = \triangle(b,a)$,

(T-4) $\triangle(\triangle(a,b),c) = \triangle(a, \triangle(b,c))$.

A triplet $(X, \mathcal{F}, \triangle)$ is called a Menger probabilistic normed space (briefly, a Menger PN-space) if X is a real vector space, $\mathcal{F}$ is a mapping from X into $\mathcal{D}$ (for $x \in X$, the distribution function $\mathcal{F}(x)$ is denoted by F_x and $F_x(t)$ is the value of F_x at $t \in R$) and the t-norm $\triangle$ satisfies the following conditions:

(PN-1) $F_x(0) = 0$,

(PN-2) $F_x(t) = H(t)$ for all $t \ge 0$ if and only if $x = 0$,

(PN-3) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for all $\alpha \in R$, $\alpha \neq 0$,

(PN-4) $F_{x+y}(t_1 + t_2) \ge \triangle(F_x(t_1), F_y(t_2))$ for all $x, y \in X$ and $t_1, t_2 \in R^+$.

A non-Archimedean Menger probabilistic normed space (briefly, a N.A. Menger PN-space) is a triplet $(X, \mathcal{F}, \triangle)$, where $(X, \mathcal{F}, \triangle)$ is a Menger PN-space and the t-norm $\triangle$ satisfies the following condition instead of (PN-4):

(PN-5) $F_{x+y}(\max\{t_1, t_2\}) \ge \triangle(F_x(t_1), F_y(t_2))$ for all $x, y \in X$ and $t_1, t_2 \in R^+$.

Note that if $(X, \mathcal{F}, \triangle)$ is a Menger PN-space with the t-norm $\triangle$ satisfying the following condition:

$$\sup_{0<t<1} \triangle(t,t) = 1,$$

then $(X, \mathcal{F}, \triangle)$ is a real metrizable Hausdorff topological vector space with the topology τ induced by the family of neighborhoods,

$$\{U_y(\epsilon, \lambda) : y \in X, \epsilon > 0, \lambda > 0\}. \tag{1.1}$$

where $U_y(\epsilon, \lambda) = \{x \in X : F_{x-y}(\epsilon) > 1 - \lambda\}$.

In the sequel, we always assume that the t-norm $\triangle$ is continuous.

Let $(E, \mathcal{F}, \triangle)$ be a Menger PN-space and Ω_E be a family of all nonempty τ-closed, probabilistically bounded subsets of E. For any given $A, B \in \Omega_E$, define the distribution functions $F_{A,B}$, F_A as follows, respectively:

$$F_{A,B}(t) = \sup_{s<t} \triangle(\inf_{a\in A} \sup_{b\in B} F_{a-b}(s), \inf_{b\in B} \sup_{a\in A} F_{a-b}(s)),$$
$$F_A(t) = \sup_{s<t} \sup_{a\in A} F_a(s), \quad t \in R.$$

From the definitions of $F_{A,B}(t)$ and $F_A(t)$, we have the following:

LEMMA 1. *Let $(E, \mathcal{F}, \triangle)$ be a Menger PN-space and $A \in \Omega_E$. Then*

(i) $F_A(0) = 0$,
(ii) $F_A(t) = 1$ *for all* $t > 0$ *if and only if* $\theta \in A$,
(iii) $F_{\lambda A}(t) = F_A(\frac{t}{|\lambda|})$ *for all* $\lambda \in R$, $\lambda \neq 0$,
(iv) *for any* $A, B \in \Omega_E$, *and* $\theta \in B$, $F_A(t) \geq F_{A,B}(t)$, $t \in R$,
(v) *for any* $A \in \Omega_E$ *and* $x \in E$, $F_{A+x}(t_1 + t_2) \geq \triangle(F_x(t_1), F_A(t_2))$ *for all* $t_1, t_2 \geq 0$.
(vi) $F_A(t_1 + t_2) \geq \triangle(F_B(t_1), F_{A,B}(t_2))$ *for all* $t_1, t_2 \geq 0$ *and* $B \in \Omega_E$.

