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A SEQUENCE OF HOMOTOPY SUBGROUPS OF A *CW*-PAIR

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Abstract. For a self-map f of a CW-pair (X, A), we introduce the $G(f)$-sequence of (X, A) which consists of subgroups of homotopy groups in the homotopy sequence of (X, A) and show some properties of the relative homotopy Jiang groups. We also show a condition for the $G(f)$-sequence to be exact.

1. Introduction

D.H.Gottlieb [1,2] introduced the subgroups $G_n(X)$ of $\pi_n(X)$. Only a few $G_n(X)$ are known despite the fact that several authors have studied and generalized $G_n(X)$. In [5,13,14], the author, J.R. Kim and K.Y. Lee introduced subgroups $G_n(X, A)$ and $G_n^{Rel}(X, A)$ of $\pi_n(X)$ and $\pi_n(X, A)$ respectively and showed that they fit together into a G-sequence

$$\cdots \xrightarrow{j_\sharp} G_{n+1}^{Rel}(X, A) \xrightarrow{\partial} G_n(A) \xrightarrow{i_\sharp} G_n(X, A) \to \cdots$$
$$\xrightarrow{\partial} G_1(A) \xrightarrow{i_\sharp} G_1(X, A)$$

where $i_\sharp, j_\sharp$ and ∂ are restrictions of the usual homomorphisms of the homotopy sequence

$$\cdots \xrightarrow{j_\sharp} \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n(A) \xrightarrow{i_\sharp} \pi_n(X) \to \cdots \xrightarrow{\partial} \pi_1(A) \xrightarrow{i_\sharp} \pi_1(X).$$

Here we extend the concept of the above G-sequence into the $G(f)$-sequence for any self-map $f : (X, A) \to (X, A)$. In [7], S.H. Lee showed

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that it is still exact in the extended concept when $i : A \to X$ has a left homotopy inverse . In this paper, we will show some properties of relative homotopy Jiang subgroups and a condition for the $G(f)$-sequence to be exact.

2. Definitions

In this paper, all spaces are finite connected CW-complexes, all topolgical pairs are CW-pairs and all subspaces mentioned contain the same base point as their total spaces. We denote by A^A the subspace of the function space X^A consisting of $f \in X^A$ such that $f(A) \subset A$. Let us take x_0 as the base point of X and its subspaces. Let I^n be the n-dimensional cube, let ∂I^n be its boundary and let J^{n-1} be the union of all $(n-1)$ faces of I^n except for the initial face. We use the same notation ω for the evaluation maps of X^X and X^A into X at the base point x_0 and use i as the inclusion map.

The Gottlieb groups $G_n(X, x_0)$ are defined by $G_n(X, x_0) = \{[h] \in \pi_n(X) | \ \exists \text{ map } H : X \times I^n \longrightarrow X \text{ such that } [H|_{x_0 \times I^n}] = [h] \text{ and } H|_{X \times u} = 1_X \text{ for } u \in \partial I^n\}$. In [2], Gottlieb showed that the Gottlieb groups (or evaluation subgroups of the homotopy groups)$G_n(X, x_0)$ is the image of $\omega_\sharp : \pi_n(X^X, 1_X) \to \pi_n(X, x_0)$. He used these groups to obtain some results about the identifications of topological spaces and to study a fixed point theory and a fibration theory. After then, many authors[4,8,9,11,16] have studed and generalized $G_n(X, x_0)$.

In [13], the author and J. Kim have generalized $G_n(X, x_0)$ to $G_n^f(X, A, x_0)$ for any map $f : (A, x_0) \to (X, x_0)$. These subgroups are defined by $G_n^f(X, A, x_0) = \{[h] \in \pi_n(X) | \ \exists \text{ map } H : A \times I^n \longrightarrow X \text{ such that } [H|_{x_0 \times I^n}] = [h] \text{ and } H|_{A \times u} = f \text{ for } u \in \partial I^n\}$. These groups are called *generalized evaluation subgroups* of the homotopy groups (or *Woo-Kim groups*). We also show that the generalized evaluation subgroup $G_n^f(X, A, x_0)$ is the image of $\omega_\sharp : \pi_n(X^A, f) \to \pi_n(X, x_0)$. Especially, if $i : A \to X$ is the inclusion, we denote $G_n^i(X, A, x_0)$ by $G_n(X, A, x_0)$. $G_n(X, A, x_0)$ has always contained $G_n(X, x_0)$ and $G_n(X, A, x_0) = G_n(X, x_0)$ if $A = X$, $G_n(X, A, x_0) = \pi_n(X, x_0)$ if $A = x_0$.

