Comm. Korean Math. Soc. **11** (1996), No. 1, pp. 209–214

TUBE VOLUMES ABOUT GEODESIC BALLS

SUNGYUN LEE

ABSTRACT. A flat space is characterized by tube volumes about geodesic balls. Similar characterizations are also given for other rank one symmetric spaces.

§1. Introduction

Let M be an n-dimensional Riemannian manifold of class C^ω. For small $r > 0$ let $V_m(r)$ denote the volume of geodesic ball $B_m(r)$ with center m and radius r. This paper is concerned with the following volume conjecture :

(I) Suppose that $V_m(r) = \omega_n r^n$ for all $m \in M$ and all sufficiently small $r > 0$. Then M is flat.

Here ω_n is the volume of the unit ball in R^n. Although the volume conjecture (I) seems quite reasonable it remains still open. However the conjecture (I) can be solved in many special cases. In particular if M has nonnegative or nonpositive Ricci curvature then (I) is true [4].

In this paper we prove (I) under a natural assumption which is related to the Weyl's formula (see (II)). To be more specific let $P \subset R^q$ be a topologically embedded n-dimensional Riemannian submanifold of R^q with compact closure, and let $V_P^q(r)$ denote the volume of a tube of radius r about P in R^q. Then the Weyl's tube formula [6] says

$$(1) \qquad V_P^q(r) = \omega_{q-n} r^{q-n} \sum_{c=0}^{[n/2]} \frac{k_{2c}(R) r^{2c}}{(q-n+2)(q-n+4)\cdots(q-n+2c)}.$$

Received July 13, 1995. Revised October 23, 1995.
1991 AMS Subject Classification: 53C20.
Key words and phrases: Weyl's tube formula, geodesic balls, rank-one symmetric spaces.

Here $k_{2c}(R)$ are integrals over P of certain polynomials in the curvature tensor R of P. For example

$$k_0(R) = Volume(P),$$

$$k_2(R) = \frac{1}{2}\int_P \tau dP, \qquad k_4(R) = \frac{1}{8}\int_P (\tau^2 - 4\|\rho\|^2 + \|R\|^2) dP,$$

where τ, ρ, and dP denote the scalar curvature, the Ricci tensor, and the voume element of P respectively (see §2). When P is flat, the Wely's tube formula implies that

(II) $V^q_{B_m(r_1)}(r_2) = \omega_n \omega_{q-n} r_1^n r_2^{q-n}$ for all $m \in P$ and sufficiently small $r_1, r_2 > 0$.

The following theorem is the converse of it. Note that (II) is stronger than the hypothesis in (I). Throughout this paper we assume that M is a manifold of class C^ω.

THEOREM 1. *Let $M \subset R^q$ be an n-dimensional Riemannian submanifold of R^q. Suppose that* (II) *holds. Then M is flat.*

In view of the Cartan-Janet-Bustin theorem or a weak version of the Nash embedding theorem which says that any Riemannian manifold has a local isometric embedding into some Euclidean space we have the following extension of Thoerem 1.

THEOREM 2. *Let M be an n-dimensional Riemannian manifold. Suppose that* (II) *holds for isometric embedding $B_m(r_1) \subset R^q$. Then M is flat.*

Moreover other rank-one symmetric spaces are characterized similarly.

THEOREM 3. *Let M be an n-dimensional Riemannian manifold. Suppose that for all $m \in M$ and sufficiently small $r_1, r_2 > 0$, $V^q_{B_m(r_1)}(r_2)$ with isometric embedding $B_m(r_1) \subset R^q$ is the same function as that of the case when M is a space of constant curvature λ. Then M has constant curvature λ.*

THEOREM 4. *Let M be a 2n-dimensional Riemannian manifold. Suppose that for all $m \in M$ and sufficiently small $r_1, r_2 > 0$, $V^q_{B_m(r_1)}(r_2)$ with isometric embedding $B_m(r_1) \subset R^q$ is the same function as that of the case when M is a space of constant holomorphic sectional curvature μ. Then M has constant holomorphic sectional curvature μ.*

THEOREM 5. *Let M be a 4n-dimensional Riemannian manifold whose holonomy group is a subgroup of $Sp(n) \cdot Sp(1)$. Suppose that for all $m \in M$ and sufficiently small $r_1, r_2 > 0$, $V^q_{B_m(r_1)}(r_2)$ with isometric embedding $B_m(r_1) \subset R^q$ is the same function as that of the case when M is a quaternionic projective space $QP(\nu)$ with maximum sectional curvature ν. Then M is locally isometric to $QP(\nu)$ or its non-compact dual.*

REMARK. For the Cayley plane a result of [1] states that a manifold whose holonomy group is contained in $Sp(9)$ is either flat or locally isometric to the Cayley plane or its non-compact dual.

§2. Proofs of Theorems

First of all we shall recall the power series expansion for the volume of geodesic balls [4].

THEOREM 6. *Let M be an n-dimensional Riemannian manifold of class C^ω. For any point $m \in M$ and all sufficiently small radius $r > 0$ we have*

$$V_m(r) = \omega_n r^n (1 + Ar^2 + Br^4 + O(r^6))_m \tag{2}$$

where the coefficients A and B are given by the following formulas:

$$A(m) = -\frac{\tau(m)}{6(n+2)},$$

$$B(m) = \frac{1}{360(n+2)(n+4)}(-3\|R\|^2 + 8\|\rho\|^2 - 5\tau^2 - 18\triangle\tau)_m.$$

Here R, ρ and τ denote the Riemannian curvature tensor, the Ricci tensor, and the scalar curvature, respectively and $\triangle$ is the Laplacian. If

$\{e_1, \cdots, e_n\}$ is a local orthonormal frame, the curvature invariants in (2) are given by the following formulas:

$$\|R\|^2 = \sum_{i,j,k,l} R_{ijkl}^2 \qquad \tau = \sum_{i,j} R_{ijij}$$
$$\|\rho\|^2 = \sum_{i,j} \rho_{ij}^2 \qquad \rho_{ij} = \sum_{k} R_{ikjk}$$

Here R_{ijkl}'s are the components of R with respect to $\{e_1, \cdots, e_n\}$.