Recall that a sequence $\{x_n\}$ in E is convergent to a point x in E in the topology τ (denoted by $x_n \xrightarrow{\tau} x$) if

$$\lim_{n\to\infty} F_{x_n - x}(t) = H(t), \quad t \geq 0.$$

A sequence $\{x_n\}$ in E is called a τ-Cauchy sequence in E if

$$\lim_{n,m\to\infty} F_{x_n - x_m}(t) = H(t), \quad t \geq 0.$$

The space E is said to be τ-compete if every τ-Cauchy sequence in E converges to a point in the topology τ.

DEFINITION 1. Let $(X, \tilde{\mathcal{F}}, \triangle)$ and $(Y, \mathcal{F}, \triangle)$ be two Menger PN-spaces. Let τ_1 and τ_2 be the topologies induced by the family of neighborhoods of type (1.1) on $(X, \tilde{\mathcal{F}}, \triangle)$ and $(Y, \mathcal{F}, \triangle)$, respectively. A set-valued mapping $P : D(P) \subset X \to \Omega_Y$ (resp., a single-valued mapping $P : D(P) \subset X \to Y$) is said t o be τ-continuous if for any $x_0 \in D(P)$, whenever $x_n \to x_0 \in D(P)$, we have

$$\lim_{n\to\infty} F_{P(x_n),P(x_0)}(t) = 1, \quad t > 0 \quad (\text{resp., } P(x_n) \to P(x_0) \text{ as } n \to \infty).$$

REMARK 1. It follows from Lemma 1 (vi) that if P is τ-continuous, then for any sequence $\{x_n\}$ in $D(P)$ with $x_n \to x \in D(P)$, we have

$$\lim_{n\to\infty} F_{P(x_n)}(t) = F_{P(x)}(t), \quad t \in R.$$

DEFINITION 2. A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called to satisfy the condition (Φ) if φ is nondecreasing, $\varphi(0) = 0$ and

$$\lim_{n\to\infty} \varphi^n(t) = +\infty, \quad t > 0. \tag{1.2}$$

REMARK 2. From the proof of Lemma 9.3.5 in [4], we can prove that if φ satisfies the condition (Φ), then $\varphi(t) > t$ for all $t > 0$.

DEFINITION 3. A t-norm $\Delta : [0,1] \times [0,1] \to [0,1]$ is said to be of a h-type if for any $\lambda \in (0,1)$, there exists a number $\delta(\lambda) \in (0,1)$ such that, as $t > \delta(\lambda)$, the following holds uniformly:

$$\Delta^k(t) > 1 - \lambda, \quad k \geq 1,$$

where $\Delta^m(\cdot) : [0,1] \to [0,1], \Delta^1(t) = \Delta(t,t)$ and

$$\Delta^m = \Delta(t, \Delta^{m-1}(t)) = \Delta(\Delta^{m-1}(t), t)$$

for all $t \in (0,1)$ and $m = 2, 3, \cdots$.

LEMMA 2. [3] *Let F_1 and F_2 be two distribution functions with $F_1(0) = F_2(0) = 0$ and $\varphi : [0, +\infty) \to [0, +\infty)$ satisfy the condition (Φ) and the following condition:*

$$F_1(\varphi(t)) \geq \min\{F_1(\varphi(t)), F_2(\varphi(t))\}, \quad t \geq 0.$$

Then we have $F_2(\varphi(t)) \leq F_1(\varphi(t))$ for all $t \geq 0$.

2. Main results

In this section, we assume that $(X, \tilde{\mathcal{F}}, \triangle)$ is a τ_1-complete N.A. PN-spaces, $(Y, \mathcal{F}, \triangle)$ is a Menger PN-space, $\triangle$ is a t-norm of h-type and Ω_Y is a family of all nonempty τ_2-closed and probabilistically bounded subsets of Y.

