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HYPERSURFACES SATISFYING $\Delta x = Rx + b$ IN S_1^{n+1} AND IN H^{n+1}

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ABSTRACT. In this paper, we classify hypersurfaces M in S_1^{n+1} or in H^{n+1} satisfying $\Delta x = Rx + b$.

1. Introduction

Let E_1^{n+2} be the $(n+2)$ - dimensional Minkowski space time with the standard flat metric given by

$$g = \sum_{i=1}^{n+1} dx_i^2 - dx_{n+2}^2,$$

where $(x_1, x_2, \ldots, x_{n+2})$ is a rectangular coordinate system of E_1^{n+1}. For a positive number r and a point $c \in E_1^{n+2}$, we denote by $S_1^{n+1}(c, r)$ and $H^{n+1}(c, -r)$, the de Sitter space time and the hyperbolic space defined respectively by

$$S_1^{n+1}(c, r) = \{x \in E_1^{n+2} | \langle x - c, x - c\rangle = r^2\},$$

$$H^{n+1}(c, -r) = \{x \in E_1^{n+2} | \langle x - c, x - c\rangle = -r^2\},$$

where $\langle\ ,\ \rangle$ denotes the indefinite inner product on E_1^{n+2}. The point c is called the center of $S_1^{n+1}(c, r)$ and $H^{n+1}(c, -r)$ respectively. We simply denote $S_1^{n+1}(0, 1)$ and $H^{n+1}(0, -1)$ by S_1^{n+1} and H^{n+1}. In [3], the second

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author considered an isometric immersion $x : M^n \longrightarrow H^{n+1} \subset E_1^{n+2}$ of an n-dimensional Riemannian Manifold M^n satisfying

$$(*) \qquad \Delta x = Rx + b,$$

where $x = (x_1, x_2, \ldots, x_{n+2})$, $\Delta x = (\Delta x_1, \ldots, \Delta x_{n+2})$, R is an $(n+2) \times (n+2)$ constant matrix, b is a constant vector in E_1^{n+2} and Δ is the Lapalcian on M^n. He classified such immersions in case that $b = 0$. The analoguous classification problem for hypersurfaces of Euclidean space, for hypersurfaces of the Minkowski space time , or for hypersurfaces of the unit sphere is studied by serveral authors and is completely solved[1,2,3,4,5]. The purpose of this paper is to study an isometric immersions $x : M^n \longrightarrow S_1^{n+1}(c, 1)$ (or $H^{n+1}(c, -1)) \subset E_1^{n+2}$ of an n-dimensional pseudo Riemannian (or Riemannian) manifold M^n satisfying $(*)$ for a constant matrix R and a constant vector b. For given such an immersion x, consider the immersion $y = x - c$. Then $\langle y, y \rangle = \pm 1$ and $\Delta y = \Delta x = Rx + b'$ $(b' = Rc + b)$. So we may assume that M^n is immersed into S_1^{n+1}(or H^{n+1}) by x without loss of generality. We will study under this assumption. Our results are:

THEOREM 1. *Let $x : M^n \longrightarrow S_1^{n+1} \subset E_1^{n+2}$ be an isometric immersion of an n-dimensional pseudo Riemannian manifold M^n satisfying $(*)$ for a constant matrix R and a constant vector b. Then M^n is minimal in S_1^{n+1} or is an open part of one of the followings:*

(1) *a flat totally umbilical hypersurface $N_1(a)$ of S_1^{n+1} defined by*

$$N_1(a) = \{x \in S_1^{n+1} | \langle x, a \rangle = \text{nonzero constant}\}$$

for a constant vector a with $\langle a, a \rangle = 0$;

(2) *a nonflat totally umbilical hypersurface $S^n(c, r)$(or $S_1^n(c, r)$) of S_1^{n+1};*

