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A RELATIVE ROOT NIELSEN NUMBER

KI-YEOL YANG

ABSTRACT. The relative Nielsen number $N(f; X, A)$ was introduced in 1986. It gives us a better, and ideally sharp, lower bound for the minimun number $MF[f; X, A]$ of fixed points in the homotopy class of the map $f : (X, A) \to (X, A)$. Similarly, we also can think about the Nielsen root theory. In this paper, we introduce a relative root Nielsen number $N(f; X, A, c)$ of $f : (X, A) \to (Y, B)$ and show some basic properties.

1. Introduction

In topological coincidence theory the coincidence problem, finding solutions to $f(x) = g(x)$ for maps $f, g : X \to Y$, can sometimes be converted to a root problem, finding solutions $f(x) = c$. That is, the special case; $g : X \to Y$ is the constant map $g(x) = c$ for some $c \in Y$. Nielsen root theory was found to possess features distinctively different from those of fixed point theory or general coincidence theory and is concerned with the determination of the minimum number of roots in the homotopy class of a given map $f : X \to Y$.

The relative Nielsen number $N(f; X, A)$ for a selfmap $f : (X, A) \to (X, A)$ of a pair of spaces is introduced by H.Schirmer [8]. $N(f; X, A)$ gives us a better, and ideally sharp, lower bound for the minimum number $MF[f; X, A]$ of fixed points in the homotopy class of the map $f : (X, A) \to (X, A)$. Similarly, we also can think about the relative case in root theory. That is, for the map $f : (X, A) \to (Y, B)$ of pairs of spaces define a relative root Nielsen number $N(f; X, A, c)$ of f.

It is the purpose of this paper to introduce such a relative root Nielsen number $N(f; X, A, c)$. This definition of $N(f; X, A, c)$ yields a positive

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integer which has the usual basic properties of the relative Nielsen number $N(f;X,A)$, and we show some examples in §3 that it can be easy to find $N(f;X,A,c)$.

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2. The relative root Nielsen number

Let $f : X \to Y$ be a map (continuous function) and $c \in Y$. Two solutions (roots) x_0 and x_1 to $f(x) = c$ are equivalent iff there is a path p in X from x_0 to x_1 such that $[f \circ p] = [c]$. (Here $[f \circ p]$ denotes the fixed-end-point homotopy class containing $f \circ p$ and c is used both to denote the point $c \in Y$ as well as the constant path at $c \in Y$).

This equivalence is induced an equivalence relation; an equivalence class of roots is calleded *root class*. The set of roots of $f(x) = c$ is denoted by $\Gamma(f,c)$, the set of root classes by $\Gamma'(f,c)$.

Let $H : f_0 \simeq f_1 : X \to Y$ be a homotopy between f_0 and f_1, c a given point in Y, and x_i a root of the equation

$$f_i(x) = c, \quad i = 0, 1$$

and R_i a root class in $\Gamma'(f_i, c)$ containing x_i. If there exists in X a path p from x_0 to x_1 such that

$$[\triangle(H,p)] = [c],$$

then x_0 and x_1 are said to be in correspondence under H and denoted by x_0Hx_1, where $\triangle(H,p)$ is a diagonal path defined by $\triangle(H,p)(t) = H(p(t),t), 0 \leq t \leq 1$. This relation x_0Hx_1 induces a correspondence from R_0 to R_1 under H, which is denoted by R_0HR_1.

Let $f : X \to Y$ be a mapping under $H : f \simeq H(\cdot,1) : X \to Y$ a homotopy. Let the root class in $\Gamma'(f,c)$ be denoted by R. If the root class R of the equation $f(x) = c$ corresponds to a root class $\in \Gamma'(H(\cdot,1),c)$ under any such H, then R is called an *essential root class*. Denote the set of essential root classes of the equation $f(x) = c$ by $\Gamma^*(f,c)$. The number $\#\Gamma^*(f,c)$ of elements in $\Gamma^*(f,c)$ is called the *Nielsen number* of $f(x) = c$ and is denoted by $N(f,c)$.

