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A CLASS OF INFINITE SERIES SUMMABLE BY MEANS OF FRACTIONAL CALCULUS

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ABSTRACT. We show how some interesting results involving series summation and the digamma function are established by means of Riemann-Liouville operator of fractional calculus. We derive the relation

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}{}_{p+2}F_{p+1}(a_1,\ldots,a_{p+1},\lambda;b_1,\ldots,b_p,\lambda+n;x/a)$$
$$=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_{p+1})_k}{(b_1)_k\cdots(b_p)_k k!}\left(\frac{x}{a}\right)^k[\psi(\lambda+k)-\psi(\lambda-\nu+k)],$$
$$\mathrm{Re}(\lambda)>\mathrm{Re}(\nu)\geq 0$$

and explain some special cases.

1. Introduction

In [8] B. Ross evaluated

$$\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n2^n n!}=\ln 4 \tag{1}$$

by using the technique of the misnomer fractional calculus (see [6], [7] and [5]). D. Callan also happened to discover the identity (1) somewhat serendipitously in considering a probability problem as noted in [1].

In [9] B. Ross and S. L. Kalla used the same technique to obtain the following:

$$\psi(\lambda)-\psi(\lambda-\nu)=\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)},\qquad \mathrm{Re}(\lambda)>\mathrm{Re}(\nu)\geq 0 \tag{2}$$

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where Γ is the well-known gamma function and the psi function $\psi(z) = \Gamma'(z)/\Gamma(z)$ is often referred to as the digamma function. Setting $\lambda = 1$ and $\nu = 1/2$ in (2) with $\psi(1) = -\gamma$, $\psi(1/2) = -\gamma - 2\ln 2$ [3, p. 34] reduces to (1), where γ is the Euler-Mascheroni's constant.

In [4] S. L. Kalla and B. Al-Saqabi used the same technique as in (2) and generalized the relation (2) by giving the following:
(3)

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}{}_2F_1(-\mu,\lambda;\lambda+n;-x/a)$$
$$=\sum_{k=0}^{\infty}\frac{(-\mu)_k}{k!}\left(-\frac{x}{a}\right)^k[\psi(\lambda+k)-\psi(\lambda-\nu+k)],\quad \mathrm{Re}(\lambda)>\mathrm{Re}(\nu)\geq 0$$

where ${}_2F_1(\alpha,\beta;\gamma;z)$ is the Gauss' hypergeometric function. We can also observe that letting $\mu = 0$ in (3) reduces to the relation (2).

In the paper we prove the following more general functional relation
(4)

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}{}_{p+2}F_{p+1}(a_1,\ldots,a_{p+1},\lambda;b_1,\ldots,b_p,\lambda+n;x/a)$$
$$=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_{p+1})_k}{(b_1)_k\cdots(b_p)_k k!}\left(\frac{x}{a}\right)^k[\psi(\lambda+k)-\psi(\lambda-\nu+k)],$$
$$\mathrm{Re}(\lambda)>\mathrm{Re}(\nu)\geq 0$$

where ${}_pF_q$ is the generalized hypergeometric series defined as

$$(5)\qquad {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}$$

provided that the b_i are not nonpositive integers and $(\lambda)_n$ denotes the Pochhammer symbol (or the generalized factorial, since $(1)_n = n!$) defined by

$$(6)\quad (\lambda)_0=1 \qquad \text{and} \qquad (\lambda)_n=\lambda(\lambda+1)\cdots(\lambda+n-1)\ (n=1,2,3,\ldots).$$

The series (5) converges for all z if $p \leq q$, converges for $|z| < 1$ if $p = q + 1$, and diverges for all nonzero z if $p > q + 1$.