Let $P_i : D \subset X \to \Omega_Y$ (resp., $P_i : D \to Y$), $i \in Z^+$, the set of all positive integers, be a set-valued mapping (resp., a single-valued mapping) and $\Gamma_i : X \to S(Y, X)$ (: the set of all operators from Y to X). Let u be a given point in Y and $\varphi : [0, +\infty) \to [0, +\infty)$ be a function satisfying the condition (Φ). Then $\{\Gamma_i\}_{i \in Z^+}$ is called a probabilistic contractor sequence of $\{P_i\}_{i \in Z^+}$ with respect to u if for any $x \in D$, $x + \Gamma_i(x) \in D$, $y \in Y$, $i, j \in Z^+$, and $t \geq 0$, we have

$$\begin{aligned} F_{P_i(x+\Gamma_i(x)y), P_j(x)+y}(t) \geq \min\{&F_y(\varphi(t)), F_{P_j(x)-u}(\varphi(t)), \\ &F_{P_i(x+\Gamma_i(x)y)-u}(\varphi(t)), \\ &F_{P_j(x)+y-u}(\varphi(t))\} \end{aligned} \tag{2.1}$$

$$\begin{aligned} (\text{resp., } F_{P_i(x+\Gamma_i(x)y)-P_j(x)-y}(t) \geq \min\{&F_y(\varphi(t)), F_{P_j(x)-u}(\varphi(t)), \\ &F_{P_i(x+\Gamma_i(x)y)-u}(\varphi(t)), \\ &F_{P_j(x)+y-u}(\varphi(t))\}). \end{aligned} \tag{2.2}$$

REMARK 3. It follows from Lemma 1 (v) that if $\triangle(t, t) \geq t$ for all $t \in [0, 1]$, then (2.1) is equivalent to the following:

$$\begin{aligned} F_{P_i(x+\Gamma_i(x)y), P_j(x)+y}(t) \geq \min\{&F_y(\varphi(t)), F_{P_j(x)-u}(\varphi(t)), \\ &F_{P_i(x+\Gamma_i(x)y)-u}(\varphi(t)), F_{P_j(x)+y-u}(\varphi(t)), \\ &F_{P_j(x-y-u}(2\varphi(t)), F_{P_i(x+\Gamma_i(x)y+y-u}(2\varphi(t)), \\ &F_{P_i(x+\Gamma_i(x)y)-y-u}(2\varphi(t))\}. \end{aligned}$$

For the single-valued mapping, we have the similar inequality.

ow we show the existence and uniqueness of solutions for the sequence of set-valued nonlinear operator equations

$$u \in P_i(x), \quad i \in Z^+. \tag{2.3}$$

THEOREM 1. *Let $P_i : D \subset X \to \Omega_Y$, $i \in Z^+$, be τ-continuous set-valued mappings, D be a τ_1-closed set and $\Gamma_i : X \to S(Y, X)$ satisfy the following conditions:*

(i) *$x + \Gamma_i(x)y \in D$ for all $x \in D$ and $y \in Y$,*

(ii) *$\{\Gamma_i\}_{i\in Z^+}$ is a probabilistic contractor sequence of $\{P_i\}_{i\in Z^+}$ with respect to u, i.e., it satisfies the condition (2.1),*

(iii) *there exists a nondecreasing function $g(t) : [0, +\infty) \to [0, +\infty)$, $g(0) = 0$, $g(t) > 0$ for all $t > 0$ such that for all $x \in D$, $y \in Y$ and $t \geq 0$,*

$$\tilde{F}_{\Gamma_i(x)y}(t) \geq F_y(g(t)),$$

(iv) *for any $A, B \in \Omega_Y$ and $a \in A$, there exists a point $b \in B$ such that*

$$F_{a-b}(t) \geq F_{A,B}(t), \quad t \geq 0.$$

Then the nonlinear set-valued operator equations (2.3) have a solution in D.