$N(f,c)$ is clearly a lower bound for the number of solutions of $f(x) = c$. If g is homotopic to f, then $N(f,c) = N(g,c)$.

Let $f : (X,A) \to (Y,B)$ be a map of pair of spaces. We shall write $\bar{f} : A \to B$ be a restriction of f to A, and $\Gamma(\bar{f},c) = \Gamma(f,c) \cap A$ if $c \in B$. Throughout this paper c will be a point lies in $B(\subset Y)$.

DEFINITION 2.1. Let $f : (X,A) \to (Y,B)$ be a map of pair of spaces. A root class $R \in \Gamma'(f,c)$ of $f : X \to Y$ is a *common root class* of f and $\bar{f}$ if $R \cap \Gamma_e(\bar{f},c) \neq \varnothing$ where $\Gamma_e(\bar{f},c)$ is the essential root set of $\bar{f} : A \to B$. It is an *essential common root class* of f and $\bar{f}$ if it is an essential root class of $f : X \to Y$ and a common root class of f and $\bar{f}$.

LEMMA 2.2. *Let $f : (X,A) \to (Y,B)$ be a map and $c \in B$ and let $R \in \Gamma'(f,c)$ and $\bar{R} \in \Gamma'(\bar{f},c)$. If $R \cap \bar{R} \neq \varnothing$, then $\bar{R} \subset R$.*

PROOF. If $a_0 \in R \cap \bar{R}$ and $a_1 \in \bar{R}$, then there exists a path p in A from a_0 to a_1 such that $[\bar{f} \circ p] = c$. So that $[f \circ p] = c$ since $A \subset X$. This implies $a_1 \in R$. □

Similarly, we define $N(\bar{f},c)$ for the number of essential root classes of $\bar{f}$ and write $N(f,\bar{f},c)$ for the number of essential common root classes of f and $\bar{f}$. If X is a compact connected ANR, then $N(f,\bar{f},c)$ is finite as $0 \leq N(f,\bar{f},c) \leq N(f,c)$.

DEFINITION 2.3. Let $(X,A),(Y,B)$ be pairs of compact connected ANR's. If $f : (X,A) \to (Y,B)$ is a map, the *relative root Nielsen number* $N(f;X,A,c)$ is defined as

$$N(f;X,A,c) = N(f,c) + N(\bar{f},c) - N(f,\bar{f},c).$$

Hence $N(f;X,A,c)$ is a finite integer ≥ 0 and equals $N(f,c)$ if $X = A$ or $A = \varnothing$. The next theorems list some other cases in which the relative root Nielsen number equals an ordinary one.

THEOREM 2.4. *Let $(X,A),(Y,B)$ be pairs of compact connected AN R's and let $f : (X,A) \to (Y,B)$ be a map and $c \in B$.*

(1) *If $N(f,c) = 0$, then $N(f;X,A,c) = N(\bar{f},c)$.*
(2) *If $N(\bar{f},c) = 0$, then $N(f;X,A,c) = N(f,c)$.*

PROOF. This is obvious from the definition, as in both case $N(f,\bar{f},c) = 0$. □

THEOREM 2.5. *Let $(X,A),(Y,B)$ be pairs of compact connected AN R's and let $f : (X,A) \to (Y,B)$ be a map. If Y is simply connected, then*

$$N(f;X,A,c) = \begin{cases} N(f,c), & if \quad N(\bar{f},c) = 0 \\ N(\bar{f},c), & if \quad N(\bar{f},c) \neq 0. \end{cases}$$

PROOF. We only have to consider the case where $N(f,c) \neq 0$ and $N(\bar{f},c) \neq 0$. If Y is simply connected, then $f : X \to Y$ has one essential root class R, and $\bar{f} : A \to B$ has at least one essential root class $\bar{R}$. But if $x \in R$ and $a \in \bar{R}$, then a is a root of $f : X \to Y$ and is in the same root class as x, so $N(f;X,A,c) = 1$. □

Now we consider some basic properties of $N(f;X,A,c)$.