(3) *a product manifold $S^k(0, r_1) \times S_1^{n-k}(0, r_2)$ in S_1^{n+1};*

(4) *a product manifold $S^k(0, r_1) \times H^{n-k}(0, -r_2)$ in S_1^{n+1}.*

THEOREM 2. *Let $x : M^n \longrightarrow H^{n+1} \subset E_1^{n+2}$ be an isometric immersion of an n-dimensional Riemannian manifold M^n satisfying $(*)$ for* *a constant matrix R and a constant vector b. Then M^n is minimal in H^{n+1} or is an open part of one of the followings:*

(1) *a flat totally umbilical hypersurface $N_1(a)$ of H^{n+1} defined by*

$$N_1(a) = \{x \in H^{n+1} | \langle x, a \rangle = \text{nonzero constant}\}$$

for a constant vector a with $\langle a, a \rangle = 0$;

(2) *a nonflat totally umbilical hypersurface $S^n(c, r)$(or $H^n(c, -r)$) of H^{n+1};*

(3) *a product manifold $S^k(0, r_1) \times H^{n-k}(0, r_2)$ in H^{n+1}.*

Since the proof of theorem2 is similar to that of theorem1, we will give only the proof of theorem1.

2. Preliminaries

Let $x : M^n \longrightarrow S_1^{n+1} \subset E_1^{n+2}$ be an isometric immersion of an n-dimensional pseudo Riemannian manifold M^n. Choose a local pseudo-orthonormal frame $e_1, e_2, \dots, e_{n+2}$ on M^n such that $e_1, e_2, \dots, e_n$ are tangent to M^n, e_{n+1}, e_{n+2} are normal to M^n and $e_{n+2} = x$. From now on, we shall use the following index conventions and notations:

$$1 \le i, j \le n, \quad n+1 \le \alpha \le n+2, \quad \epsilon_i = \langle e_i, e_i \rangle, \quad \epsilon = \langle e_{n+1}, e_{n+1} \rangle.$$

The pseudo Euclidean connection $\overline{\nabla}$ on E_1^{n+1} induces the Levi-Civita connection ∇ on M^n,

$$\overline{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + \mathbf{h}(e_i, e_j) \quad \text{for} \quad i, j = 1, 2, \dots, n,$$

where $\mathbf{h}$ is the second fundamental form of M^n in E_1^{n+2}. Then there exists an $n \times n$ symmetric matrix $\{h_{ij}\}$ such that $\mathbf{h}(e_i, e_j) = h_{ij} e_{n+1} - \epsilon_i \delta_{ij} e_{n+2}$. The Weingarten equations are given by

$$\overline{\nabla}_{e_i} e_{n+1} = -Ae_i = -\epsilon \sum_{j=1} \epsilon_j h_{ij} e_j,$$

$$\overline{\nabla}_{e_i} e_{n+2} = e_i \quad \text{for} \quad i = 1, \dots, n,$$

where A is the weingarten map associated with e_{n+1}.

Let $f \in C^{\infty}(M^n)$. Then ∇f, the gradient f and Δf, the Laplacian of f are given by

$$\nabla f = \sum_i \epsilon_i (e_i f) e_i = \sum_i \epsilon_i f_i e_i,$$

$$\Delta f = \sum_i \epsilon_i \{ e_i e_i f - (\nabla_{e_i} e_i f) \}$$

So the following holds

$$\begin{aligned} \Delta x &= \sum_i \epsilon_i (\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} x - \overline{\nabla}_{\nabla_{e_i} e_i} x) \\ &= \sum_i \epsilon_i \mathbf{h}(e_i, e_i) \\ &= \sum_i \epsilon_i h_{ii} e_{n+1} - nx \\ &= H e_{n+1} - nx, \end{aligned} \tag{2.1}$$

where H is the mean curvature function of M^n in S_1^{n+1}. By similar calculation to that appeared in [3], we have the follwing.

