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A CLASS OF CONDITIONAL ANALYTIC FEYNMAN INTEGRALS

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ABSTRACT. In this paper we establish the existence of the conditional Feynman integral of certain functions which are not in the Banach algebra **S** of functions on Wiener space which are a kind of stochastic Fourier transform of complex Borel measures on $L^2[a,b]$. This result is used to provide the fundamental solution for the Schrödinger equation for the forced harmonic potential.

1. Introduction

In[3] Cameron and Storvick treated a Banach algebra **S** of functions on Wiener space which are a kind of stochastic Fourier transform of complex Borel measures on $L^2[a,b]$. They established the existence of the analytic Feynman integral for all functions F in **S** and give a formula for their Feynman integrals. In [10] Johnson has shown that **S** is isometrically isomorphic to the class $\mathcal{F}(H)$ of Fresnel integrable functions as given by Albeverio and Hoegh-Krohn[1].

In[6], Chung and Skoug introduced the concept of a conditional analytic Feynman integral of a function F on Wiener space given a function X. For a certain choice of X , they established the existence of the conditional Feynman integral for all functions F in the Banach algebra **S** and used the concept of conditional Feynman integral to provide a method of getting the fundamental solution to the Schrödinger equation and to obtain the kernel of various operator valued Feynman integrals[7].

In this paper we establish the existence of the conditional Feynman integral of certain functions which are not in the Banach algebra **S**. This

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result is used to provide the fundamental solution for the Schrödinger equation for the forced harmonic potential.

2. Definitions and Preliminaries

Let $(C_0[0,T], \mathcal{B}(C_0[0,T]), m_w)$ denote Wiener space where $C_0[0,T]$ is the space of all continuous functions x on $[0,T]$ with $x(0)=0$. For the partition $\tau = \tau_n = \{t_1, t_2, ..., t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, let $X_\tau : C_0[0,T] \to \mathbf{R}^n$ be defined by $X_\tau(x) = (x(t_1), \cdots, x(t_n))$. Let F be a complex-valued ($\mathbf{C}$-valued) integrable function on $C_0[0,T]$. Let $\mathcal{F}_\tau$ be the σ-algebra generated by the set $\{X_\tau^{-1}(B) : B \in \mathcal{B}(\mathbf{R}^n)\}$. Then, by the definition of conditional expectation, the conditional expectation of F given $\mathcal{F}_\tau$, written $E[F|X_\tau]$, is any real valued $\mathcal{F}_\tau$-measurable function on $C_0[0,T]$ such that

$$\int_E F dm_w = \int_E E[F|X_\tau] dm_w \quad \text{for} \quad E \in \mathcal{F}_\tau.$$

It is well known that there exists a Borel measurable and P_{X_τ}-integrable function Ψ on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), P_{X_\tau})$ such that $E[F|X_\tau] = \Psi \circ X_\tau$, where P_{X_τ} is the probability distribution of X_τ defined by $P_{X_\tau}(A) = m_w(X_\tau^{-1}(A))$ for $A \in \mathcal{B}(\mathbf{R}^n)$. Following Yeh [16], the function $\Psi(\vec{\xi})$, written $E[F|X_\tau = \vec{\xi}]$, is called *the conditional Wiener integral of F given* X_τ.

We note that a real valued function Y on $[0,T] \times C_0[0,T]$ defined by

$$Y(t,x) \equiv y(t) = x(t) - \frac{t}{T} x(T)$$

is a pinned Wiener process on $(C_0[0,T], \mathcal{B}(C_0[0,T]), m_w)$ and $[0,T]$ with $y(0) = 0$ and $y(T) = 0$. This process $\{y(t), 0 \le t \le T\}$ induces the Gaussian measure, called the pinned Wiener measure m_p, on $C_0^0[0,T] = \{x \in C_0[0,T] | x(T) = 0\}$, which is uniquely determined by mean function $E[y(t)] = 0$ for every t $\in$ [0,T] and covariance function $k(s,t) = E[y(s)y(t)] = \min\{s,t\} - \frac{st}{T}$.

The following proposition gives a convenient formula for evaluating conditional Wiener integral of functions involving quadratic potential $V(s,\xi)$(see [4]).

PROPOSITION 2.1. *Let F be an integrable function on $C_0[0,T]$ defined by $F(x) = \exp\{\int_{t_1}^T V(s,x(s))ds\}$. Then for $0 < t_1 < T$ and $\xi, \xi_1 \in \mathbf{R}$,*

$$E[F(x)|x(t_1) = \xi_1, x(T) = \xi] = \int_{C_0^0[0,T-t_1]} F(y+g)dm_p(y),$$

where $g(t) = \dfrac{\xi - \xi_1}{T - t_1}t + \xi_1, t \in [0, T - t_1]$.

In particular, if $t_1 = 0$, then

$$E[F(x)|x(T) = \xi] = \int_{C_0^0[0,T]} F(y+h)dm_p(y)$$

where $h(t) = \frac{t}{T}\xi, t \in [0,T]$.

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable([5],[11]) provided ρE is Wiener measurable for each $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere.

