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EQUIVARIANT VECTOR BUNDLE STRUCTURES ON REAL LINE BUNDLES

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ABSTRACT. Let G be a topological group and X a G space. For a given nonequivariant vector bundle over X there does not always exist a G equivariant vector bundle structure. In this paper we find some sufficient conditions for nonequivariant real line bundles to have G equivariant vector bundle structures.

1. Introduction

Let G be a topological group, and let X be a G space. Let ξ be a nonequivariant real vector bundle over X. We say that ξ has a G *equivariant vector bundle structure* if there is a G vector bundle over X which is nonequivariantly isomorphic to ξ. The question we are interested in here is when does ξ have a G equivariant real vector bundle structure. It is obvious that not every nonequivariant vector bundle has a G equivariant vector bundle structure, see Example 1.2. In this note we study the question for real line bundles.

It is well known in vector bundle theory that the isomorphism classes of real line bundles over X are in one to one correspondence with the first cohomology class $\alpha \in H^1(X, \mathbb{Z}_2)$ of X, where the isomorphism class of a line bundle ξ corresponds to its first Stiefel-Whitney class $w^1(\xi) \in H^1(X, \mathbb{Z}_2)$[2, p 236 Theorem 3.4].

Since G acts on X the action induces an action of G on $H^*(X, \mathbb{Z}_2)$. Let ξ be a G equivariant real line bundle over X. Then by the naturality of Stiefel-Whitney classes we have

$$w^i(\xi) = g^* w^i(\xi)$$

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for all $g \in G$ [4, p37]. This implies that $w^i(\xi) \in H^i(X, \mathbb{Z}_2)^G$. In other words one of the necessary conditions for a nonequivariant real line bundle ξ to have a G equivariant vector bundle structure is that its Stiefel-Whitney classes $w^i(\xi)$ lie in $H^i(X, \mathbb{Z}_2)^G$. Sometimes this necessary condition is sufficient for a nonequivariant vector bundle to have a G equivariant vector bundle structure. In fact we have the following theorem. Since trivial nonequivariant bundles over G spaces have obvious G-equivariant vector bundle structures we only consider nontrivial vector bundles.

THEOREM 1.1. *Let G be a topological group and X a connected G space. A given nontrivial nonequivariant real line bundle ξ over X has a G equivariant vector bundle structure if*

1. *the first Stiefel-Whitney class $w^1(\xi) \in H^1(X, \mathbb{Z}_2)^G$, and*
2. *if one of the following conditions is satisfied:*
   (a) *X is locally pathconnected and $X^G \neq \emptyset$.*
   (b) *G is of odd order, or more generally G is a discrete group whose second group cohomology $H^2(G, \mathbb{Z}_2) = 0$.*

The above theorem is in a sense 'complete' because the following example shows that there exists a nonequivariant real line bundle over a G circle without a G equivariant vector bundle structure for even order cyclic group G in which case $H^2(G, \mathbb{Z}_2) \neq 0$.

EXAMPLE 1.2. Let G be the cyclic group of order $2d$. Let V be the orthogonal 2 dimensional G representation space where the generator of G acts on V as the rotation by π/d. Let $S(V)$ be the unit circle, and let ξ be the nontrivial nonequivariant real line bundle over $S(V)$. Then ξ does not have a G equivariant real line bundle structure.

PROOF. It is proved by Kim and Masuda in [5] that for any compact Lie group K every K equivariant real line bundle over K circle is equivariantly isomorphic to either a trivial bundle $S(U) \times \delta \to S(U)$ or a nontrivial bundle $S(U) \times_{\mathbb{Z}_2} \delta \to S(U)/\mathbb{Z}_2 = P(U)$ according as the G line bundle $L \to S^1$ is nonequivariantly trivial or not for some orthogonal representation space U of K. Here $P(U)$ is the real projective space of U. Thus if V is as above, then it is easy to see that there is no orthogonal representation space U of G such that $S(V)$ is G homeomorphic to

$P(U)$. This implies that ξ does not have a G equivariant vector bundle structure.

It should be noted that the real line bundle ξ in the above example does have a G' equivariant real line bundle structure where G' is the cyclic group of order $4d$ and V is viewed as an orthogonal representation of G' where the generator acts as the rotation by π/d. Such phenomenon is not accidental. In fact we have the following:

PROPOSITION 1.3. *Let X be a connected G space and ξ a nonequivariant real line bundle over X. If the first Stiefel-Whitney class $w^1(\xi)$ lies in $H^1(X, \mathbb{Z}_2)^G$, then there exists an extension G' of $\mathbb{Z}_2$ by G such that ξ has a G' equivariant vector bundle structure. Here we consider X as a G' space with the obvious G' action.*

In Chapter 1 Section 9 of [1] liftings of actions of Lie groups on base spaces to covering spaces are treated, and we could possibly reach a similar conclusion if we treat our problem as a pure covering space theoretic one as Bredon did. However our approach is simpler, and we do not assume the acting group to be a Lie group.