PROPOSITION 2.1. *For a hypersurface M in S_1^{n-1}, $\Delta H e_{n+1}$ is calculated as follows,*

$$\Delta(H e_{n+1}) = -(2A + \epsilon H I)(\nabla H) + (\Delta H - \epsilon H tr A^2) e_{n+1} + \epsilon H^2 x.$$

Now suppose that the immersion x satisfies $(*)$ for a constant matrix R and a constant vector b. Let R^* and C denote the adjoint matrix of R with respect to the indefinite metric $\langle\, , \rangle$ and the skew self adjoint matrix $\frac{1}{2}(R - R^*)$ respectively. Defferentiating the following equation in the direction e_j,

$$\langle Rx + b, e_i \rangle = 0, i = 1, 2, \dots, n$$

we can get

$$\langle R e_j, e_i \rangle = -\langle Rx + b, \mathbf{h}(e_i, e_j) \rangle, \quad i, j = 1, \dots, n. \tag{2.2}$$

(2.1) implies that $\langle \Delta x, x \rangle = \langle Rx + b, x \rangle = -n$. From this one can deduce that $R^* x$ is normal to M^n. Hence we can get

$$\langle R^* e_j, e_i \rangle = -\langle R^* x, \mathbf{h}(e_i, e_j) \rangle, \quad i, j = 1, \dots, n. \tag{2.3}$$

Since $\langle R e_i, e_i \rangle = \langle R^* e_i, e_i \rangle$,by (2.2) and (2.3) we obtain

LEMMA 2.2. *$2Cx + b$ is normal to M^n and $\langle Rx + b, 2Cx + b\rangle = 0$.*

The symmetry of the second fundamental form $\mathbf{h}$ implies the following.

LEMMA 2.3. *CX is normal to M^n for every tangent vector field X of M^n.*

3. Proof of theorem1

At first, we will give the proof under the assumption that $b = 0$ and R is self adjoint. Start with the equation $Rx + nx = He_{n+1}$. This equation implies

$$\langle Re_i, e_{n+1}\rangle = \epsilon H_i, i = 1, \dots, n, \tag{3.1}$$

$$\Delta(He_{n+1}) = R(He_{n+1}). \tag{3.2}$$

From proposition2.1 and (3.2), we have

$$H\langle Re_{n+1}, e_i\rangle = -2\epsilon \sum_j \epsilon_j H_j h_{ji} - \epsilon H H_i. \tag{3.3}$$

Since R is self adjoint, from (3.1) and (3.3) we find

$$HH_i = -\sum_j \epsilon_j H_j h_{ji}, i = 1, \dots, n.$$

This implies that

$$A(\nabla H) = -\epsilon H \nabla H.$$

Suppose that $\nabla H \neq 0$, i.e, H is nonconstant. Then ∇H is an eigen vector of A with eigen value $-\epsilon H$ If $\langle \nabla H, \nabla H\rangle \neq 0$, then by choosing $e_n = \frac{\nabla H}{|\nabla H|}$, we get

$$A = \begin{pmatrix} A' & 0 \\ 0 & -\epsilon H \end{pmatrix},$$

where A' is self adjoint. And $R + nI$ can be written as

$$\begin{pmatrix} -HA' & & 0 & \\ & \epsilon H^2 & * & * \\ 0 & * & * & * \\ & * & * & * \end{pmatrix}.$$

Hence the eigenvalues of $-HA'$ are constants and so is $-HtrA' = -HtrA - \epsilon H^2 = -2\epsilon H^2$, which is a contradiction. If $\langle \nabla H, \nabla H \rangle = 0$ locally, then one can choose a frame $e_1, \ldots, e_{n-2}, u, v$ of M^n with all scalar products zero except $\langle u, v \rangle = 1 = \langle e_i, e_i \rangle$ such that

$$A = \begin{pmatrix} D_{n-k} & 0 \\ 0 & A'_k \end{pmatrix} \text{ and } v \text{ is parallel to } \nabla H,$$

where D_{n-k} is an $(n-k)$ diagonal matrix and

$$A'_2 = \begin{pmatrix} -\epsilon H & 0 \\ +1 & -\epsilon H \end{pmatrix}, A'_3 = \begin{pmatrix} -\epsilon H & 1 & 0 \\ 0 & -\epsilon H & 0 \\ 1 & 0 & -\epsilon H \end{pmatrix}.$$