THEOREM 2.6 (LOWER BOUND PROPERTY). *If $(X,A),(Y,B)$ are pairs of compact connected ANR's, then every map $f : (X,A) \to (Y,B)$ has at least $N(f;X,A,c)$ roots.*

PROOF. Let $\bar{f} : A \to B$ have the essential root classes $\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_m$ and let $f : X \to Y$ has the essential root classes $R_1, R_2, \ldots, R_n, R_{n+1}, \ldots, R_s$ which are indexed so that the essential common root class of f and $\bar{f}$ are $R_{n+1}, R_{n+2}, \ldots, R_s$. Then

$$N(f;X,A,c) = m + s - (s - n) = m + n.$$

Each root class $\bar{R}_i$ contains at least one root a_i of $\bar{f}$, and each root class R_j contains at least one root x_j of f. If $j = 1, 2, \ldots, n$, then $R_j \cap A$ is distinct from $\Gamma_c(\bar{f},c)$ and so the set $\{a_1, a_2, \ldots, a_m, x_1, x_2, \ldots, x_n\}$ consists of $m + n$ distinct points which are all roots of $f : (X,A) \to (Y,B)$. □

THEOREM 2.7. *If $(X,A),(Y,B)$ are pairs of compact ANR's and $f : (X,A) \to (Y,B)$ is a map, then $N(f;X,A,c) \geq N(\bar{f},c)$ and $N(f;X,A,c) \geq N(f,c)$.*

PROOF. By the fact, each essential common root class of f and $\bar{f}$ contains at least one essential root class of $\bar{f}$, we have $N(f,\bar{f},c) \leq N(\bar{f},c)$ and hence

$$N(f;X,A,c) = N(f,c) + [N(\bar{f},c) - N(f,\bar{f},c)] \geq N(f,c).$$

From $N(f,\bar{f},c) \leq N(f,c)$ follows $N(f;X,A,c) \geq N(\bar{f},c)$. □

THEOREM 2.8 (HOMOTOPY INVARIANCE). *If $(X,A),(Y,B)$ are pairs of compact connected ANR's and if the maps $f_0, f_1 : (X,A) \to (Y,B)$ are homotopic, then*

$$N(f_0;X,A,c) = N(f_1;X,A,c).$$

PROOF. As it is well known that $N(f_0,c) = N(f_1,c)$ and $N(\bar{f}_0,c) = N(\bar{f}_1,c)$ (By [6],p.129 Theorem 4.4). It suffices to show that $N(f,\bar{f},c)$ is invariant under homotopies $H : (X \times I, A \times I) \to (Y,B)$. i.e., $N(f_0,\bar{f}_0;c) = N(f_1,\bar{f}_1,c)$. Let R_0 be an essential common root class of f_0 and $\bar{f}_0$. Then R_0 contains an essential root class $\bar{R}_0$ of $\bar{f}_0$

$$(\text{i.e., } {}^{\exists}\bar{R}_0 \in \Gamma^*(\bar{f}_0,c) \text{ s.t. } \bar{R}_0 \subset R_0).$$

Since $\bar{f}_0$ and $\bar{f}_1$ are homotopic under the restriction $\bar{H}$ of H to $A \times I$, and by definition of essential root class of $\bar{f}$ there exist an essential root class $\bar{R}_1$ s.t. $\bar{R}_0\bar{H}\bar{R}_1(\bar{R}_1 \in \Gamma^*(\bar{f}_1,c))$ i.e., for $y_0 \in \bar{R}_0, y_1 \in \bar{R}_1$

$${}^{\exists}\text{a path } p_t \text{ in } A \text{ from } y_0 \text{ to } y_1 \text{ s.t., } y_0\bar{H}y_1$$