DEFINITION 2.1. Let X be an $\mathbf{R}^n$-valued scale-invariant measurable function on $C_0[0,T]$ and let F be a scale-invariant measurable function on

$$\int_{C_0[0,T]} F(\lambda^{-1/2}x)dm(x)$$

exists as a finite number for all $\lambda > 0$. For $\lambda > 0$, let

$$J_\lambda(\vec{\eta}) = E(F(\lambda^{-1/2}\cdot)|X(\lambda^{-1/2}\cdot) = \vec{\eta})$$

denote the conditional Wiener integral of $F(\lambda^{-1/2}\cdot)$ given $X(\lambda^{-1/2}\cdot)$. If for almost every $\vec{\eta} \in \mathbf{R}^n$, there exists a function $J_\lambda^*(\vec{\eta})$, analytic in λ on $\mathbf{C}^+ \equiv \{\lambda \in \mathbf{C} : \text{Re } \lambda > 0\}$ such that $J_\lambda^*(\vec{\eta}) = J_\lambda(\vec{\eta})$ for all $\lambda > 0$, then J_λ^* is defined to be *the conditional analytic Wiener integral of F given X* with parameter λ and for $\lambda \in \mathbf{C}^+$ we write

$$E^{\text{anw}_\lambda}(F|X = \vec{\eta}) = J_\lambda^*(\vec{\eta}).$$

If for fixed real $q \neq 0$, the limit

$$\lim_{\lambda \to -iq} E^{\text{anw}_\lambda}(F|X = \vec{\eta})$$

exists for almost every $\vec{\eta} \in \mathbf{R}^n$ where λ approaches $-iq$ through $\mathbf{C}^+$, we will denote the value of this limit by $E^{\text{anf}_q}(F|X)$ and call it *the conditional analytic Feynman integral of F given X* with parameter q.

3. Conditional analytic Feynman integrals

Let k be the covariance function of the pinned Wiener process $\{y(t) : t \in [0,T]$, that is, k is the function on $[0,T] \times [0,T]$ defined by

$$k(s,t) = \min\{s,t\} - \frac{st}{T}. \tag{3.1}$$

Let A be the integral operator on $L^2[0,T]$ (the space of real valued square integrable function on [0,T]) defined by

$$Af(s) = \int_0^T k(s,t)f(t)dt, \quad s \in [0,T], \quad f \in L^2[0,T]. \tag{3.2}$$

Then it can be shown that the orthonormal eigen-functions $\{e_n\}$ of A and the corresponding eigen-value $\{\alpha_n\}$ are given by, respectively

$$e_n(s) = \sqrt{\frac{T}{2}}\sin(\frac{n\pi}{T}s) \qquad \text{and} \qquad \alpha_n = \frac{T^2}{n^2\pi^2}. \tag{3.3}$$

Furthermore, it can be shown that $\{e_n\}$ is a basis of $L^2[0,T]$, and that A is a trace class operator on $L^2[0,T]$. The Karhunen - Loeve theorem [2] shows that the Fourier series representation of the pinned Wiener process $\{y(t) : t \in [0,T]\}$ is given by

$$y(t) = \sum_{n=1}^{\infty} z_n e_n(t), \qquad 0 \leq t \leq T \tag{3.4}$$

where z_n's are independent Gaussian random variables with mean 0 and variance α_n.

We need the following lemmas(see [4]).

LEMMA 3.1. *For* $\alpha > 0, t \in [0,T]$,

$$(3.5)\qquad \sum_{n=1}^{\infty} \frac{T}{n^2\pi^2 + \alpha T^2} cos(\frac{n\pi}{T} t) = \frac{cosh\sqrt{\alpha}(T-t)}{2\sqrt{\alpha} sinh\sqrt{\alpha} t} - \frac{1}{2\alpha T}.$$

LEMMA 3.2. *For* $\alpha > 0$, *let*

$$R(s,t,\alpha) = \sum_{n=1}^{\infty} \frac{\alpha_n}{1+\alpha\alpha_n} e_n(s)e_n(t), \quad s,t \in [0,T]$$

where α_n *and* e_n *are as in (3.3). Then for each* $t \in [0,T]$,

$$(3.6)\qquad R(s,t,\alpha) = \begin{cases} \dfrac{sinh\sqrt{\alpha}(T-t) sinh\sqrt{\alpha} s}{\sqrt{\alpha} sinh\sqrt{\alpha} T}, & 0 \le s \le t; \\ \dfrac{sinh\sqrt{\alpha}(T-s) sinh\sqrt{\alpha} t}{\sqrt{\alpha} sinh\sqrt{\alpha} T}, & t \le s \le T. \end{cases}$$

THEOREM 3.3. *Let* F *be a measurable function on* $C_0[0,T]$ *defined by*

$$F(x) = exp\Big\{-\frac{1}{2}\alpha \int_{t_1}^{T} x^2(s)ds + \alpha\beta \int_{t_1}^{T} v(s)x(s)ds\Big\},$$

where $Re\ \alpha \ge 0$, $\beta \in \mathbf{C}$, *and* $v \in L^2[0,T]$. *Let* $X(x) = (x(t_1), x(T))$. *Then for all* $\lambda \in \mathbf{C}^+$, $E^{\mathrm{anw}_\lambda}(F|X)$ *exists and for all* $(\xi_1, \xi) \in \mathbf{R}^2$ *is given by the formula*