PROOF. 1. The case of $u = \theta$: Then (2.1) can be written as follows:

$$\begin{aligned} F_{P_i(x+\Gamma_i(x)y),P_j(x)+y}(t) \geq \min\{&F_y(\varphi(t)), F_{P_j(x)}(\varphi(t)), \\ &F_{P_i(x+\Gamma_i(x)y)}(\varphi(t)), \\ &F_{P_j(x)+y}(\varphi(t))\}. \end{aligned} \tag{2.4}$$

For any given $x_1 \in D$, taking $y_1 \in P_1(x_1)$ and letting $x_2 = x_1 + \Gamma_2(x_1)(-y_1)$, by the condition (i), we have $x_2 \in D$. Taking $i = 2, j = 1$ in (2.4) and replacing x and y by x_1 and $-y_1$, respectively, from (iv) of Lemma 1 and $\theta \in P_i(x_1) - y_1$, we have

$$\begin{aligned} F_{P_2(x_2)}(t) &\geq F_{P_2(x_2),P_1(x_1)+(-y_1)}(t) \\ &= F_{P_2(x_1+\Gamma_i(x_1)(-y_1)),P_1(x_1)+(-y_1)}(t) \\ &\geq \min\{F_{y_1}(\varphi(t)), F_{P_1(x_1)}(\varphi(t)), F_{P_2(x_2)}(\varphi(t)), \\ &\qquad F_{P_1(x_1)-y_1}(\varphi(t))\} \\ &\geq \min\{F_{y_1}(\varphi(t)), F_{P_2(x_2)}(\varphi(t))\} \end{aligned}$$

for all $t \geq 0$. Thus, it follows from Lemma 2 that

$$F_{y_1}(\varphi(t)) \geq F_{P_2(x_2)}(\varphi(t)), \quad t \geq 0. \tag{2.5}$$

By the condition (iv), since we have $\theta \in P_1(x_1) - y_1$, there exists a point $y_2 \in P_2(x_2)$ such that

$$F_{y_2}(t) \geq F_{P_2(x_2), P_1(x_1)-y_1}(t), \quad t \geq 0.$$

In view of (2.4), (2.5) and the preceding inequality, we have

$$F_{y_2}(t) \geq F_{y_1}(\varphi(t)), \quad t \geq 0.$$

Again letting $x_3 = x_2 + \Gamma_3(x_2)(-y_2)$, then it follows from the condition (i) that $x_3 \in D$. Taking $i = 3$, $j = 2$ in (2.4) and replacing x and y by x_2 and $-y_2$, respectively, from the similar method as stated above, that there exists a point $y_3 \in P_3(x_3)$ such that

$$F_{y_3}(t) \geq F_{y_2}(\varphi(t)) \geq F_{y_1}(\varphi^2(t)), \quad t \geq 0.$$

Inductively, we can obtain two sequences $\{x_n\}$ in D and $\{y_n\}$ in Y such that

$$x_{n+1} = x_n + \Gamma_{n+1}(x_n)(-y_n), \tag{2.6}$$

$$y_n \in P_n(x_n), \tag{2.7}$$

$$F_{y_{n+1}}(t) \geq F_{y_1}(\varphi^n(t)) \tag{2.8}$$

for all $t \geq 0$. From the conditions (iii), (2.6) and (2.7), we have

$$\tilde{F}_{x_{n+1}-x_n}(t) \geq \tilde{F}_{\Gamma_{n+1}(x_n)(-y_n)}(t) \geq F_{y_n}(g(t)) \geq F_{y_1}(\varphi^{n-1}(g(t)))$$

for all $t \geq 0$. Since $(X, \mathcal{F}, \Delta)$ is a τ_1-complete N.A. PN-space and $\varphi(t) > t$ for all $t > 0$, by the same method as stated in the proof Theorem 1 of [2], we can prove that for any $m, n \in Z^+$, $m > n$.

$$\tilde{F}_{x_m - x_n}(t) \geq \Delta^{m-n-1}(F_{y_1}(\varphi^{n-1}(g(t))))$$

for all $t \geq 0$. Since Δ is a t-norm of h-type and for all $t > 0$, $\varphi^{n+1}(g(t)) \to \infty$ as $n \to \infty$, it follows that $\{x_n\}$ is a τ_1-Cauchy sequence in D. Let $x_n \xrightarrow{\tau_1} x^*$. Since D is τ_1-closed, $x^* \in D$.