so that the path $\triangle(\bar{H},p)$ and c are homotopic in A. Now let R_1 be the root class of f_1 containing y_1. Since $\{p_t\}$ is a path in X from y_0 to y_1 and $\triangle(\bar{H},p) = \triangle(H,p)$. This means that y_0Hy_1 and $R_0HR_1 \iff R_1H^{-1}R_0$. That is, R_1 corresponds to a root class $R_0 \in \Gamma'(H^{-1}(\cdot,1),c)$ under H^{-1}. Hence R_1 is an essential root class (by def.) and $R_1 \cap \bar{R}_1 \neq \varnothing$. So R_1 is lass of f_1 and $\bar{f}_1$. i.e., $R_1 \in \Gamma^*(f_1,\bar{f}_1,c)$. Hence $N(f_0,\bar{f}_0,c) = N(f_1,\bar{f}_1,c)$. □

3. Some examples

EXAMPLE 1. Let $X = Y = \mathbb{R}^1$ and $A = [0,1], B = [0,3]$. If $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a map given by $f(x) = x^2$ and take $c = 1 \in B$. Then $\Gamma(f,1) = \{-1,1\}$ and $\Gamma(\bar{f},1) = \{1\}$. Since $f \simeq g$ where $g(x) = 0, \forall x \in X$ and g has no roots at 1, so $N(g,1) = 0$.

By the homotopy invariance, we know that $N(f,1) = 0$. Similarly, we can show that $N(\bar{f},1) = 0$.

EXAMPLE 2. Let $X = Y = B^2 = \{z|\ |z| \leq 1\}$ and $A = B = S^1$ (the boundary of B^2). If $f : (B^2, S^1) \to (B^2, S^1)$ is a map given by $f(z) = z^2$ and we take $c = 1 \in S^1$.

Then $\Gamma(\bar{f},1) = \{e^{\pi i}, e^{2\pi i}\}$ and each root constitute a root class by itself and since $\deg\bar{f} = 2$ is not zero, every root class is essential. Hence $N(\bar{f},1) = 2$. (it follows from [6], Example 3.3 p.127 and Corollary 7.3 p.138). Now $f : B^2 \to B^2$ is homotopic to g s.t. $g(x) = x_0, x_0 \neq 1, \forall x \in B^2$. Hence $N(g,1) = 0$ and $N(f,1) = 0$ and easily we know the fact by definition $N(f,\bar{f},1) = 0$. Thus we have

$$N(f; X, A, c) = N(f; B^2, S^1, 1) = 2.$$

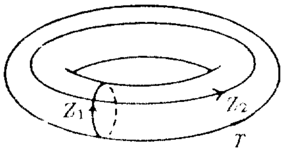
EXAMPLE 3. (Compare with Example 2) Let $f : (X,A) \to (Y,B)$ be a map given by Example 2. Now, we consider the case of $c \notin B$.

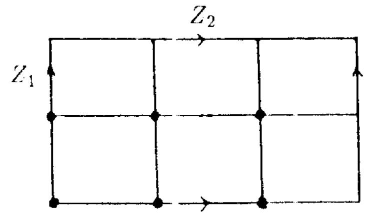
If $f : (B^2, S^1) \to (B^2, S^1)$ is a map which has the property $\deg\bar{f} \neq 0$, where $\bar{f} = f|_{S^1} : S^1 \to S^1$. Then since $c \notin S^1, N(\bar{f},c) = 0$. Also $N(f,c) = 0$ as Example 2. Thus we have $N(f; X, A, c) = 0$. But every $g(\simeq f) : (X,A) \to (Y,B)$ must have a root as g must be onto.