In [5,14], the author and K.R. Lee introduced the subgroups $G_n^{Rel}(X, A, x_0)$ of the relative homotopy groups $\pi_n(X, A, x_0)$ which are defined by

$G_n^{Rel}(X, A, x_0) = \{[h] \in \pi_n(X, A, x_0) | \exists \text{ map } H : (X \times I^n, A \times \partial I^n) \longrightarrow (X, A)$ such that $[H|_{x_0 \times I^n}] = [h]$ and $H|_{X \times u} = 1_X$ for $u \in J^{n-1}\}$. Equivalently, $G_n^{Rel}(X, A, x_0)$ is the image of $\omega_\sharp : \pi_n(X^A, A^A, i) \to \pi_n(X, A, x_0)$, where A^A is the subspace of X^A which consists of maps from A into itself.

In [7], the relative homotopy Jiang group $G_n^{Rel}(f, x_0)$ is defined by $G_n^{Rel}(f, x_0) = \{[h] \in \pi_n(X, A, x_0) | \exists \text{ map } H : (X \times I^n, A \times \partial I^n) \longrightarrow (X, A)$ such that $[H|_{x_0 \times I^n}] = [h]$ and $H|_{X \times u} = f$ for $u \in J^{n-1}\}$ for $n \geq 2$ for a self-map f of a pair (X, A).

We leave the base points out of the notations when the simplification will not lead to confusion.

3. Relative homotopy Jiang groups

Let $f : (X, A) \to (X, A)$ be a self-map, $\bar{f} : A \to A$ be the restriction of f and $f_A = i \circ \bar{f} : A \to X$ be the restriction of f to A. We use X^A and A^A as the path-components of the function spaces X^A and A^A containing f_A and $\bar{f}$ respectively. Consider the evaluation map $\omega : X^A \longrightarrow X$ given by $\omega(h) = h(x_0)$, then ω induces a homomorphism $\omega_\sharp : \pi_n(X^A, A^A, f) \longrightarrow \pi_n(X, A, x_0)$.

THEOREM 3.1. *Let f be a self-map of (X, A). Then we have $G_n^{Rel}(f) = \omega_\sharp(\pi_n(X^A, A^A, f_A))$ for $n \geq 2$.*

PROOF. Let α be an arbitrary element of $\pi_n(X^A, A^A, f_A)$ for $n \geq 2$. Then α is represented by a map $h : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X^A, A^A, f_A)$. Let $F : A \times I^n \longrightarrow X$ be a map given by $F(x, u) = h(u)(x)$. Then $F(A \times \partial I^n) \subset A$ and $F(x, u) = f(x)$ for all $x \in A$ and $u \in J^{n-1}$. Define

$$G : X \times I^{n-1} \times 1 \cup (A \times I^{n-1} \cup X \times \partial I^{n-1}) \times I \longrightarrow X$$

by

$$G(x, v, s) = \begin{cases} f(x), & (\text{if } x \in X, v \in I^{n-1}, s = 1) \\ F(x, v, s), & (\text{if } x \in A, v \in I^{n-1}, s \in I) \\ f(x), & (\text{if } x \in X, v \in \partial I^{n-1}, s \in I). \end{cases}$$

This is well defined and continuous. Since $(X \times J^{n-1}, A \times I^{n-1} \cup X \times \partial I^{n-1})$ is a CW-pair, there exists an extension $\bar{G} : X \times I^n \longrightarrow X$ of G by the absolute homotopy extension property. Then

$$\bar{G}(A \times \partial I^n) = G(A \times \partial I^n) = F(A \times \partial I^n) \subset A.$$

Next, we have

$$\bar{G}(x_0, u) = \omega h(u)$$

for all $u \in I^n$ and

$$\bar{G}(x, u) = f(x)$$

for all $x \in X$ and $u \in J^{n-1}$. Thus $\bar{G} : (X \times I^n, A \times \partial I^n) \to (X, A)$ is an affiliated map with trace ωh in $G_n^{Rel}(f)$.