Using the elementary property of the gamma function it is sometimes convenient to write the definition (5) as the following:

$$
\begin{aligned}
&{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) \\
(7) \qquad &= \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\cdots\Gamma(a_p+k)}{\Gamma(b_1+k)\cdots\Gamma(b_q+k)} \frac{z^k}{k!}.
\end{aligned}
$$

For easy reference we also give the β-type integral

(8)

$$\int_0^x (x-t)^d t^a dt = \frac{\Gamma(d+1)\Gamma(a+1)}{\Gamma(d+a+2)} x^{d+a+1}, \quad \mathrm{Re}(d) > -1, \ \mathrm{Re}(a) > -1.$$

The integral

$$(9) \qquad \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt, \qquad \mathrm{Re}(\nu) \geq 0$$

is called the Riemann-Liouville integral of order ν and is of fundamental importance in the fractional calculus. This integral defines differentiation and integration to an arbitrary order. The operator notation which best describes this integral, invented by H. T. Davis [2], is

$$(10) \qquad {}_0D_x^{-\nu} f(x),$$

where the subscripts on D are the terminals of integration and ν is arbitrary.

2. The Functional Relation

We begin with the following relation

$$
\begin{aligned}
&\frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^{\lambda-1} {}_{p+1}F_p(a_1, \dots, a_{p+1}; b_1, \dots, b_p; t/a) dt \\
(11) \quad &= x^{\nu+\lambda-1} \frac{\Gamma(\lambda)}{\Gamma(\lambda+\nu)} {}_{p+2}F_{p+1}(a_1, \dots, a_{p+1}, \lambda; b_1, \dots, b_p, \lambda+\nu; x/a) \\
&\qquad \mathrm{Re}(\lambda) > \mathrm{Re}(\nu) \geq 0,
\end{aligned}
$$

which can readily be deduced by using the definition ${}_pF_q$ and the formula (8).

Differentiating both sides of (11) with respect to λ according to Leibniz's rule yields

$$\begin{aligned}&\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}\ln t\, t^{\lambda-1}\,{}_{p+1}F_p(a_1,\dots,a_{p+1};b_1,\dots,b_p;t/a)dt\\ =&x^{\nu+\lambda-1}\frac{\Gamma(\lambda)}{\Gamma(\lambda+\nu)}[\ln x\,{}_{p+2}F_{p+1}(a_1,\dots,a_{p+1},\lambda;b_1,\dots,b_p,\lambda+\nu;x/a)\\ &+\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_{p+1})_k(\lambda)_k}{(b_1)_k\cdots(b_p)_k(\lambda+\nu)_k k!}\left(\frac{x}{a}\right)^k\{\psi(\lambda+k)-\psi(\lambda+\nu+k)\}].\end{aligned}$$

For convenience, we denote the right-hand side above

$$H(x,\lambda,\nu,a_1,\dots,a_{p+1},b_1,\dots,b_p,a).$$

The above formula, by definition (10) of Riemann-Liouville operator, can be written as

$$\begin{aligned}&{}_0D_x^{-\nu}(x^{\lambda-1}\ln x\,{}_{p+1}F_p(a_1,\dots,a_{p+1};b_1,\dots,b_p;x/a))\\ =&H(x,\lambda,\nu,a_1,\dots,a_{p+1},b_1,\dots,b_p,a).\end{aligned}\tag{12}$$

Due to the property of analyticity and continuity at $\nu=0$, we can interchange the roles of $-\nu$ and ν (see [5], Chap. IV).

Hence, for differentiation of $x^{\lambda-1}\ln x\,{}_{p+1}F_p(a_1,\dots,a_{p+1};b_1,\dots,b_p;x/a)$ to an arbitrary order we have

$$\begin{aligned}&{}_0D_x^{\nu}(x^{\lambda-1}\ln x\,{}_{p+1}F_p(a_1,\dots,a_{p+1};b_1,\dots,b_p;x/a))\\ =&H(x,\lambda,-\nu,a_1,\dots,a_{p+1},b_1,\dots,b_p,a).\end{aligned}\tag{13}$$