EXAMPLE 4. Let $T = S_1^1 \times S_2^1$ be an ordinary torus and $f : T \to T$ a map of pairs given by $f(z_1, z_2) = (z_1^m, z_2^n), (m \neq 0, n \neq 0)$. Take $c = 1 \in S_1^1$. As we know, the fundamental group of T is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Let A denote the matrix of $f_\# : \pi_1(T) \to \pi_1(T)$ relative the basis; we require that $\det A \neq 0$. Using the Proposition 6 1 of [3], we see that $N(f,c) = |\det\ A| = |mn|$.

Also, since T is orientable, we know that $N(f,c) = |\deg\ f|$. And the Nielsen number $N(\bar{f},c)$ of $\bar{f} : S_1^1 \to S_1^1$ is $|\deg\ \bar{f}| = |m|$. Now, to calculate the number $N(f,\bar{f},c)$, consider the special case $m = 2, n = 3$.

We know that $N(f,1) = 6$, and in this case we can realize the locations of roots of f as follows;

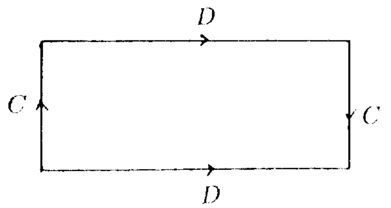


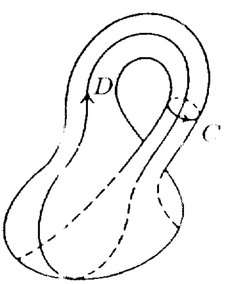


Since $\Gamma(\bar{f},1) = \{e^{2\pi ri/|m|}, r = 1,2,\ldots,|m|\}$ and each root consists a root class by itself. Also $\deg\bar{f} = m$ is not zero, each root class is essential. That is, $\Gamma(\bar{f},1)$ coincides with $\Gamma_e(\bar{f},1)$. As we shcwn in the special case, we know that $N(f,\bar{f},c) = N(\bar{f},c)$ by definition of $N(f,\bar{f},c)$. Thus we have

$$N(f;X,A,c) = N(f;T,S^1,1) = |\det A|.$$

EXAMPLE 5. Let K be a Klein bottle. We know that K is non-orientable 2-dimensional close manifold. It is the quotient space obtained from rectangle by pasting the edges as figure.





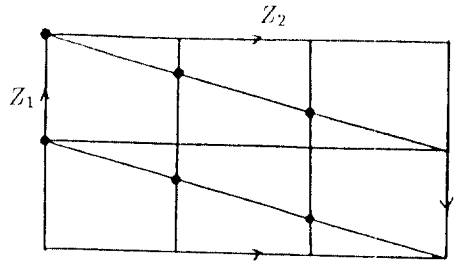
Then in terms of generators and relations

$$\pi_1(K) = < D, C | DCD^{-1}C = 1 >$$

The following property is shown from Proposition 6.4 in [4]; Let $f : K \to K$ be a map. Then there are integers b, d and e such that $f_\#(D) = D^bC^d, f_\#(C) = C^e$, and either b is odd or $e = 0$. In either event, $N(f,c) = |be|$.

So, if we let $X = Y = K$ and $A = B = C(\cong S^1)$ and we regard $\bar{f}$ as the map $\bar{f} : S^1 \to S^1$ of degree $\bar{f}(\neq 0)$. Then $N(\bar{f}, c) = |\deg \bar{f}|$. To calculate the number $N(f, \bar{f}, c)$, also consider the special case. Now we regard $f : K \to K$ given by $f(z_1, z_2) = (z_1^e, z_2^b z_1^d)$ The interesting case is b odd and $\deg \bar{f} = e \neq 0$.

Let $b = 3, d = 0$ and $e = 2$. Then we know $N(f, c) = |be| = 6$, also we can realize the location of roots of f as follows;



And the same argument of Example 4 we obtain $N(f, \bar{f}, c) = N(\bar{f}, c)$. Hence $N(f; X, A, c) = N(f, c) = |be|$.

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Department of Mathematics Education
Sunchon National University
Sunchon 540-742, Korea