$$(3.7)\qquad \begin{aligned} E^{\mathrm{anw}_\lambda}[F|X = (\xi_1,\xi)] &= \left(\frac{\sqrt{\frac{\alpha}{\lambda}}(T-t_1)}{\sinh\sqrt{\frac{\alpha}{\lambda}}(T-t_1)} \right)^{\frac{1}{2}} \cdot \exp\left\{ \frac{\lambda(\xi-\xi_1)^2}{2(T-t_1)} \right\} \\ &\cdot \exp\left\{ -\frac{\sqrt{\alpha\lambda}}{2}(\xi^2 + \xi_1{}^2)\coth\sqrt{\frac{\alpha}{\lambda}}(T-t_1) + \frac{\sqrt{\alpha\lambda}\xi\xi_1}{\sinh\sqrt{\frac{\alpha}{\lambda}}(T-t_1)} \right\} \\ &\cdot \exp\left\{ \alpha\beta \int_{t_1}^{T} \frac{v(t)}{\sinh\sqrt{\frac{\alpha}{\lambda}}(T-t_1)} \Big(\xi \sinh\sqrt{\frac{\alpha}{\lambda}}(t-t_1) \right. \\ &\qquad \left. + \xi_1 \sinh\sqrt{\frac{\alpha}{\lambda}}(T-t) \Big) dt \right\} \\ &\cdot \exp\left\{ \frac{\alpha^2\beta^2}{2\lambda} \int_0^{T-t_1} \int_0^{T-t_1} R\Big(s,t,\frac{\alpha}{\lambda}\Big) v(s+t_1)v(t+t_1)dsdt \right\} \end{aligned}$$

Furthermore, $E^{\mathrm{anf}_q}(F|X)$ exists for all real $q \neq 0$ and for all $(\xi_1, \xi) \in \mathbf{R}^2$ is given by the formula

$$
\begin{aligned}
E^{\mathrm{anf}_q}[F|X=(\xi_1,\xi)] &= \left(\frac{\sqrt{\frac{\alpha i}{q}}(T-t_1)}{\sinh\sqrt{\frac{\alpha i}{q}}(T-t_1)} \right)^{\frac{1}{2}} \exp\left\{ \frac{q(\xi-\xi_1)^2}{2i(T-t_1)} \right\} \\
&\cdot \exp\left\{ -\frac{\sqrt{-iq\alpha}}{2}(\xi^2+{\xi_1}^2)\coth\sqrt{\frac{\alpha i}{q}}(T-t_1) + \frac{\sqrt{-iq\alpha}\,\xi\xi_1}{\sinh\sqrt{\frac{\alpha i}{q}}(T-t_1)} \right\} \\
&\cdot \exp\left\{ \alpha\beta \int_{t_1}^{T} \frac{v(t)}{\sinh\sqrt{\frac{\alpha i}{q}}(T-t_1)} \left(\xi \sinh\sqrt{\frac{\alpha i}{q}}(t-t_1) \right.\right. \\
&\qquad \left.\left. + \xi_1 \sinh\sqrt{\frac{\alpha i}{q}}(T-t) \right) dt \right\} \\
&\cdot \exp\left\{ \frac{\alpha^2\beta^2 i}{2q} \int_0^{T-t_1}\int_0^{T-t_1} R(s,t,\frac{\alpha i}{q}) v(s+t_1)v(t+t_1)\,ds\,dt \right\}
\end{aligned} \tag{3.8}
$$

PROOF. Note that for any $z \in \mathbf{C}$ and $\lambda > 0$, $\exp\{\frac{z}{\sqrt{\lambda}}\int_0^T v(s)x(s)ds\}$ is Wiener integrable. Hence $F(\lambda^{-\frac{1}{2}}\cdot)$ is Wiener integrable for $\lambda > 0$, $\mathrm{Re}\alpha \geq 0$ and any $\beta \in \mathbf{C}$. So $F(\lambda^{-\frac{1}{2}}\cdot)$ is conditional Wiener integrable for the given $x(t_1) = \xi_1$ and $x(T) = \xi$. By Proposition 2.1, we have, for $\xi, \xi_1 \in \mathbf{R}$ and $\lambda > 0$

$$
\begin{aligned}
& E\big[F(\lambda^{-\frac{1}{2}}\cdot)\big|\lambda^{-\frac{1}{2}}x(t_1)=\xi_1, \lambda^{-\frac{1}{2}}x(T)=\xi\big] \\
&= \int_{C_0^0[0,T-t_1]} \exp\left\{ -\frac{\alpha}{2\lambda}\int_0^{T-t_1} (y(s)+\sqrt{\lambda}g(s))^2 ds \right. \\
&\qquad \left. + \frac{\alpha\beta}{\sqrt{\lambda}} \int_0^{T-t_1} v(s+t_1)(y(s)+\sqrt{\lambda}g(s))ds \right\} dm_p(y)
\end{aligned}
$$

where $g(t) = \dfrac{\xi-\xi_1}{T-t_1}t + \xi_1, t \in [0, T-t_1]$. The previous quantity equals

(3.9)

$$
\int_{C_0^0[0,T-t_1]} \exp\left\{ -\frac{1}{2}\alpha \sum_{n=1}^{\infty} [\frac{1}{\lambda}(z_n+\sqrt{\lambda}g_n)^2 - \frac{2\beta}{\sqrt{\lambda}}v_n(z_n+\sqrt{\lambda}g_n)] \right\} dm_p
$$

where $y(t)=\sum_{n=1}^{\infty} z_n e_n(t)$ is the Fourier series representation of function y in $C_0^0[0,T-t_1]$ as in (3.4), $g(t)=\sum_{n=1}^{\infty} g_n e_n(t)$, and $v(t+t_1)=\sum_{n=1}^{\infty} v_n e_n(t)$. Since z_n's are independent Gaussian random variables with mean 0 and variance α_n, (3.10) equals

$$\prod_{n=1}^{\infty}\int_{C_0^0[0,T-t_1]}\exp\left\{-\frac{\alpha}{2\lambda}{z_n}^2+\frac{\alpha}{\sqrt{\lambda}}(\beta v_n-g_n)z_n+\alpha\beta g_n v_n-\frac{\alpha}{2}{g_n}^2\right\}dm_p$$