Next we prove that for any $k \in Z^+$, $\theta \in P_k(x^*)$.

In fact, from (2.8) and (1.1), we know that $y_n \xrightarrow{\tau_2} \theta$. Since $\{x_n\}$ is a sequence in D, $x_n \xrightarrow{\tau_1} x^* \in D$ and P_k is τ-continuous, from (v) of Lemma 1, it is easy to deduce that

$$(2.9)\qquad F_{P_k(x^*)}(t) = \lim_{n\to\infty} F_{P_k(x_n)-y_n}(t) = \lim_{n\to\infty} F_{P_k(x_n)-y_n-y_{n+1}}(t)$$

for all $t \geq 0$. Since $y_{n+1} \in P_{n+1}(x_{n+1})$, we have

$$F_{(P_n(x_n)-y_n)-y_{n+1}}(t) \geq F_{P_{n+1}(x_{n+1}),P_n(x_n)-y_n}(t).$$

Taking$i = n+1$, $j = k$ in (2.4) and replacing x and y by x_n and $-y_n$, respectively, from the above inequality, we obtain

$$\begin{aligned} & F_{P_k(x_n)-y_n-y_{n+1}}(t) \\ & \geq F_{P_{n+1}(x_n+\Gamma_{n+1}(x_n)(-y_n)),P_k(x_n)+(-y_n)}(t) \\ & \geq \min\{F_{y_n}(\varphi(t)), F_{P_k(x_n)}(\varphi(t)), F_{P_{n+1}(x_{n+1})}(\varphi(t)), F_{P_k(x_n)-y_n}(\varphi(t))\} \\ & \geq \min\{F_{y_n}(\varphi(t)), F_{P_k(x_n)}(\varphi(t)), F_{y_{n+1}}(\varphi(t)), F_{P_k(x_n)-y_n}(\varphi(t))\} \end{aligned}$$

for all $t \geq 0$. Letting $n \to \infty$, from $y_n \xrightarrow{\tau_2} \theta$ and (2.9), we have

$$F_{P_k(x^*)}(t) \geq F_{P_k(x^*)}(\varphi(t)) \geq \cdots \geq F_{P_k(x^*)}(\varphi^m(t))$$

for all $t \geq 0$. Letting $m \to \infty$, it follows from (1.1) that $F_{P_k(x^*)}(t) = 1$ for all $t > 0$. From (ii) of Lemma 1, we have

$$\theta \in P_k(x^*), \quad k \in Z^+.$$

2. The case of $u \neq 0$: Letting $Q_i(x) = P_i(x) - u$, $x \in D$, $i \in Z^+$, then $D(Q_i) = D$, and P_i, $i \in Z^+$, satisfying the condition (2.1) is equivalent to Q_i, $i \in Z^+$, satisfying the condition (2.4). Therefore, by using the case of $u = \theta$, we can show the existence of solutions for the sequence of nonlinear set-valued operator equations

$$\theta \in Q_i(x), \quad i \in Z^+.$$

This completes the proof.