Conversely, let $[g]$ be an element of $\pi_n(X, A)$ such that there is an affiliated map

$$G : (X \times I^n, A \times \partial I^n) \to (X, A)$$

such that $[G|_{x_0 \times I^n}] = [g]$ and $G(x, u) = f(x)$ for all $x \in X$ and $u \in J^{n-1}$. Let $F = G|_{A \times I^n} : A \times I^n \to X$ be the restriction of G to $A \times I^n$. Define $h : I^n \to X^A$ by $h(u)(x) = F(x, u)$. Then $h(\partial I^n) \subset A^A$ and $h(u)(x) = F(x, u) = f(x)$ for all $x \in A$ and $u \in J^{n-1}$. Thus $[h]$ is an element in $\pi_n(X^A, A^A, f_A)$ such that $\omega_\sharp([h]) = [\omega h] = [g]$. This completes the proof of the theorem.

THEOREM 3.2. *Let $f_i : (X, A, x_0) \to (X, A, x_0)$ be maps for $i = 1, 2$. If f_1 is homotopic to f_2, then $G_n^{Rel}(f_1)$ is isomorphic to $G_n^{Rel}(f_2)$.*

PROOF. Let $H : A \times I \to X$ be a homotopy between f_1 and f_2. Then $\bar{H} : I \to X^A$ given by $\bar{H}(s)(a) = H(a, s)$ is a path from f_1 to f_2 and $\omega \circ \bar{H} : I \to X$ is a loop at x_0. Thus $\bar{H}_\sharp : \pi_n(X^A, A^A, f_1) \longrightarrow \pi_n(X^A, A^A, f_2)$ and $(\omega \circ \bar{H})_\sharp : \pi_n(X, A, x_0) \longrightarrow \pi_n(X, A, x_0)$ are isomorphisms. If we consider the following commutative diagram

$$\begin{array}{ccc}
\pi_n(X^A, A^A, f_1) & \xrightarrow{\bar{H}_\sharp} & \pi_n(X^A, A^A, f_2) \\
\downarrow \omega_\sharp & & \downarrow \omega_\sharp \\
\pi_n(X, A, x_0) & \xrightarrow{(\omega \circ \bar{H})_\sharp} & \pi_n(X, A, x_0),
\end{array}$$

then we have

$$\begin{aligned}(\omega \circ \bar{H})_\sharp(G_n^{Rel}(f_1)) &= (\omega \circ \bar{H})_\sharp \omega_\sharp(\pi_n(X^A, A^A, f_1)) \\ &= \omega_\sharp \bar{H}_\sharp(\pi_n(X^A, A^A, f_1)) \\ &= \omega_\sharp(\pi_n(X^A, A^A, f_2)) \\ &= G_n^{Rel}(f_2).\end{aligned}$$

Therefore $G_n^{Rel}(f_1)$ is isomorphic to $G_n^{Rel}(f_2)$.

In [1], Gottlieb showed that $G_1(X)$ is contained in the center of $\pi_1(X)$.