$$=\prod_{n=1}^{\infty}\Big[\Big\{\frac{1}{\sqrt{2\pi\alpha_n}}\int_{\mathbf{R}}\exp\Big\{-\frac{\alpha}{2\lambda}u^2+\frac{\alpha}{\sqrt{\lambda}}\omega_n u-\frac{u^2}{2\alpha_n}\Big\}du\Big\}\cdot\exp\Big\{\alpha\beta g_n v_n-\frac{\alpha}{2}g_n^2\Big\}\Big],$$

where $\omega_n=\beta v_n-g_n$. Hence (3.10) equals

$$\prod_{n=1}^{\infty}\left[\frac{1}{\sqrt{2\pi\alpha_n}}\left\{\int_{\mathbf{R}}\exp\left\{-\frac{1}{2}\Big(\frac{\alpha}{\lambda}+\frac{1}{\alpha_n}\Big)\Big(u-\frac{\sqrt{\lambda}\alpha\alpha_n\omega_n}{\alpha\alpha_n+\lambda}\Big)^2+\frac{\alpha^2\alpha_n\omega_n^2}{2(\alpha\alpha_n+\lambda)}\right\}du\right\}+\alpha\beta g_n v_n-\frac{\alpha}{2}g_n^2\right]$$

(3.10)
$$=\prod_{n=1}^{\infty}\left[\Big(1+\frac{\alpha}{\lambda}\alpha_n\Big)^{-\frac{1}{2}}\exp\left\{\frac{\alpha^2\beta^2\alpha_n v_n^2}{2(\alpha\alpha_n+\lambda)}+\frac{\alpha^2\alpha_n g_n^2}{2(\alpha\alpha_n+\lambda)}-\frac{\alpha^2\beta\alpha_n g_n v_n}{\alpha\alpha_n+\lambda}+\alpha\beta g_n v_n-\frac{\alpha}{2}g_n^2\right\}\right]$$

$$=\left[\prod_{j=1}^{\infty}(1+\frac{\alpha}{\lambda}\alpha_n)\right]^{-\frac{1}{2}}\exp\left\{\frac{\alpha^2\beta^2}{2}\sum_{n=1}^{\infty}\frac{\alpha_n}{\alpha\alpha_n+\lambda}v_n^2+\frac{\alpha^2}{2}\sum_{n=1}^{\infty}\frac{\alpha_n}{\alpha\alpha_n+\lambda}g_n^2-\alpha^2\beta\sum_{n=1}^{\infty}\frac{\alpha_n}{\alpha\alpha_n+\lambda}v_n g_n+\alpha\beta\sum_{n=1}^{\infty}g_n v_n-\frac{\alpha}{2}\sum_{n=1}^{\infty}g_n^2\right\}.$$

Using

$$\prod_{n=1}^{\infty}\Big(1-\frac{z^2}{n^2\pi^2}\Big)=\frac{\sin z}{z},\qquad \alpha_n=\frac{(T-t_1)^2}{n^2\pi^2},$$

we have

(3.11)
$$\prod_{n=1}^{\infty}\Big(1+\frac{\alpha}{\lambda}\alpha_n\Big)=\frac{\sinh\sqrt{\frac{\alpha}{\lambda}}(T-t_1)}{\sqrt{\frac{\alpha}{\lambda}}(T-t_1)}.$$

Observing that

(3.12)
$$\sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda+\alpha\alpha_n} v_n^2 = \frac{1}{\lambda}\int_0^{T-t_1}\int_0^{T-t_1} R\Big(s,t,\frac{\alpha}{\lambda}\Big) v(s+t_1)v(t+t_1)dsdt$$

(3.13)
$$\sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda+\alpha\alpha_n} g_n^2 = \frac{1}{\lambda}\int_0^{T-t_1}\int_0^{T-t_1} R\Big(s,t,\frac{\alpha}{\lambda}\Big) g(s)g(t)dsdt$$

(3.14)
$$\sum_{n=1}^{\infty} \frac{\alpha_n}{\lambda+\alpha\alpha_n} v_n g_n = \frac{1}{\lambda}\int_0^{T-t_1}\int_0^{T-t_1} R\Big(s,t,\frac{\alpha}{\lambda}\Big) v(s+t_1)g(t)dsdt$$

(3.15)
$$\sum_{n=1}^{\infty} g_n v_n = \int_0^{T-t_1} g(t)v(t+t_1)dt$$

(3.16)
$$\sum_{n=1}^{\infty} g_n^2 = \int_0^{T-t_1} g^2(t)dt$$

and using Lemmas 3.1 and 3.2 with replacing T by $T-t_1$, one can show that

$$\frac{\alpha^2}{2}\sum_{n=1}^{\infty}\frac{\alpha_n}{\alpha\alpha_n+\lambda}g_n^2 - \frac{\alpha}{2}\sum_{n=1}^{\infty} g_n^2$$

(3.17)
$$= -\frac{1}{2}\Bigg\{\sqrt{\alpha\lambda}(\xi^2+\xi_1^2)\coth\sqrt{\frac{\alpha}{\lambda}}(T-t_1) - \frac{2\sqrt{\alpha\lambda}\xi\xi_1}{\sinh\sqrt{\frac{\alpha}{\lambda}}(T-t_1)} - \lambda\frac{(\xi-\xi_1)^2}{T-t_1}\Bigg\},$$