We proceed to solve an integral equation of the Voltera type

$$\begin{aligned}&\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)dt\\ =&x^{\lambda-1}\ln x\,{}_{p+1}F_p(a_1,\dots,a_{p+1};b_1,\dots,b_p;x/a),\quad \mathrm{Re}(\lambda)>\mathrm{Re}(\nu)\geq 0.\end{aligned}\tag{14}$$

This is an integral equation of convolution type which may be solved by Laplace transform and we would like to solve it by the use of fractional calculus, showing the power, elegance and simplicity of the method used. By definition (10), equation (14) can be written as

$$(15)\qquad {}_0D_x^{-\nu} f(x) = x^{\lambda-1} \ln x \; {}_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; x/a).$$

Operating on both sides with ${}_0D_x^{\nu}$ leads to

$$(16)\qquad f(x) = {}_0D_x^{\nu} \left(x^{\lambda-1} \ln x \; {}_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; x/a)\right).$$

Hence, the result (13) gives us at once the solution to (14)

(17)

$$f(x) = \frac{x^{\lambda-\nu-1}\Gamma(\lambda)}{\Gamma(\lambda-\nu)}[\ln x \; {}_{p+2}F_{p+1}(a_1, \ldots, a_{p+1}, \lambda; b_1, \ldots, b_p, \lambda-\nu; x/a)$$
$$+ \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{p+1})_k (\lambda)_k}{(b_1)_k \cdots (b_p)_k (\lambda-\nu)_k k!} \left(\frac{x}{a}\right)^k \{\psi(\lambda+k) - \psi(\lambda-\nu+k)\}].$$

We verify this result by substituting (17) into (14) in terms of the argument t. Write a series expansion for $\ln t$ as follows:

$$(18)\qquad t = x + t - x = x\left(1 + \frac{t-x}{x}\right),$$

where x and t are real and $x > 0$. Then

$$(19)\qquad \ln t = \ln x + \ln\left(1 + \frac{t-x}{x}\right).$$

When $|(t-x)/x| < 1$, we can expand $\ln(1 + (t-x)/x)$ into a Taylor series expansion. Thus

$$(20)\qquad \ln t = \ln x - \sum_{n=1}^{\infty} \frac{(x-t)^n}{nx^n}$$

with the interval of convergence $0 < t \leq 2x$.

When (20) and (17) are substituted in (14) and after simplification, we obtain the desired result (4) by use of the β-type integrals (8).

3. Special Cases

Setting $p = 1$ in (4) with a_1, a_2 and b_1 replaced by α, β and γ respectively yields

$$\begin{aligned}&\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}\,{}_3F_2(\alpha,\beta,\lambda;\gamma,\lambda+n;x/a)\\ (21)\qquad &=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}\left(\frac{x}{a}\right)^k[\psi(\lambda+k)-\psi(\lambda-\nu+k)],\\ &\mathrm{Re}(\lambda)>\mathrm{Re}(\nu)\geq 0.\end{aligned}$$

Letting $\beta = \gamma$ in (21) with α and a replaced by $-\mu$ and $-a$ respectively reduces to (3).

If $\alpha \to 0$ in (21), then it becomes the result of Kalla and Ross (2).

Setting $\gamma = \lambda$ in (21) and taking the limit as $x \to a$ in the resulting equation yields

$$\begin{aligned}&\frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)\Gamma(\lambda+n-\alpha-\beta)}{n\Gamma(\lambda+n-\alpha)\Gamma(\lambda+n-\beta)}\\ (22)\qquad &=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\lambda)_k k!}[\psi(\lambda+k)-\psi(\lambda-\nu+k)],\quad \mathrm{Re}(\lambda)>\mathrm{Re}(\nu)\geq 0.\end{aligned}$$

In fact, setting $p = 0$ in (4) with a_1 and a replaced by $-\mu$ and $-a$ respectively reduces to the relation (3).

By specializing the parameters lots of special cases can be derived from our general formula (4).

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