This gives the matrix for $R + nI$

$$\begin{pmatrix} -HD_{n-k} & 0 & 0 \\ 0 & -HA'_k & * \\ 0 & * & * \end{pmatrix}.$$

This implies that $HtrD_{n-k} = H(trA - trA'_k) = (k+1)\epsilon H^2$ is constant, which is a contradiction. Hence we have $\nabla H = 0$, i.e., H is constant. Suppose M^n is not a minimal hypersurface of S_1^{n+1}. This means that $H \neq 0$. From proposition2.1 and $\Delta(He_{n+1}) = HRe_{n+1}$, we get

$$Re_{n+1} = -\epsilon trA^2 e_{n+1} + \epsilon Hx. \tag{3.4}$$

And we know that trA^2 is constant. Differentiating (3.4) and the equation $Rx = He_{n+1} - nx$ in the direction e_i $(i = 1, \ldots, n)$, we get

$$RAe_i = -\epsilon trA^2 Ae_i - \epsilon He_i, \tag{3.5}$$

$$Re_i = -HAe_i - ne_i. \tag{3.6}$$

Combining (3.5) with (3.6), we obtain

$$\{R^2 + (n + \epsilon trA^2)R + (\epsilon ntrA^2 - \epsilon H^2)I\}e_i = 0.$$

And by a direct computation we know that

$$\{R^2 + (n + \epsilon tr A^2)R + (\epsilon ntr A^2 - \epsilon H^2)I\}e_\alpha = 0.$$

These imply that

$$R^2 + (n + \epsilon tr A^2)R + (\epsilon ntr A^2 - \epsilon H^2)I = 0. \tag{3.7}$$

Since R is self adjoint, R has at least one real eigenvalue. So the left side of (3.7) can be factorized into linear terms such that

$$(R - \lambda_1 I)(R - \lambda_2 I) = 0 \tag{3.8}$$

for some real numbers λ_1,λ_2. If $R = \lambda_1 I$ or $R = \lambda_2 I$, then $\langle Rx, x\rangle = -n$ implies that $Rx = -nx$. In this case M^n is minimal in S_1^{n+1}, which contradicts to our assumption. We proceed with two cases seperately.

CASE1. $\lambda_1 \neq \lambda_2$

Then E_1^{n+2} is decomposed into the direct sum of two linear subspaces V_1, V_2, which are eigen spaces of R corresponding to λ_1 and λ_2, such that $E_1^{n+2} = V_1 \oplus V_2$. Since R is self adjoint, V_1 and V_2 are mutually orthogonal. Let $E_1, E_2, \ldots, E_{n+2}$ be a pseudo orthonormal frame of E_1^{n+2} such that $E_1, \ldots, E_i$ span V_1 and $E_{i+1}, \ldots, E_{n+2}$ span V_2. Then R can be written as

$$\begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix}.$$

And the position vector x can be expressed as $x = x_1 + x_2$, where $x_1 \in span\{E_1, \ldots, E_i\}$ and $x_2 \in span\{E_{i+1}, \ldots, E_{n+2}\}$. $\langle x, x\rangle = 1$ and $\langle Rx, x\rangle = \lambda_1\langle x_1, x_1\rangle + \lambda_2\langle x_2, x_2\rangle = -n$ imply that $\langle x_1, x_1\rangle = constant$ and $\langle x_2, x_2\rangle = constant$. So we can conclude that M^n is contained in $S^n(c, r_1)$, in $S_1^n(c, r_1)$, in $S^k(0, r_1) \times S_1^{n-k}(0, r_2)$ or in $S^k(0, r_1) \times H^{n-k}(0, -r_2)$ for some positive numbers r_1, r_2 and a point $c \in E_1^{n+2}$.