(3.18)
$$-\alpha^2\beta\sum_{n=1}^{\infty}\frac{\alpha_n}{\lambda+\alpha\alpha_n}g_n v_n + \alpha\beta\sum_{n=1}^{\infty} g_n v_n$$
$$= \alpha\beta\int_{t_1}^{T}\left(\frac{\xi\sinh\sqrt{\frac{\alpha}{\lambda}}(t-t_1)+\xi_1\sinh\sqrt{\frac{\alpha}{\lambda}}(T-t)}{\sinh\sqrt{\frac{\alpha}{\lambda}}(T-t_1)}\right)v(t+t_1)dt.$$

Putting (3.11),(3.12),(3.17) and (3.18) in the last equation in (3.10) we obtain the formula in (3.7). But one can show the right-hand side in (3.7) is an analytic function of λ throughout $\mathbf{C}^+$ and is a continuous function of λ for $\mathrm{Re}\lambda \geq 0, \lambda \neq 0$. Hence (3.7) and (3.8) are established, which concludes the proof of theorem.

COLLOARY 3.4. *Let* $Re\,\alpha > 0$ *and* F *be an integrable function on* $C_0[0,T]$ *defined by* $F(x) = exp\{-\frac{\alpha}{2}\int_0^T x^2(s)ds + \alpha\beta\int_0^T v(s)x(s)ds\}$. *Let* $0 = t_0 < t_1 < \cdots < t_n = T$ *and* $X(t) = (x(t_1), x(t_2), \cdots, x(t_n))$. *Then we have, for* $\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbf{R}^n$ *and real* $q \neq 0$,

$$E^{\mathrm{anf}_q}[F|X = \vec{\xi}] = \prod_{k=1}^{n}\Bigg[\left(\frac{\sqrt{\frac{\alpha i}{q}}(t_k - t_{k-1})}{\sinh\sqrt{\frac{\alpha i}{q}}(t_k - t_{k-1})}\right)^{\frac{1}{2}} \cdot \exp\left\{\frac{q(\xi_k - \xi_{k-1})^2}{2i(t_k - t_{k-1})}\right\}$$

$$\cdot \exp\left\{-\frac{\sqrt{-iq\alpha}}{2}(\xi_k^2 + \xi_{k-1}^2)\coth\sqrt{\frac{\alpha i}{q}}(t_k - t_{k-1}) + \frac{\sqrt{-iq\alpha}\,\xi_k\xi_{k-1}}{\sinh\sqrt{\frac{\alpha i}{q}}(t_k - t_{k-1})}\right\}$$

$$\cdot \exp\left\{\alpha\beta\int_{t_{k-1}}^{t_k}\frac{v(t)}{\sinh\sqrt{\frac{\alpha i}{q}}(t_k - t_{k-1})}\left(\xi_k \sinh\sqrt{\frac{\alpha i}{q}}(t - t_{k-1})\right)\right.$$

$$\left. + \xi_{k-1}\sinh\sqrt{\frac{\alpha i}{q}}(t_k - t))dt\right\}$$

$$\cdot \exp\left\{\frac{\alpha^2\beta^2 i}{2q}\int_0^{t_k - t_{k-1}}\int_0^{t_k - t_{k-1}} R(s, t, \frac{\alpha i}{q})v(s + t_{k-1})v(t + t_{k-1})dsdt\right\}\Bigg],$$

where $t_0 = 0, \xi_0 = 0$ *and* $R(s,t,\alpha)$ *is as in (3.6, with replacing* T *by* $t_k - t_{k-1}$.

PROOF. Let $V(s,\xi) = \alpha\xi^2 - 2\alpha\beta v(s)\xi$. Since the Wiener process $\{x(s) : 0 \leq s \leq T\}$ is additive, it can be shown that

$$E\left[\exp\left\{-\frac{1}{2}\int_0^T V(s, \lambda^{-\frac{1}{2}}x(s))ds\right\}|\lambda^{-\frac{1}{2}}x(t_k) = \xi_k, k = 1, 2, \cdots, n\right]$$

$$= E\Bigg[\exp\left\{-\frac{1}{2}\sum_{k=1}^{n}\left\{\int_{t_{k-1}}^{t_k} V(s, \lambda^{-\frac{1}{2}}x(s))ds\right\}\right\}|\lambda^{-\frac{1}{2}}x(t_k) = \xi_k,$$

$$k = 1, 2, \cdots, n\Bigg]$$

$$= \prod_{k=1}^{n} E\Big[\exp\Big\{-\frac{1}{2}\int_{t_{k-1}}^{t_k} V(s,\lambda^{-\frac{1}{2}}x(s))ds\Big\}|\lambda^{-\frac{1}{2}}x(t_{k-1}) = \xi_{k-1}, \lambda^{-\frac{1}{2}}x(t_k) = \xi_k\Big].$$

Hence this, together with Theorem 3.3, gives the desired result.