The author would like to express his appreciation to the referee for generalizing the original results.

2. Proof of the main results

We first prove Proposition 1.3. Suppose ξ is a nontrivial nonequivariant real line bundle over X such that $w^1(\xi) \in H^1(X, \mathbb{Z}_2)^G$. Let $\tilde{X} = S(\xi)$ be the corresponding unit sphere bundle of ξ, which is a double cover of X. For each $g \in G$ we want to construct a map $\tilde{g} : \tilde{X} \to \tilde{X}$ which covers the map $g : X \to X$ to as follows. Let $g^*(\xi) = \{(x, v) \in X \times \xi \mid gx = pv\}$ be the total space of the pull back bundle of ξ by the map $g : X \to X$. Let $g' : g^*(\xi) \to \xi$ be the induced bundle map. By the naturality of characteristic classes

$$\begin{aligned} w^1(g^*(\xi)) =& g^* w^1(\xi) \\ =& w^1(\xi) \qquad \text{(because } w^1(\xi) \in H^1(X, \mathbb{Z}^2)^G\text{)} \end{aligned}$$

Thus the vector bundle $g^*(\xi)$ is isomorphic to ξ. Let $\phi_g : \xi \to g^*(\xi)$ be a bundle isomorphism. Now define $\tilde{g} : \xi \to \xi$ be the composition

$$\tilde{g} = g' \circ \phi_g.$$

Then $\tilde{g}$ is a bundle map which covers the map $g : X \to X$.

Note that if $\phi_g : \xi \to g^*(\xi)$ is a bundle isomorphism then $-\phi_g : \xi \to g^*(\xi)$ defined by $(-\phi_g)(v) = -\phi_g(v)$ is also a bundle isomorphism. Therefore for each $g : X \to X$ there are two maps $\tilde{g}$ and $-\tilde{g}$ which cover g where $-\tilde{g} = g' \circ (-\phi_g)$. It is clear that $-\tilde{g}$ is nothing but $\tilde{g}$ composed with map induced from the nontrivial deck transformation of the covering space $\tilde{X}$ of X. Now let $G' = \{\tilde{g}, -\tilde{g} \mid g \in G\}$. Then since X is connected it is clear that G' has a group structure so that we have the following exact sequence of groups:

$$0 \to \Delta \to G' \to G \to 0$$

where $\Delta \cong \mathbb{Z}_2$ is the deck transformation group of the double covering space $\tilde{X}$. The covering maps $\tilde{g}$ and $-\tilde{g}$ clearly extends to bundle maps $\xi \to \xi$ which we still call $\tilde{g}$ and $-\tilde{g}$. Thus G' can be considered as a group of bundle automorphisms of ξ. This proves Proposition 1.3.

In general the above exact sequence of groups does not split, and Example 1.2 is one of such kind. A nonequivariant real line bundle has a G vector bundle structure if and only if the above exact sequence splits. Therefore Theorem 1.1 is proved if the exact sequence splits under the given hypotheses.

We now prove Theorem 1.1. First assume that X is locally path-connected and $X^G \neq \emptyset$. Let $x_0 \in X^G$. Choose $\tilde{x}_0 \in p^{-1}(x_0)$ where $p : \tilde{X} \to X$ is the covering projection. From the elementary theory of covering spaces there exists unique map $\tilde{g} : \tilde{X} \to \tilde{X}$ which covers the map $g : X \to X$ such that $\tilde{g}(\tilde{x}_0) = \tilde{x}_0$. This defines a splitting map $\gamma : G \to G'$ in the above exact sequence. This proves the first part. For the second part consider the following [5, p 151]:

LEMMA 2.1. [Schur-Zassenhaus] *If Δ and G are finite group of order m and n, respectively, and if $(m,n) = 1$, then every extension G' of Δ by G is a semidirect product.*

Since the order of G is odd and Δ is of order 2 the group G' is a semidirect. Hence the above exact sequence splits by a property of semidirect product.

For the general case all we need is the following, see [5, p150]

LEMMA 2.2. *Let K be abelian. If a discrete group G has $H^2(G,K) = 0$ then any extension of K by G is a semidirect product.*

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