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COEXISTENCE IN COMPETITIVE LOTKA-VOLTERRA SYSTEMS

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ABSTRACT. In this paper we consider n-species autonomous competitive Lotka-Volterra systems. We exhibit here simple algebraic criteria on the parameters which guarantee the coexistence of all the species.

1. Introduction

Consider a community of n mutually competing species modeled by the autonomous Lotka-Volterra system

$$\dot{x}_i = x_i\left(b_i - \sum_{j=1}^{n} a_{ij}x_j\right), \qquad i = 1, 2, \cdots, n, \tag{1}$$

where x_i is the population size of the ith species at time t, and $\dot{x}_i = \dfrac{dx_i}{dt}$. The mutual competition between the species dictates that $a_{ij} > 0$ for all $i \neq j$. We assume that, for each i, $b_i > 0$ and $a_{ii} > 0$, meaning that each species, in isolation, would exhibit logistic growth. As usual we restrict our attention to the closed positive cone $\overline{\mathbb{R}^n_+}$. We denote the open positive cone by $\mathbb{R}^n_+$. It is well known that for the two-species competitive Lotka-Volterra model with no fixed points in the open positive cone $\mathbb{R}^n_+$, one of the species is driven to extinction, whilst the other population stabilises at its own carrying capacity. Moreover, M.L. Zeeman[Z1] proved the following

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THEOREM 1(Z). *If system (1) satisfies the inequalities*

$$\frac{b_j}{a_{jj}} < \frac{b_i}{a_{ij}} \quad \forall i < j \quad \text{and} \quad \frac{b_j}{a_{jj}} > \frac{b_i}{a_{ij}} \quad \forall i > j,$$

then the axial fixed point

$$R_1 = (\frac{b_1}{a_{11}}, 0, 0, \cdots, 0)$$

is globally attracting on $\mathbb{R}^n_+$.

In other words, for all strictly positive initial conditions, species $x_2, x_3, \cdots, x_n$ are driven to extinction, whilst species x_1 stabilises at its own carrying capacity. Hence it is natural to consider the conditions which guarantee the coexistence of all the species. In this paper we give a slightly different simple algebraic criteria on the parameters which guarantee the coexistence of all the species.

2. Statement of Result

Our main result is the following

THEOREM 2. *If the system (1) satisfies the inequalities*

$$\frac{b_j}{a_{jj}} < \frac{b_i}{a_{ij}} \qquad \forall i \neq j, \tag{2.1}$$

then there is a unique point $p \in \mathbb{R}^n_+$ *which is globally attracting on* $\mathbb{R}^n_+$.

This means that for all strictly positive initial conditions, all the species $x_1, x_2, \cdots, x_n$ are stabilise to some positive constants $p_1, p_2, \cdots, p_n$ respectively at their own carrying capacities.

Let us first discuss the two dimensional case. The general n-dimensional case follows easily from this particular case. Now if $n = 2$, the inequalities (2.1) reduce to

$$\frac{b_1}{a_{11}} < \frac{b_2}{a_{21}}, \qquad \frac{b_2}{a_{22}} < \frac{b_1}{a_{12}}. \tag{2.2}$$

Consider the vector field $V(x) := \Big(v_1(x), v_2(x)\Big)$ corresponding to the system (1). Here

$$v_1(x) = x_1(b_1 - a_{11}x_1 - a_{12}x_2), \qquad (2.3)$$
$$v_2(x) = x_2(b_2 - a_{21}x_1 - a_{22}x_2).$$

Hence

$$v_1(x) = 0 \iff x_1(b_1 - a_{11}x_1 - a_{12}x_2) = 0 \qquad (2.4)$$
$$\iff \begin{cases} x_1 = 0 & \text{or} \\ a_{11}x_1 + a_{12}x_2 = b_1 , \end{cases}$$

$$v_2(x) = 0 \iff x_2(b_2 - a_{21}x_1 - a_{22}x_2) = 0 \qquad (2.5)$$
$$\iff \begin{cases} x_2 = 0 & \text{or} \\ a_{21}x_1 + a_{22}x_2 = b_2 . \end{cases}$$