If we let $v(t) \equiv 0$ in Corollary 3.4, we then have

COROLLARY 3.5. *Let α be a complex number with $Re\alpha > 0$. Let $0 = t_0 < t_1 < \cdots < t_n = T$. Then for $\vec{\xi} = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbf{R}^n$ and real $q \neq 0$*

$$E^{\mathrm{anf}_q}\left[\exp\Big\{-\frac{1}{2}\alpha\int_0^T x^2(s)ds\Big\}|x(t_k) = \xi_k, k = 1,2,\cdots n\right]$$

$$= \prod_{k=1}^{n}\left[\left(\frac{\sqrt{\frac{\alpha i}{q}}(t_k - t_{k-1})}{\sinh\sqrt{\frac{\alpha i}{q}}(t_k - t_{k-1})}\right)^{\frac{1}{2}} \cdot \exp\left\{\frac{q(\xi_k - \xi_{k-1})^2}{2i(t_k - t_{k-1})}\right\}\right.$$

$$\cdot \exp\left\{-\frac{\sqrt{-iq\alpha}}{2}(\xi_k^2 + \xi_{k-1}^2)\coth\sqrt{\frac{\alpha i}{q}}(t_k - t_{k-1})\right.$$

$$\left.\left. + \frac{\sqrt{-iq\alpha}\,\xi_k\xi_{k-1}}{\sinh\sqrt{\frac{\alpha i}{q}}(t_k - t_{k-1})}\right\}\right]$$

COROLLARY 3.6[9, P.64]. *The solution of the Schrödinger equation for an external force $f(t)$:*

$$\frac{\partial U}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 U}{\partial \xi^2} + \frac{i}{\hbar}\Big(-\frac{mw^2}{2}\xi^2 + f(t)\xi\Big)U \tag{3.19}$$

satisfying $U(\xi_2, t_2; \xi_1, t_1) \longrightarrow \delta(\xi_2 - \xi_1)$ as $t_2 \downarrow t_1$ is given by

(3.20)

$$U(\xi_2,t_2;\xi_1,t_1) = \sqrt{\frac{mw}{2\pi i\hbar \sin w(t_2 - t_1)}}\ \exp\Big\{\frac{imw}{2\hbar \sin w(t_2 - t_1)}$$

$$\times \Big[\cos w(t_2 - t_1)(\xi_2^2 + \xi_1^2) - 2\xi_2\xi_1 + \frac{2\xi_2}{mw}\int_{t_1}^{t_2} f(t)\sin w(t - t_1)dt$$

$$+\frac{2\xi_1}{mw}\int_{t_1}^{t_2} f(t)\sin w(t_2-t)dt$$
$$-\frac{2}{m^2w^2}\int_{t_1}^{t_2}\int_{t_1}^{t_2} f(s)f(t)\sin w(t_2-t)\sin w(s-t_1)dsdt\Big]\Big\}.$$

PROOF. It can be shown that for $(\xi_2, t_2, \xi_1, t_1, q) \in \mathbf{R} \times [0,T] \times \mathbf{R} \times [0,T] \times (\mathbf{R}-\{0\})$ with $t_1 < t_2$

(3.21)
$$\Gamma(\xi_2, t_2; \xi_1, t_1; -iq) = \sqrt{\frac{q}{2\pi i t}}\exp\Big\{-\frac{q(\xi_2-\xi_1)^2}{2it}\Big\}E^{\text{anf}_q}[F|X=(\xi_1,\xi_2)]$$

satisfies the Schrödinger equation

$$\frac{\partial\Gamma}{\partial t} = \frac{i}{2q}\frac{\partial^2\Gamma}{\partial\xi^2} - \frac{\alpha}{2}\xi^2\Gamma + \alpha\beta\xi q(t)\Gamma$$

with the condition $\Gamma(\xi_2, t_2; \xi_1, t_1; -iq) \to \delta(\xi_2-\xi_1)$ as $t_2 \downarrow t_1$. But by using (3.8) in Theorem 3.3 with $q=\frac{m}{\hbar}$, $\alpha = \frac{imw^2}{\hbar}$ and $\beta = \frac{1}{mw^2}$ we can easily obtain (3.20) from (3.21). Hence (3.20) is the solution of (3.19).

Let S be the operator on $L^2[0,T]$ defined by

$$Sv(s) = \int_s^T v(u)du, s \in [0,T], v \in L^2[0,T]. \tag{3.22}$$

Then S is a bounded linear operator from $L^2[0,T]$ to $L^2[0,T]$.

THEOREM 3.7. *Let F be a measurable function on $C_0[0,T]$ defined by*

$$F(x) = \int_{L^2[0,T]} \exp\Big\{i\int_{t_1}^T v(s)x(s)ds\Big\}d\sigma(v), \quad v \in L^2[0,T],$$

where $\sigma \in M(L^2[0,T])$, the space of the $\mathbf{C}$-valued, countably additive Borel measure on $L^2[0,T]$. Let $X(x) = (x(t_1), x(T))$. Then for all $\lambda \in$

$\mathbf{C}^+$, *the conditional analytic Wiener integral* $E^{\mathrm{anw}_\lambda}(F|X)$ *exists and for all* $(\xi_1,\xi)\in\mathbf{R}^2$ *is given by the formula*
(3.23)

$$\begin{aligned}&E^{\mathrm{anw}_\lambda}[F|X=(\xi_1,\xi)]\\&=\int_{L^2[0,T]}\exp\left\{-\frac{1}{2\lambda}\int_{t_1}^{T}\left(\int_s^T v(u)du\right)^2ds+\frac{1}{2\lambda(T-t_1)}\left(\int_{t_1}^{T}\int_s^T v(u)duds\right)^2\right.\\&\qquad\left.+i\frac{\xi-\xi_1}{T-t_1}\int_{t_1}^{T}\int_s^T v(u)duds\right\}d\sigma(v).\end{aligned}$$

Furthermore, $E^{\mathrm{anf}_q}(F|X)$ *exists for all real* $q\neq 0$ *and for all* $(\xi_1,\xi)\in\mathbf{R}^2$ *is given by the formula*
(3.24)

$$\begin{aligned}&E^{\mathrm{anf}_q}[F|X=(\xi_1,\xi)]\\&=\int_{L^2[0,T]}\exp\left\{-\frac{i}{2q}\int_{t_1}^{T}\left(\int_s^T v(u)du\right)^2ds+\frac{i}{2q(T-t_1)}\left(\int_{t_1}^{T}\int_s^T v(u)duds\right)^2\right.\\&\qquad\left.+i\frac{\xi-\xi_1}{T-t_1}\int_{t_1}^{T}\int_s^T v(u)duds\right\}d\sigma(v).\end{aligned}$$