CASE2. $\lambda_1 = \lambda_2 = \lambda$

In this case, from(3.7) we get

$$(n + \epsilon tr A^2)^2 - 4(\epsilon ntr A^2 - \epsilon H^2) = 0. \tag{3.9}$$

If $\epsilon = 1$, we get from this $\{(trA^2)^2 - n\}^2 + 4H^2 = 0$, which is impossible. So $\epsilon = -1$. This means that M^n is Riemannian. Therefore the weingarten map $A = -\frac{1}{H}(R + nI)$ can be diagonalizable and hence there exists an orthonormal frame $e_1, e_2, \dots, e_n$ of M^n such that

$$Ae_i = \mu_i e_i, \quad \mu_i \in C^\infty(M^n).$$

This implies that $Re_i = (-H\mu_i - n)e_i$. Since the only eigen value of R is λ, $-H\mu_i - n = \lambda$. Thus $\mu_i = -\frac{1}{H}(\lambda + n)$. And hence

$$H^2 = n(\lambda + n), trA^2 = \lambda + n.$$

Replacing these in (3.9), we get $\lambda = 0$. From $Re_i = 0$ and $R^2 = 0$, we can see that Rx is a null constant vector field. Subsequently M^n is an open part of a flat totally umbilical hypersurface. Now suppose that R is not self adjoint or b is nonzero. If $2Cx + b = 0$ locally, then M^n is contained in an n dimensional affine subspace of E_1^{n+2}. This is a contradiction. Hence $U = \{p \in M^n | 2Cx(p) + b \neq 0\}$ is an open dense subset of M^n. We will work in U. Let M_0, M_1 be the sets

$$\{p \in U | \langle Rx(p) + b, Rx(p) + b\rangle = 0\},$$
$$\{p \in U | \langle Rx(p) + b, Rx(p) + b\rangle \neq 0\},$$

respectively. If M_0 has nonempty interior, then we may assume $\langle Rx + b, Rx + b\rangle = 0$ locally. Then Re_i is tangential for a local orthonormal frame $e_1, e_2, \dots, e_n$. And if $Rx + b$ and $2Cx + b$ are linearly independent, then there exist functions α, β such that

$$x = \alpha(Rx + b) + \beta(2Cx + b).$$

Then $-n = \langle Rx + b, x\rangle = 0$, which is a contradiction. Hence we know that $Rx + b = \gamma(2Cx + b)$ for some function γ. Differentiating this formula in the direction e_i, we get

$$Re_i = (e_i\gamma)(2Cx + b) + \gamma(2Ce_i).$$

This formula and lemma2.3 imply that Re_i is normal to M^n. Therefore $Re_i = 0$ for $i = 1, 2, \dots, n$. So $Rx + b$ is a constant null vector in the every component of M_0. If M_1 is nonempty, then we may assume that $Rx+b, 2Cx+b$ span the normal space of M_1. So there exist differentiable functions α, β such that

$$x = \alpha(Rx + b) + \beta(2Cx + b).$$

Differentiating this in the direction e_i, we get

$$e_i = (e_i\alpha)(Rx + b) + \alpha Re_i + (e_i\beta)(2Cx + b) + \beta(2Ce_i).$$

Hence we get $\epsilon_i\delta_{ij} = \alpha\langle Re_i, e_j\rangle$. This implies that the weingarten map associated with $Rx + b$ is $-\frac{1}{\alpha}I$, where I is the identity map. And we know that the weingarten map associated with $2Cx + b$ is the zero map. So every component of M_1 is an open part of a nonflat totally umbilical hypersurface of S_1^{n+1}. If M_0 has nonempty interior, then, since $Re_i = 0$ for a local orthonormal frame $e_1, e_2, \cdots, e_n$, we can see that $rankR$ is 0 or 1. And if M_1 is nonempty, then $rankR$ is not less than n. Therefore, by continuity we can conclude that M^n is contained a flat totally umbilical hypersurface of S_1^{n+1} or in a nonflat totally umbilical hypersurface of S_1^{n+1}.

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