THEOREM 3.3. *Let f be a self-map of (X, A). Then $G_2^{Rel}(f)$ is contained in the centralizer of a subgroup $f_\sharp(\pi_2(X, A))$ of $\pi_2(X, A)$.*

PROOF. Let $[h] \in G_2^{Rel}(f)$ and $[g] \in \pi_2(X, A)$. It is sufficient to show $[h]f_\sharp([g]) = f_\sharp([g])[h]$. Since $[h] \in G_2^{Rel}(f)$, there exists an affiliated map

$$F : (X \times I^2, A \times \partial I^2) \longrightarrow (X, A)$$

such that $[F|_{x_0 \times I^2}] = [h]$ and $F(x, u) = f(x)$ for all $x \in X$ and $u \in J^1$. Since (X, A) is a CW-pair, we may assume $F(x_0, u) = h(u)$ for all $u \in I^2$. Define a homotopy

$$G : (I^2 \times I, \partial I^2 \times I, J^1 \times I) \longrightarrow (X, A, x_0)$$

by

$$G((t_1, t_2), s) = \begin{cases} F(g(2t_1(1-s), t_2), (2t_1 s, t_2)), \\ \qquad \text{if } 0 \le t_1 \le \frac{1}{2} \\ F(g(1-(2-2t_1)s, t_2), ((2-2t_1)s + 2t_1 - 1, t_2)), \\ \qquad \text{if } \frac{1}{2} \le t_1 \le 1. \end{cases}$$

This is well defined and continuous. Since

$$G((t_1, t_2), 0) = (fg * h)(t_1, t_2)$$

and

$$G((t_1, t_2), 1) = (h * fg)(t_1, t_2)$$

for $(t_1, t_2) \in I^2$, we have $[fg][h] = [h][fg]$.

In the above theorem, if we take f as the identity map of (X, A), then we obtain the following corollary.

COROLLARY 3.4. *Let (X, A) be a CW-pair. Then $G_2^{Rel}(X, A)$ is contained in the center of $\pi_2(X, A)$.*

4. $G(f)$-sequences and exactness

Let f be a self-map of (X, A) , $\bar{f} : A \to A$ and $f_A = i \circ \bar{f} : A \to X$ be the restriction of f to A.

LEMMA 4.1. *Let $f : (X, A) \to (X, A)$ be a self-map. Then $G_n^{\bar{f}}(A, A)$, $G_n^{f_A}(X, A)$ and $G_n^{Rel}(f)$ form a sequence :*

$$\cdots \xrightarrow{j_\sharp} G_{n+1}^{Rel}(f) \xrightarrow{\partial} G_n^{\bar{f}}(A, A) \xrightarrow{i_\sharp} G_n^{f_A}(X, A) \to \cdots$$
$$\xrightarrow{j_\sharp} G_2^{Rel}(f) \xrightarrow{\partial} G_1^{\bar{f}}(A, A) \xrightarrow{i_\sharp} G_1^{f_A}(X, A).$$

PROOF. The evaluation map $\omega : (X^A, A^A, f_A) \longrightarrow (X, A, x_0)$ gives rise to the following commutative ladder of the homotopy groups :

$$\begin{array}{ccccccccc}
\cdots \longrightarrow \pi_n(A^A, \bar{f}) & \xrightarrow{\bar{i}_\sharp} & \pi_n(X^A, f_A) & \xrightarrow{\bar{j}_\sharp} & \pi_n(X^A, A^A, f_A) & \xrightarrow{\bar{\partial}} & \pi_{n-1}(A^A, \bar{f}) & \longrightarrow \cdots \\
\downarrow \omega_\sharp & & \downarrow \omega_\sharp & & \downarrow \omega_\sharp & & \downarrow \omega_\sharp & \\
\cdots \longrightarrow \pi_n(A) & \xrightarrow{i_\sharp} & \pi_n(X) & \xrightarrow{j_\sharp} & \pi_n(X, A) & \xrightarrow{\partial} & \pi_{n-1}(A) & \longrightarrow \cdots .
\end{array}$$

By the fact $G_n^{\bar{f}}(A, A) = \omega_\sharp \pi_n(A^A, \bar{f})$, we have

$$\begin{aligned}
i_\sharp(G_n^{\bar{f}}(A, A)) &= i_\sharp \omega_\sharp \pi_n(A^A, \bar{f}) \\
&= \omega_\sharp \bar{i}_\sharp \pi_n(A^A, \bar{f}) \subset \omega_\sharp \pi_n(X^A, f_A) = G_n^{f_A}(X, A).
\end{aligned}$$

Similarly, we have $j_\sharp(G_n^{f_A}(X, A)) \subset G_n^{Rel}(f)$ and $\partial(G_n^{Rel}(f)) \subset G_{n-1}^{\bar{f}}(A, A)$. Thus $G_n^{\bar{f}}(A, A)$, $G_n^{f_A}(X, A)$ and $G_n^{Rel}(f)$ form a sequence.