Observe that the axial fixed points $\frac{b_1}{a_{11}}, \frac{b_2}{a_{22}}$ are in the positive part of the nullclines $b_2 - a_{21}x_1 - a_{22}x_2 = 0$ and $b_1 - a_{11}x_1 - a_{12}x_2 = 0$ respectively. Let p be the solution of the following linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases} \qquad (2.6)$$

In this situation we easily see that p is the unique globally attracting fixed point in $\mathbb{R}^2_+$ of the system (1).

3. Proof of Theorem 2

Consider the vector field $V(x) := \Big(v_1(x), v_2(x), \cdots, v_n(x)\Big)$. Here

$$v_i(x) = x_i(b_i - \sum_{j=1}^{n} a_{ij}x_j), \qquad i = 1, 2, \cdots, n.$$

We first show that under the inequalities (2.1) V has a unique singular point p in $\mathbb{R}^n_+$. To do this we have only to examine the solutions of the following linear system

$$(3.1)\qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n. \end{cases}$$

Since $b_i > 0, \quad i = 1, 2, \cdots, n$, the above system (3.1) is equivalent to the following system

$$(3.2)\qquad \begin{cases} \dfrac{a_{11}}{b_1}x_1 + \dfrac{a_{12}}{b_1}x_2 + \cdots + \dfrac{a_{1n}}{b_1}x_n = 1, \\ \dfrac{a_{21}}{b_2}x_1 + \dfrac{a_{22}}{b_2}x_2 + \cdots + \dfrac{a_{2n}}{b_2}x_n = 1, \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ \dfrac{a_{n1}}{b_n}x_1 + \dfrac{a_{n2}}{b_n}x_2 + \cdots + \dfrac{a_{nn}}{b_n}x_n = 1. \end{cases}$$

We claim that the inequalities (2.1) imply that the coefficient matrix

$$A = \begin{pmatrix} \dfrac{a_{11}}{b_1} & \dfrac{a_{12}}{b_1} & \cdots & \dfrac{a_{1n}}{b_1} \\ \dfrac{a_{21}}{b_2} & \dfrac{a_{22}}{b_2} & \cdots & \dfrac{a_{2n}}{b_2} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{a_{n1}}{b_n} & \dfrac{a_{n2}}{b_n} & \cdots & \dfrac{a_{nn}}{b_n} \end{pmatrix}$$

is nonsingular. To see this, we divide the ith column vector A^i

$$\left(\frac{a_{1i}}{b_1}, \frac{a_{2i}}{b_2}, \cdots, \frac{a_{ni}}{b_n}\right)$$

of A by $\frac{a_{ii}}{b_i}$. Then we have

$$\frac{b_i}{a_{ii}} A^i = (c_{i1}, c_{i2}, \cdots, c_{i-1\,i}, 1, c_{i+1\,i}, \cdots, c_{ni}). \tag{3.3}$$

Now the inequalities (2.1) imply that

$$0 < c_{ki} < 1, \qquad \forall k \neq i. \tag{3.4}$$

Hence the vector $\frac{b_i}{a_{ii}} A^i$ lies on the ith face of the n-dimensional unit cube. Therefore the column vectors of A are linearly independent and hence A is nonsingular. Moreover the fact that each $\frac{b_i}{a_{ii}} A^i$ lies on the ith face of the n-dimensional unit cube implies that the system (3.1) has a unique positive solution p in $\mathbb{R}^n_+$ which is a singular point of the vector field V corresponding to the system (1). Inequalities (2.1) further implies that the axial fixed points of the vector field V lies either positive or negative side of the other nullclines. Hence they are all repelling fixed points. On the other hand p is positive and lies on each of the nullclines $b_i - \sum_{j=1}^n a_{ij} x_j = 0$. Therefore p is globally attracting in $\mathbb{R}^n_+$. □

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