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CAPACITY OF A SET OF HÖLDER α-CONTINUOUS FUNCTIONS IN WIENER SPACE

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ABSTRACT. Let C_α be the collection of those members of $C[0,1]$ which are Hölder α-continuous functions on $[0,1]$. In this paper, I will show that $\mu(C_{\frac{1}{2}}) = 0$ for Wiener measure μ.

1. Introduction

Let $C[0,1]$ denote the set of all real-valued continuous functions ω in the unit interval $[0,1]$ with $\omega(0) = 0$. For $\alpha > 0$, let C_α be the collection of those members of $C[0,1]$ which are Hölder α-continuous functions on $[0,1]$. If μ is the Wiener measure on $C[0,1]$, it is well known that $\mu(C_\alpha) = 1$ for $0 < \alpha < \frac{1}{2}$ and $\mu(C_\alpha) = 0$ for $\frac{1}{2} < \alpha$ ([4][11]).

It is natural to ask oneselves what happens if $\alpha = \frac{1}{2}$. The purpose of this paper is to calculate $\mu(C_{\frac{1}{2}})$.

Let W_0 be the set of all continuous paths $\omega : [0,\infty) \to R$ vanishing at 0 with the compact uniform topology and μ be the Wiener measure on W_0. Given $n \geq 1$ and $t > 0$, set $\Delta_n(t) = \{(t_1, t_2, ..., t_n) \in R^n; 0 \leq t_1 \leq ... \leq t_n \leq t\}$ and $\Delta_n = \bigcup_{t \geq 0} \Delta_n(t)$. Given $f_1, ..., f_n \in L^2([0,\infty))$, define

$$\int_{\Delta_n(t)} f_1(t_1)...f_n(t_n) d^n\omega$$

inductively by

$$\int_{\Delta_{n+1}(t)} f_1(t_1)...f_{n+1}(t_{n+1}) d^{n+1}\omega$$
$$= \int_0^t f_{n+1}(t_{n+1}) \Big(\int_{\Delta_n(t_{n+1})} f_1(t_1)...f_n(t_n) d^n\omega \Big) d\omega(t_{n+1})$$

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Here and throughout, $d\omega(t)$-integrals are taken in the sense of *Itô*. Define $Z_0 = R$ and $Z_n = \{\int_{\Delta_n} f d^n\omega : f \in L^2(\Delta_n)\}$ for each $n \geq 1$. It can be shown that $\{Z_n\}_{n=1}^{\infty}$ are mutually orthogonal closed linear subspaces of $L^2(\mu)$ ([1][9]). The remarkable Theorem which Wiener proved is that $L^2(\mu) = \bigoplus_{n=0}^{\infty} Z_n$.

For each n, let $\Pi_n : L^2(\mu) \to Z_n$ be the orthogonal projection onto Z_n. The Ornstein-Uhlenbeck operator is a self-adjoint operator on $L^2(\mu)$ defined by $\mathcal{L} = -\sum_{n=0}^{\infty} \frac{n}{2}\Pi_n$. This linear operator $\mathcal{L}$ is a symmetric diffusion operator ([1][9]). The associated Dirichlet form is given by

$$\begin{cases} \mathcal{F} = \{u \in L^2(\mu) : \Sigma_{n=0}^{\infty} n(\Pi_n u, \Pi_n u)_{L^2} < \infty\} \\ \mathcal{E}(u,v) = \frac{1}{2}\Sigma_n(\Pi_n u, \Pi_n v)_{L^2}. \end{cases}$$

We introduce a capacity $Cap(A)$ for all subsets A of W_0 as follows; for an open set $G \subset W_0$,

$$Cap(G) = \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{F}, u \geq 1 \ \mu - a.e. \ on \ G\}$$

and for any set $E \subset W_0$,

$$Cap(E) = \inf\{Cap(G) : G \text{ is open}, E \subset G\},$$

where $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2}$, $u, v \in \mathcal{F}$. We use the term "quasi-everywhere" or "q.e." to mean "except on a subset of W_0 of capacity zero".

In this paper, I will show that $Cap(C_{\frac{1}{2}}) = 0$ and $\mu(C_{\frac{1}{2}}) = 0$.

2. Capacitary Estimates

Following P.A. Meyer [8], we introduce the space W_0' of all functions from $[0, \infty)$ to R with finite variation and compact support, and then the paring $\{\alpha, \omega\}$ defined by

$$\{\alpha, \omega\} = -\int_0^{\infty} \omega(s) d\alpha(s), \ \alpha \in W'_0, \ \omega \in W_0.$$

In what follows, we denote $\sqrt{\mathcal{E}(u,u)}$ and $\sqrt{\mathcal{E}_1(u,u)}$ by $\|u\|_{\mathcal{E}}$ and $\|u\|_{\mathcal{E}_1}$, respectively, $u \in \mathcal{F}$.

THEOREM 1. *If $\alpha_1, \alpha_2, ..., \alpha_n \in W'_0$ and $F \in C_0^\infty(\mathbb{R}^n \to R)$, then*

$$F(\{\alpha_1, \cdot\}, \{\alpha_2, \cdot\}, ..., \{\alpha_n, \cdot\}) \in \mathcal{F}$$

and

$$\|F(\{\alpha_1, \cdot\}, \{\alpha_2, \cdot\}, \cdots, \{\alpha_n, \cdot\})\|_{\mathcal{E}}^2$$
$$= \frac{1}{2} \sum_{i,j=1}^{n} E[F_{x_i}(\{\alpha_1, \cdot\}, \cdots, \{\alpha_n, \cdot\}) F_{x_j}(\{\alpha_1, \cdot\}, \cdots, \{\alpha_n, \cdot\})]$$
$$\cdot \int_0^\infty \alpha_i(s) \alpha_j(s) ds.$$

PROOF. This formula is an elementary application of the Malliavin Calculus but also follows from the chain rule for the local Dirichlet form discovered by Y.Lejan in his thesis (see[3]).