PROOF. We note that

$$\begin{aligned}F(x)&=\int_{L^2[0,T]}\exp\left\{i\int_{t_1}^{T}v(s)x(s)ds\right\}d\sigma(v)\\&=\int_{L^2[0,T]}\exp\left\{i\int_{t_1}^{T}Sv(s)dx(s)\right\}d\sigma(v)\\&=\int_{L^2[0,T]}\exp\left\{i\int_{t_1}^{T}w(s)dx(s)\right\}d\sigma S^{-1}(w).\end{aligned}$$

Using Proposition 2.1, Fubini theorem and the change of variable formula we obtain that for $\lambda>0$ and all $(\xi_1,\xi)\in\mathbf{R}^2$
(3.25)

$$\begin{aligned}&E[F(\lambda^{-\frac{1}{2}}\cdot|X(\lambda^{-\frac{1}{2}}\cdot)=(\xi_1,\xi)]\\&=\int_{C_0^0[0,T-t_1]}\int_{L^2[0,T]}\exp\left\{i\int_0^{T-t_1}w(s+t_1)d(\lambda^{-\frac{1}{2}}y+g)(s)\right\}d\sigma S^{-1}(w)dm_p(y)\\&=\int_{L^2[0,T]}\exp\left\{i\frac{\xi-\xi_1}{T-t_1}\int_0^{T-t_1}w(s+t_1)ds\right\}\\&\qquad\cdot\int_{C_0^0[0,T-t_1]}\exp\left\{\frac{i}{\sqrt{\lambda}}\int_0^{T-t_1}w(s+t_1)dy(s)\right\}dm_p(y)d\sigma S^{-1}(w)\end{aligned}$$

$$= \int_{L^2[0,T]} \exp\left\{i\frac{\xi-\xi_1}{T-t_1}\int_0^{T-t_1} w(s+t_1)ds\right\} \left[\frac{1}{\sqrt{2\pi\rho^2}}\int_{\mathbf{R}} \exp\left\{\frac{i}{\sqrt{\lambda}}u\right\} \cdot \exp\left\{-\frac{u^2}{2\rho^2}\right\}du\right] d\sigma S^{-1}(w)$$

$$= \int_{L^2[0,T]} \exp\left\{i\frac{\xi-\xi_1}{T-t_1}\int_0^{T-t_1} w(s+t_1)ds\right\} \exp\left\{-\frac{\rho^2}{2\lambda}\right\} d\sigma S^{-1}(w)$$

$$= \int_{L^2[0,T]} \exp\left\{-\frac{1}{2\lambda}\int_{t_1}^T w(s)^2 ds + \frac{1}{2\lambda(T-t_1)}\left(\int_{t_1}^T w(s)ds\right)^2 + i\frac{\xi-\xi_1}{T-t_1}\int_{t_1}^T w(s)ds\right\} d\sigma S^{-1}(w)$$

$$= \int_{L^2[0,T]} \exp\left\{-\frac{1}{2\lambda}\int_{t_1}^T Sv(s)^2 ds + \frac{1}{2\lambda(T-t_1)}\left(\int_{t_1}^T Sv(s)ds\right)^2 + i\frac{\xi-\xi_1}{T-t_1}\int_{t_1}^T Sv(s)ds\right\} d\sigma(v).$$

where $\rho^2 = \int_{t_1}^T w(s)^2 ds - \frac{1}{T-t_1}\left(\int_{t_1}^T w(s)ds\right)^2$ is the variance of $\int_0^{T-t_1} w(s+t_1)dy(s)$. Since $\sigma \in M(L_2[0,T])$, we see that the last expression on the right-hand side of (3.25) is an analytic function of λ throughout $\mathbf{C}^+$ and is a continuous function λ for $\mathrm{Re}\lambda \geq 0$, $\lambda \neq 0$. Thus (3.23) and (3.24) are established, which concludes the proof of Theorem 3.7.

THEOREM 3.8. *Let F be a measurable function on $C_0[0,T]$ defined by*

$$F(x) = exp\left\{-\frac{i}{2}\int_{t_1}^T x^2(s)ds\right\}\int_{L^2[0,T]} \exp\left\{i\int_{t_1}^T v(s)x(s)ds\right\}d\sigma(v),$$

where $x \in C_0[0,T]$, $v \in L^2[0,T]$ and $\sigma \in M(L^2[0,T])$. Let $X(x) = (x(t_1), x(T))$. Then for all $\lambda \in \mathbf{C}^+$, the conditional analytic Feynman integral $E^{\mathrm{anf}_q}(F|X)$ exists and for all $(\xi_1,\xi) \in \mathbf{R}^2$ and real $q \neq 0$ is given by the formula

$$E^{\mathrm{anf}_q}[F|X=(\xi_1,\xi)] = \left(\frac{\sqrt{-\frac{1}{q}}(T-t_1)}{\sinh\sqrt{-\frac{1}{q}}(T-t_1)}\right)^{\frac{1}{2}} \cdot \exp\left\{\frac{q(\xi-\xi_1)^2}{2i(T-t_1)}\right\} \cdot \exp\left\{-\frac{\sqrt{q}}{2}(\xi^2+\xi_1{}^2)\coth\sqrt{-\frac{1}{q}}(T-t_1) + \frac{\sqrt{q}\xi\xi_1}{\sinh\sqrt{-\frac{1}{q}}(T-t_1)}\right\} \tag{3.26}$$