For the sequence of single-valued operator equations

$$(2.10)\qquad u = P_i(x), \quad i \in Z^+,$$

we have the following:

THEOREM 2. *Let $P_i : D \subset X \to Y$, $i \in Z^+$, be τ-continuous single-valued mappings, D be a τ_1-closed set and $\Gamma_i : X \to S(Y, X)$ satisfy the following conditions:*

(i) *$x + \Gamma_i(x)y \in D$ for any $x \in D$ and $y \in Y$,*

(ii) *$\{\Gamma_i\}_{i \in Z^+}$ is a probabilistic contractor sequence of $\{P_i\}_{i \in Z^+}$ with respect to u, i.e., it satisfies the condition (2.2),*

(iii) *there exists a nondecreasing function $g(t) : [0, \infty) \to [0, \infty)$, $g(0) = 0$, $g(t) > 0$ for all $t > 0$ such that for all $x \in D$ and $y \in Y$, the following holds:*

$$\tilde{F}_{\Gamma_i(x)y}(t) \geq F_y(g(t)), \quad t \geq 0.$$

Then the sequence of single-valued nonlinear operator equations (2.10) has a solution in D, and for any given $x_1 \in D$, the following iterative sequence:

$$x_{n+1} = x_n + \Gamma_{n+1}(x_n)(-P_n(x_n) - u)) \tag{2.11}$$

τ_1-converges to the solution x^* of the sequence of single-valued nonlinear operator equations (2.10). Especially, if there exists $n_0 \in Z^+$ such that $\Gamma_{n_0}(x^*)$ is a surjective mapping from Y to X, then x^* is the unique solution of the sequence of the equations (2.10) in D.

PROOF. From the proof of Theorem 1, we can assume that $u = \theta$. In this case, (2.2) can be written as follows:

$$\begin{aligned} F_{P_i(x+\Gamma_i(x)y)-P_j(x)-y}(t) \geq \min\{&F_y(\varphi(t)), F_{P_j(x)}(\varphi(t)), \\ &F_{P_i(x+\Gamma_i(x)y)}(\varphi(t)), \\ &F_{P_j(x)+y}(\varphi(t))\}. \end{aligned} \tag{2.12}$$

By the assumption (i) and (2.11), we have $x_n \in D$, $n = 1, 2, \cdots$. From (2.12), we have

$$\begin{aligned} F_{P_{n+1}(x_{n+1})}(t) &\geq F_{P_{n+1}(x_n+\Gamma_{n+1}(x_n)(-P_n(x_n)))-P_n(x_n)-(-P_n(x_n))}(t) \\ &\geq \min\{F_{(-P_n(x_n))}(\varphi(t)), F_{P_n(x_n)}(\varphi(t)), \\ &\qquad F_{P_{n+1}(x_{n+1})}(\varphi(t)), F_{P_n(x_n)+(-P_n(x_n))}(\varphi(t))\} \\ &\geq \min\{F_{P_n(x_n)}(\varphi(t)), F_{P_{n+1}(x_{n+1})}(\varphi(t))\} \end{aligned}$$

for all $t \geq 0$. In view of Lemma 2, we have

$$F_{P_{n+1}(x_{n+1})}(t) \geq F_{P_n(x_n)}(\varphi(t)), \quad t \geq 0.$$

By using the induction, we can prove that

$$F_{P_{n+1}(x_{n+1})}(t) \geq F_{P_1(x_1)}(\varphi^n(t)), \quad t \geq 0. \tag{2.13}$$

From the condition (iii), (2.11) and (2.13), we have

$$\tilde{F}_{x_{n+1}-x_n}(t) \geq F_{P_1(x_1)}(\varphi^{n-1}(g(t))), \quad t \geq 0.$$

Hence from (1.1) and (2.13), we have $P_n(x_n) \xrightarrow{\tau_2} \theta$. Imitating the proof of Theorem 1, we can prove that $\{x_n\}$ is a τ_1-Cauchy sequence in D. Letting $x_n \xrightarrow{\tau_1} x^* \in D$, by using the τ-continuity of P_i, we can obtain $\theta = P_i(x^*)$, $i \in Z^+$.