This sequence is called the $G(f)$-*sequence* of (X, A) for the self-map f. In general, the $G(f)$-sequence is not exact(see Theorem 3.4 in [5]). The G-sequence in [5,14] is just the $G(1)$-sequence,where 1 is the identity map of (X, A).

Now we show a condition for the $G(f)$-sequence to be exact. In [7], Lee showed that for any self-map f of (X, A), the $G(f)$-sequence of (X, A) is exact when the inclusion map $i : A \to X$ has a left homotopy inverse.

Let B^n be the n-dimensional ball and S^{n-1} the boundary of B^n. Then the inclusion map $i : S^{n-1} \to B^n$ does not have a left homotopy inverse, but the pair (B^n, S^{n-1}) has the exact $G(1)$-sequence[15]. We will generalize this result.

LEMMA 4.2. *Let $f : (A, x_0) \to (X, x_0)$ be a map such that $f(A) \subset A$. If f is homotopic to a constant map, then $G_n^f(X, A) = \pi_n(X)$ for $n \geq 1$.*

PROOF. Let $f : A \to X$ be homotopic to the constant map $c_{x_0} : A \to x_0 \in X$. Since $G_n^f(X, A) \subset \pi_n(X)$, it is sufficient to show that $\pi_n(X) \subset G_n^f(X, A)$ for $n \geq 1$. Let $\alpha \in \pi_n(X)$ and $h : (I^n, \partial I^n) \longrightarrow (X, x_0)$ be a representative of α. Since $f : A \longrightarrow X$ is homotopic to c_{x_0}, there exists a homotopy $H : A \times I \to X$ such that

$$H(x, 1) = f(x), H(x, 0) = x_0 \text{ and } H(x_0, t) = x_0$$

for all $x \in A$ and $t \in I$. Define $K : A \times \partial I^n \times I \cup A \times I^n \times 0 \longrightarrow X$ by

$$K(x, u, t) = \begin{cases} H(x, t), & (\text{if } x \in A, u \in \partial I^n, t \in I) \\ h(u), & (\text{if } x \in A, u \in I^n, t = 0). \end{cases}$$

This is well-defined and continuous. Since $(A \times I^n, A \times \partial I^n)$ is a CW-pair, there exists an extension $\bar{K} : A \times I^n \times I \to X$ of K by the absolute homotopy extension property. Define $F : A \times I^n \to X$ by

$$F(x, u) = \bar{K}(x, u, 1),$$

then $F(x, u) = f(x)$ for all $x \in A$ and $u \in \partial I^n$. Let $g = F|_{x_0 \times I^n}$. Then F is an affiliated map with trace g. Thus $[g] \in G_n^f(X, A)$. Since g is homotopic to h relative to ∂I^n by the homotopy $\bar{K}|_{x_0 \times I^n \times I}$, we have $\alpha = [h] = [g] \in G_n^f(X, A)$.

THEOREM 4.3. *Let f be a self-map of (X, A). If the inclusion $i : A \to X$ is homotopic to a constant map, then the $G(f)$-sequence of (X, A) is exact.*