For an interval $I = (s, t) \subset R$, we put

$$X_I(\omega) = \frac{\omega(t) - \omega(s)}{\sqrt{t - s}}$$

THEOREM 2. *For disjoint intervals $I_1, I_2, ..., I_n \subset R$, we have*

$$Cap\Big(\bigcap_{i=1}^{n} \{a_i < X_{I_i} < b_i\}\Big) \le \Big(\frac{n}{2c^2} + 1\Big) \mu\Big(\bigcap_{i=1}^{n} \{a_i - c_i < X_{I_i} < b_i + c_i\}\Big)$$

where $a_i < b_i, i = 1, 2, ..., n,\ c = \min_i\ c_i > 0$.

PROOF. Fix $\varepsilon > 0$. For each $i \le 1 \le n$, choose a real valued C^∞ function F_i such that

$$\begin{cases} 0 \le F_i(x) \le 1 \\ F_i(x) = 1 \quad x \in (a_i, b_i) \\ F_i(x) = 0 \quad x \notin (a_i - c_i, b_i + c_i) \\ |F'_i(x)| \le \dfrac{1}{c - \varepsilon}\ . \end{cases}$$

Put $F(x_1, x_2, ..., x_n) = F_1(x_1)F_2(x_2)...F_n(x_n)$, $F(X_{I_1}, X_{I_2}, ..., X_{I_n})$ $(= u \in \mathcal{F})$ is equal to 1 on the open set $\bigcap_{i=1}^n \{a_i < X_{I_i} < b_i\}$. By Theorem 1,

$$\|F(X_{I_1}, X_{I_2}, ..., X_{I_n})\|_{\mathcal{E}}^2 = \frac{1}{2}\sum_{i=1}^n E[F_{x_i}(X_{I_1}, X_{I_2}, ..., X_{I_n})^2].$$

We have, by the definition of capacity,

$$\begin{aligned} Cap\Big(\bigcap_{i=1}^n a_i < X_{I_i} < b_i\Big) &\le \mathcal{E}_1(u,u) \\ &\le \Big(\frac{n}{2(c-\varepsilon)^2} + 1\Big)\mu\Big(\bigcap_{i=1}^n \{a_i - c_i < X_{I_i} < b_i + c_i\}\Big). \end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, the proof is complete.

DEFINITION 1. For $\alpha > 0$, let C_α be the collection of those members of $C[0,1]$ which are Hölder α-continuous functions on $[0,1]$, i.e. ,

$$\begin{aligned} C_\alpha = \{\omega \in C[0,1] : \exists h = h(\omega) \\ \text{such that } |\omega(t) - \omega(s)| \le h|t-s|^\alpha \text{ for } t, s \in [0,1]\} \end{aligned}$$

Let S be the collection of binary rationals in $[0,1]$. For $\alpha > 0$, let $H_\alpha = \{\omega \in C[0,1] : \forall h > 0, \exists s, t \in S \text{ such that } |\omega(t) - \omega(s)| > h|t-s|^\alpha\}$.
As immediate consequences of the above definitions, we have $H_\alpha = C[0,1] \backslash C_\alpha$.

THEOREM 3. $Cap(C_{\frac{1}{2}}) = 0$

PROOF. We choose $0 < \varepsilon < 1$, $0 < \eta < \epsilon$ and put

$$\begin{aligned} B_n = \Big\{\omega \in W_0 : \Big|\frac{\omega(k2^{-n}) - \omega((k-1)2^{-n})}{\sqrt{2^{-n}}}\Big| \\ \le (1-\epsilon)\sqrt{2n\log 2},\ k = 1, 2, .., 2^n\Big\} \end{aligned}$$

$$Cap(B_n) \le \Big(\frac{2^n}{4\eta^2 n\log 2}+1\Big)\mu\Big\{\omega\in W_0 : \Big|\frac{\omega(k2^{-n})-\omega((k-1)2^{-n})}{\sqrt{2^{-n}}}\Big| \le (1-\epsilon+\eta)\sqrt{2n\log 2}, \qquad k=1,2,3,...,2^n\Big\}$$
$$\le \Big(\frac{2^n}{4\eta^2 n\log 2}+1\Big)\exp(-2^{n(\epsilon-\eta)}).$$

The first inequality follows from Theorem 2 and the second inequality holds for sufficiently large n [7]. Since $\sum_{n=1}^{\infty}(\frac{2^n}{4\eta^2 n\log 2}+1)\exp(-2^{n(\epsilon-\eta)})$ is finite, $\sum_{n=1}^{\infty} Cap(B_n)<\infty$.

By the capacitary version of Borel Cantelli lemma[3], $Cap(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} B_n)=0$. If $\omega\notin\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} B_n$, $\omega\in H_{\frac{1}{2}}$. $Cap(C_{\frac{1}{2}})=1-Cap(H_{\frac{1}{2}})=0$.

COROLLARY 1. $\mu(C_{\frac{1}{2}})=0$

PROOF. $\mu(A)\le Cap(A)$ for any Borel set A.

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