$$\cdot \int_{L_2[0,T]} \exp\left\{ i \int_{t_1}^{T} \frac{v(t)}{\sinh\sqrt{-\frac{1}{q}}(T-t_1)} \left(\xi \sinh\sqrt{-\frac{1}{q}}(t-t_1) + \xi_1 \sinh\sqrt{-\frac{1}{q}}(T-t) \right) dt \right\}$$

$$\cdot \exp\left\{ -\frac{i}{2q} \int_0^{T-t_1} \int_0^{T-t_1} R\left(s,t,-\frac{1}{q}\right) v(s+t_1) v(t-t_1) ds dt \right\} d\sigma(v).$$

PROOF. By Proposition 2.1, we have, for $\xi, \xi_1 \in \mathbf{R}$ and $\lambda > 0$

$$\begin{aligned} &E\left[F(\lambda^{-\frac{1}{2}}\cdot) \middle| \lambda^{-\frac{1}{2}} x(t_1) = \xi_1, \lambda^{-\frac{1}{2}} x(T) = \xi\right] \\ &= \int_{C_0^0[0,T-t_1]} \exp\left\{ -\frac{i}{2\lambda} \int_0^{T-t_1} (y(s) + \sqrt{\lambda} g(s))^2 ds \right\} \\ &\qquad \int_{L_2[0,T]} \exp\left\{ \frac{i}{\sqrt{\lambda}} \int_0^{T-t_1} v(s+t_1)(y(s) + \sqrt{\lambda} g(s)) ds \right\} d\sigma(v) dm_p(y) \\ &= \int_{L_2[0,T]} \int_{C_0^0[0,T-t_1]} \exp\left\{ -\frac{i}{2\lambda} \int_0^{T-t_1} (y(s) + \sqrt{\lambda} g(s))^2 ds \right. \\ &\qquad \left. + \frac{i}{\sqrt{\lambda}} \int_0^{T-t_1} v(s+t_1)(y(s) + \sqrt{\lambda} g(s)) ds \right\} dm_p(y) d\sigma(v) \end{aligned}$$

where $g(t) = \dfrac{\xi - \xi_1}{T - t_1} t + \xi_1, t \in [0, T - t_1]$ and the second equality is justified by the Fubini theorem. But we know that the integrand of integral on $L_2[0,T]$ in the last equality is given by the right side of (3.7) with $\lambda > 0, \alpha = i$ and $\beta = 1$. Since σ is in $M(L_2[0,T])$ and the right side of (3.7) is an analytic function of λ through $\mathbf{C}^+$ and a continuous function for $\mathrm{Re}\lambda \geq 0, \lambda \neq 0$, we obtain, for $\lambda \in \mathbf{C}^+$ and $(\xi_1, \xi) \in \mathbf{R}^2$

$$E^{\mathrm{anw}_\lambda}[F|X = (\xi_1, \xi)] = \left(\frac{\sqrt{\frac{i}{\lambda}}(T-t_1)}{\sinh\sqrt{\frac{i}{\lambda}}(T-t_1)} \right)^{\frac{1}{2}} \cdot \exp\left\{ \frac{\lambda(\xi - \xi_1)^2}{2(T-t_1)} \right\}$$
$$\cdot \exp\left\{ -\frac{\sqrt{i\lambda}}{2} (\xi^2 + \xi_1{}^2) \coth\sqrt{\frac{i}{\lambda}}(T-t_1) + \frac{\sqrt{i\lambda}\xi\xi_1}{\sinh\sqrt{\frac{i}{\lambda}}(T-t_1)} \right\} \tag{3.27}$$

$$\cdot\int_{L_2[0,T]} \exp\left\{ i\int_{t_1}^{T} \frac{v(t)}{\sinh\sqrt{\frac{i}{\lambda}}(T-t_1)}\left(\xi \sinh\sqrt{\frac{i}{\lambda}}(t-t_1) + \xi_1 \sinh\sqrt{\frac{i}{\lambda}}(T-t)\right)dt \right\}$$

$$\cdot \exp\left\{-\frac{1}{2\lambda}\int_0^{T-t_1}\int_0^{T-t_1} R\left(s,t,\frac{i}{\lambda}\right)v(s+t_1)v(t+t_1)dsdt\right\} d\sigma(v).$$

Hence as $\lambda \to -iq, q \neq 0$, the result (3.26) is obtained as desired.

4. Correction to "A class of conditional Wiener integrals"

In the first author's paper[4] with S.J.Chang, the hypothesis for Theorem 2.2 should have been : Let F be an integrable function on $C_0[0,T]$ defined by $F(x) = \exp\{\int_{t_1}^{T} V(s,x(s))ds\}$.

The expression appearing in Theorem 3.3 and its proof :

$$\alpha\beta(\xi-\xi_1)\int_0^{T-t_1}\left(\frac{\sinh\sqrt{\alpha}t}{\sinh\sqrt{\alpha}(T-t_1)} + \frac{\xi_1}{\xi-\xi_1}\right)q(t+t_1)dt$$

should read as

$$\alpha\beta\left[\int_0^{T-t_1}\frac{q(t+t_1)}{\sinh\sqrt{\alpha}(T-t_1)}\left(\xi\sinh\sqrt{\alpha}t + \xi_1\sinh\sqrt{\alpha}(T-t_1-t)\right)dt\right].$$

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