PROOF. Consider the following commutative ladder

$$\begin{array}{ccccccccccc}
\cdot & \longrightarrow & \pi_n(A) & \xrightarrow{i_*} & \pi_n(X) & \xrightarrow{j_*} & \pi_n(X,A) & \xrightarrow{\partial} & \pi_{n-1}(A) & \longrightarrow & \cdot \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdot & \longrightarrow & G_n^f(A,A) & \longrightarrow & G_n^{f_A}(X,A) & \longrightarrow & G_n^{Rel}(f) & \longrightarrow & G_{n-1}^f(A,A) & \longrightarrow & \cdot
\end{array}$$

for $n > 1$. Since the inclusion $i : A \to X$ is homotopic to a constant map, $i_\sharp$ is the 0-homomorphism and hence $j_\sharp$ is a monomorphism. Thus, $i_\sharp|_{G_n^f(A,A)}$ is the 0-homomorphism and $j_\sharp|_{G_n^{f_A}(X,A)}$ is a monomorphism. Since $i : A \to X$ is homotopic to a constant map, there exists a homotopy $H : A \times I \to X$ such that $H(,0) = i$ and $H(,1) = c_{x_0}$. Define a homotopy $K : A^A \times I \to X^A$ given by $K(h,t)(a) = H(h(a),t)$. Then K is a homotopy between the inclusion $\bar{i} : (A^A, \bar{f}) \to (X^A, f_A)$ given by $\bar{i}(h) = i \circ h$ and the constant map. Consider the homotopy sequence

$$\cdots \longrightarrow \pi_n(X^A, A^A, f_A) \xrightarrow{\bar{\partial}} \pi_{n-1}(A^A, \bar{f}) \xrightarrow{\bar{i}_\sharp} \pi_{n-1}(X^A, f_A) \longrightarrow \cdots$$

of the triplet (X^A, A^A, f_A), then $\bar{i}_\sharp$ is the 0-homomorphism and hence $\bar{\partial}$ is an epimorphism. Since the following diagram

$$\begin{array}{ccc}
\pi_n(X^A, A^A, f_A) & \xrightarrow{\bar{\partial}} & \pi_{n-1}(A^A, \bar{f}) \\
\downarrow \omega_\sharp & & \downarrow \omega_\sharp \\
\pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0)
\end{array}$$

commutes, we have

$$\begin{aligned}
\partial(G_n^{Rel}(f)) &= \partial(\omega_\sharp(\pi_n(X^A, A^A, f))) \\
&= \omega_\sharp(\bar{\partial}(\pi_n(X^A, A^A, f))) \\
&= \omega_\sharp(\pi_{n-1}(A^A, \bar{f})) \\
&= G_{n-1}^f(A, A).
\end{aligned}$$

This implies $\partial|_{G_n^{Rel}(f)}$ is an epimorphism. Thus the lower sequence on the commutative ladder is exact at $G_n^{f_A}(X,A)$ and $G_n^f(A,A)$ for $n \geq 1$.

We must show that the $G(f)$-sequence is exact at $G_n^{Rel}(f)$ for $n \geq 2$. It is clear that

$$\text{image of } j_\sharp|_{G_n^{f_A}(X,A)} \subset \text{kernel of } \partial|_{G_n^{Rel}(f)}.$$

Since $i : A \to X$ is homotopic to a constant map f_A is homotopic to a constant map and hence $G_n^{f_A}(X, A) = \pi_n(X)$ for $n \geq 1$ by Lemma 4.2. Thus we have

$$\begin{aligned}\text{kernel of } \partial|_{G_n^{Rel}(f)} &\subset \text{kernel of } \partial = j_\sharp(\pi_n(X))\\ &= j_\sharp(G_n^{f_A}(X, A)) = \text{image of } j_\sharp|_{G_n^{f_A}(X,A)}.\end{aligned}$$

This proves the $G(f)$-sequence of (X, A) is exact.

COROLLARY 4.4. *Let f be a self-map of (X, A). Then the $G(f)$-sequence of (X, A) is exact if X or A is contractible.*

EXAMPLE. If we identify S^k with $\{(x_1, .., x_i, .., x_{n+1}) \in S^n \mid x_i = 0$ for $i > k+1\}$ for $k < n$, then the inclusion map $i : S^k \to S^n$ is homotopic to a constant map. By Theorem 4.3, (S^n, S^k) has the exact $G(f)$-sequence for $n > k > 0$ for any self-map f.

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