Next, we prove the uniqueness of x^*. Suppose that $x^{**} \in D$ is also a solution of (2.10). By the surjectivity of $\Gamma_{n_0}(x^*)$, there exists a point $y \in Y$ such that

$$x^{**} - x^* = \Gamma_{n_0}(x^*)y.$$

For any $j \in Z^+$, $j \neq n_0$, taking $i = n_0$ in (2.12), from $P_{n_0}(x^{**}) = P_j(x^*) = \theta$, we can obtain

$$\begin{aligned}
F_y(t) &= F_{P_{n_0}(x^{**})-P_j(x^*)-y}(t) \\
&= F_{P_{n_0}(x^*+\Gamma_{n_0}(x^*)y)-P_j(x^*)-y}(t) \\
&\geq \min\{F_y(\varphi(t)), F_{P_j(x^*)}(\varphi(t)), F_{P_{n_0}(x^{**})}(\varphi(t)), \\
&\qquad F_{P_j(x^*)+y}(\varphi(t))\} \\
&= F_y(\varphi(t)) \geq \cdots \geq F_y(\varphi^n(t))
\end{aligned}$$

for all $t \geq 0$. Letting $n \to \infty$, it follows from (1.1) that $F_y(t) = 1$ for all $t > 0$. By the condition (iii), we have $F_{\Gamma_{n_0}(x^*)y}(t) = 1$ for all $t > 0$. Therefore, we have $\Gamma_{n_0}(x^*)y = \theta$, i.e., $x^* = x^{**}$. This completes the proof.

3. Applications

As applications, in this section, we shall use Theorem 1 to obtain some fixed point theorems for the sequence of mappings.

THEOREM 3. *Let $(X, \mathcal{F}, \triangle)$ be a τ-complete N.A. PN-space, $\triangle$ be a t-norm of h-type. Let $\{T_i\}$ be a sequence of mappings $T_i : X \to \Omega_Y, i \in Z^+$, such that for any $x, y \in X$ and $i, j \in Z^+, i \neq j$, the following holds:*

$$(3.1)\quad F_{T_i(x),T_j(y)}(t) \geq \min\{F_{x-y}(\varphi(t)), F_{x-T_i(x)}(\varphi(t)), F_{y-T_j(y)}(\varphi(t))\}$$

for all $t \geq 0$. Suppose further that for any A, $B \in \Omega_Y$ and $a \in A$, there exists $b \in B$ such that

$$F_{a-b}(t) \geq F_{A,B}(t), \quad t \geq 0.$$

Then there exists a point $x^* \in X$ such that $x^* \in T_i(x^*), i \in Z^+$, i.e., x^* is a common fixed point of the sequence $\{T_i\}$ of set-valued mappings T_i.

PROOF. Let $P_i(x) = x - T_i(x)$ and $\Gamma_i(x) = I_X$ for all $x \in X$ and $i \in Z^+$, where I_X denotes the identity mapping on X. From (3.1), we have

$$\begin{aligned} F_{P_i(x+y),P_j(x)+y}(t) &= F_{x+y-T_i(x+y),x-T_j(x)+y}(t) \\ &= F_{T_i(x+y),T_j(x)}(t) \\ &\geq \min\{F_y(\varphi(t)), F_{P_i(x+y)}(\varphi(t)), F_{P_j(x)}(\varphi(t))\} \\ &\geq \min\{F_y(\varphi(t)), F_{P_i(x+y)}(\varphi(t)), F_{P_j(x)}(\varphi(t)), \\ &\qquad F_{P_j(x)+y}(\varphi(t))\}, \end{aligned}$$

which means that P_i, $i \in Z^+$, satisfies the condition (2.4). By Theorem 1, there exists a point $x^* \in X$ such that $\theta \in P_i(x^*)$, i.e., $x^* \in T_i(x^*), i \in Z^+$. This completes the proof.