For each of the symmetric spaces of rank one, the curvature invariants τ, $\|\rho\|^2$, $\|R\|^2$ are given in the table below.

type	real dimension	τ	$\|\rho\|^2$	$\|R\|^2$
$S^n(\lambda)$	n	$n(n-1)\lambda$	$n(n-1)^2\lambda^2$	$2n(n-1)\lambda^2$
$CP^n(\mu)$	$2n$	$n(n+1)\mu$	$n(n+1)^2\mu^2/2$	$2n(n+1)\mu^2$
$QP^n(\nu)$	$4n$	$4n(n+2)\nu$	$4n(n+2)^2\nu^2$	$4n(5n+1)\nu^2$
$CalP^2(\zeta)$	16	$16 \cdot 36 \cdot \zeta$	$16 \cdot 36^2 \cdot \zeta^2$	$16^2 \cdot 36 \cdot \zeta^2$

Here $S^n(\lambda)$, $CP^n(\mu)$, $QP^n(\nu)$, and $CalP^2(\zeta)$ denote the sphere with constant sectional curvature λ, the complex projective space with constant holomorphic sectional curvature μ, the quaternionic projective space with maximum sectional curvature ν, and the Cayley plane with maximum sectional curvature ζ respectively.

PROOF OF THOEREM 1. From the hypothesis we have $V_m(r_1) = \omega_n r_1^n$ for all small $r_1 > 0$ which implies $\tau = 0$ and $-3\|R\|^2 + 8\|\rho\|^2 = 0$ by (2). From the hypothesis and (1) we also have $k_4(R) = 0$ on $B_m(r_1)$. Hence we get

$$\int_{B_m(r_1)} \{\tau^2 - 4\|\rho\|^2 + \|R\|^2\} d(B_m(r_1)) = \int_{B_m(r_1)} -\frac{1}{2}\|R\|^2 d(B_m(r_1)) = 0$$

for all $m \in M$ and all small $r_1 > 0$. By the continuity we have $\|R\|^2 = 0$ which shows that M is flat. □

PROOF OF THEOERM 3. From the hypothesis and the table we have $\tau = n(n-1)\lambda$ and $-3\|R\|^2 + 8\|\rho\|^2 + 5\tau^2 = n(n-1)(n+2)(5n-7)\lambda^2$. From the hypothesis and (1) we also have

$$\int_{B_m(r_1)} \{\tau^2 - 4\|\rho\|^2 + \|R\|^2 - n(n-1)(n-2)(n-3)\lambda^2\} \, d(B_m(r_1)) = 0$$

which gives

$$\frac{-2(n-1)}{4n-7} \int_{B_m(r_1)} \left\{ \|R\|^2 - \frac{2}{n-1}\|\rho\|^2 \right\} d(B_m(r_1)) = 0.$$

By the continuity we have $\|R\|^2 = 2\|\rho\|^2/(n-1)$. According to a result of [2] this implies that M has constant curvature λ. □

PROOF OF THEOREM 4. From the hypothesis and the table we have $\tau = n(n+1)\mu$, $-3\|R\|^2 + 8\|\rho\|^2 + 5\tau^2 = n(n+2)(n+5)(5n-1)\mu^2$. From the hypothesis and (1) we also have

$$\int_{B_m(r_1)} \{\tau^2 - 4\|\rho\|^2 + \|R\|^2 - n^2(n+1)(n-1)\mu^2\} \, d(B_m(r_1)) = 0$$

which gives

$$\frac{1}{(-n-1)(2n-1)} \int_{B_m(r_1)} \left\{ \|R\|^2 - \frac{4}{n+1}\|\rho\|^2 \right\} d(B_m(r_1)) = 0.$$

By the continuity we have $\|R\|^2 = 4\|\rho\|^2/(n+1)$. According to a result of [3] this implies that M has constant holomorphic sectional curvature μ. □

PROOF OF THEOREM 5. From the hypothesis we have $\tau = 4n(n+2)\nu$, $-3\|R\|^2 + 8\|\rho\|^2 + 5\tau^2 = 4n(20n^3 + 88n^2 + 97n + 29)\nu^2$. From the hypothesis we also have

$$\int_{B_m(r_1)} \{\tau^2 - 4\|\rho\|^2 + \|R\|^2 \\ -4n(4n^3 + 12n^2 + 5n - 15)\nu^2\} \, d(B_m(r_1)) = 0$$

which implies

$$-\frac{1}{2}\int_{B_m(r_1)}\left\{\|R\|^2-\frac{(5n+1)\tau^2}{4n(n+2)^2}\right\}d(B_m(r_1))=0.$$

By the continuity we have $\|R\|^2=(5n+1)\tau^2/4n(n+2)^2$. According to a result of [5] this implies that M is locally isometric to $QP^n(\nu)$. □

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Department of Mathematics
Korea Advanced Institute of Science and Technology
Taejon, 305-701, KOREA