Since for any $x \in X$, $\Gamma_i(x) = I_X$ is surjective, by the same mothod as stated in the proof of Theorem 3, we can obtain the following:

THEOREM 4. *Let $(X, \mathcal{F}, \triangle)$ be a τ-complete N.A. PN-space and $\triangle$ be a t-norm of h-type. Let $T_i : X \to X$ and $m_i : X \to Z^+, i \in Z^+$, be mappings such that for any $x, y \in X$ and $i, j \in Z^+, i \neq j$, the following holds: for all $t \geq 0$,*

$$
(3.2)\qquad \begin{aligned} F_{T_i^{m_i(x)}(x), T_j^{m_j(y)}(y)}(t) \geq \min\{&F_{x-y}(\varphi(t)), F_{x-T_i^{m_i(x)}(x)}(\varphi(t)),\\ &F_{y-T_j^{m_j(y)}(y)}(\varphi(t))\}. \end{aligned}
$$

Then the sequence $\{T_i\}$ of mappings has a unique common fixed point in X, and for any $x_1 \in X$, the iterative sequence

$$
x_{n+1} = T_n^{m_n(x_n)}(x_n), \quad n = 1, 2, \cdots,
$$

τ-converges to this fixed point.

PROOF. Let $P_i(x) = x - T_i^{m_i(x)}(x)$ for all $x \in X$ and $i \in Z^+$. Then, from (3.2), we obtain

$$
\begin{aligned}
F_{P_i(x+y), P_j(x)-y}(t) &= F_{x+y-T_i^{m_i(x+y)}(x+y)-x+T_j^{m_j(x)}(x)-y}(t) \\
&= F_{T_i^{m_i(x+y)}(x+y)-T_j^{m_j(x)}(x)}(t) \\
&\geq \min\{F_y(\varphi(t)), F_{P_i(x+y)}(\varphi(t)), F_{P_j(x)}(\varphi(t))\} \\
&\geq \min\{F_y(\varphi(t)), F_{P_i(x+y)}(\varphi(t)), F_{P_j(x)}(\varphi(t)), \\
&\qquad F_{P_j(x)+y}(\varphi(t))\}
\end{aligned}
$$

for all $t \geq 0$, which implies that P_i, $i \in Z^+$, satisfies the condition (2.12). Therefore, all the conditions in Theorem 2 are satisfied (in which $u = \theta$). The conclusion of Theorem 4 follows from Theorem 2 immediately. This completes the proof.

ACKNOWLEDGEMENTS. The Present Studies were supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1995, Project No. BSRI-95-1405.

References

1. M. Altman, *Contractors and Contractor Directions, Theory and Applications*, Marcel Dekker, New York, 1977.
2. S. S. Chang, *Probabilistic contractors and the solutions of nonlinear equations in probabilistic normed spaces*, Chinese Sci. Bull. **15** (1990), 1451-1454.
3. S. S. Chang, Y. J. Cho and F. Wang, *On the existence and uniqueness problems of solutions for set-valued and single-valued nonlinear operator equations in probabilistic normed spaces*, Internat. J. Math. Math. Sci. **17** (1994), 389-396.
4. S. S. Chang, *Fixed Point Theory and Applications*, Chongqing Publishing House, Chongqing, 1984.
5. A. C. Lee and W. J. Padgett, *Random contractors and the solutions of random nonlinear equations*, Nonlinear Analysis TMA **1** (1977), 173-185.
6. A. C. Lee and W. J. Padgett, *Random contractors with random nonlinear majorant functions*, Nonlinear Analysis TMA **3** (1979), 707-715.
7. A. C. Lee and W. J. Padgett, *Solutions of random operator equations by random stepcontractor*, Nonlinear Analysis TMA **4** (1980), 145-151.

S. S. Chang
Department of Mathemaics
Sichuan University
Chengdu, Sichuan 610064
People's Republic of China

Y. J. Cho
Department of Mathematics
Gyeongsang National University
Jinju 660-701, Korea

J. S. Jung
Department of Mathematics
Dong-A University
Pusan 604-714, Korea

F. Wang
Department of Mathematics
Jiangsu Industrial and Commercial Managerial School
Nantong, Jiangsu 226004
People's Republic of China