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NOTES ON CORRECT MODULES

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ABSTRACT. In this paper we will define correct module and strongly correct module. We can have some basic results about those modules. And we will show that M is a graded correct R-module if and only if M_e is a correct R_e-module.

1. Introduction

In infinite abelian group theory, the following problems were suggested by I. Kaplansky, are well known.

Kaplansky's Test Problems:

Let G be an infinite abelian group. Are the following statements true or false ?

I If G is isomorphic to a direct summand of a group H and H is isomorphic to a direct summand of G, then G and H are isomorphic.

II If the direct sums $G \oplus G$ is isomorphic to $H \oplus H$, then G and H are isomorphic.

III If the direct sums of G and K are isomorphic to the direct sums of G and L where K and L are abelian groups, then K and L are isomorphic.

It is also well known that all of the above problems are false in general case. Since every abelian group is a Z-moudule, we know that all of the same problems in module case are also false. But in some special modules those are will be true. In 1965, Bumby studied some modules

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which are isomorphic to submodules of each other. In this paper we can define some new modules called correct module and strongly correct module by using Bumby's idea. And we will study some properties of the structure of those modules. At first we will give sufficient conditions that a module is correct or strongly correct. And we will find the necessary and sufficient conditions that R is correct or strongly correct under the condition that R has no nonzero zero divisors where R is a ring with 1. Also we will define graded correct modules and graded strongly correct modules in graded ring theory. There is an example of a module which is not correct but is graded correct. And we will show that a graded module M is graded correct (or strongly correct) if and only if an R_e module M_e is correct (or strongly correct) where R is a strongly graded ring of type a group G and e is the identity of G. Throughout this paper R is an associative ring with 1 and a module is an unitary right R-module.

2. Correct and strongly correct modules

At first we will define correct and strongly correct modules as follows;

DEFINITION. (1) An R- module M is called correct if the following condition is satisfied. For every R-module N, if there exist a submodule N' of N and a submodule M' of M such that N' is isomorphic to M and M' is isomorphic to N then N and M are isomorphic.

(2) An R-module M is called strongly correct if the following condition is satisfied. For every R-module N, if there exist a monomorphism $f : N \to M$ and a monomorphism g :M$\to N$ then f is isomorphism (or g is isomorphism).

Instantly we know that a strongly correct module is correct. And we know that if a module M is strongly correct then every submodule N of M, which is isomorphic to M is M itself since inclusion monomorphism is isomorphism. From this fact we also know that Z is correct Z-module where Z is the integer ring but is not strongly correct. We can find some sufficient conditons that a module is correct or strongly correct.

PROPOSITION 1. *Let R be a ring and M be an R-module.*

(1) *Every semisimple R-module is correct.*

(2) *Every artinian R-module is strongly correct.*
(3) *Every noetherian quasi-injective module M is strongly correct.*

PROOF. (1) Let M be a semisimple R-module and N be an arbitrary module where a submodule N' of N is isomorphic to M and a submodule M' of M is isomorphic to N. Then N is also semisimple module since every submodule of a semisimple module is semisimple. At first we assume that M is the direct sums of simple modules all of which are isomorphic. Then clearly M is isomorphic to N since N is also direct sums of simple modules which are isomorphic to a simple submodule of M. In general case we also can know that M is isomorphic to N since M is direct sums of simple modules all of which are isomorphic.

(2) Let f and g be monomorphisms of M to N and of N to M respectively where M is artinian module. Then the composition homomorphism gf is monomorphism of M to M itself. By Fitting's lemma gf is isomorphism. So g is onto and g is isomorphism.

(3) Let M' be a submodule of M such that M' is isomorphic to M. Let ϕ be an isomorphism of M' to M. Then we can draw the following diagram where $E(M)$ is the injective hull of M and $i_{M'}$ is the inclusion map of M' into M , and i_M is the inclusion map of M into $E(M)$.

$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \xrightarrow{i_{M'}} & M \\ & & \phi \downarrow & & \\ & & M & \swarrow \psi & \\ & & i_M \downarrow & & \\ & & E(M) & & \end{array}$$

Then we can find a homomorphism ψ of M into $E(M)$ such that $\psi i_{M'} = i_M \phi$. Then $\psi \mid_M i_{M'} = \phi$ since i_M is the inclusion map. So $\psi \mid_M$ is onto homomorphsim of M into M. By Fitting's Lemma $\psi \mid_M$ is isomorphism since M is noetherian. Thus $i_{M'}$ is also isomorphism that is $M' = M$.

THEOREM 2. *Let R have no nonzero zero divisors. Then (1) R is correct if and only if R is right principal ideal ring. (2) R is strongly correct if and only if R is a division ring.*

PROOF. (1) Suppose that there exists an ideal which is not principal ideal. Say $aR+bR$. Since R has no nonzero zero divisors aR is isomorphic to R and clearly $aR + bR$ is a submodule of R. This contradicts to the fact that R is correct. Conversely if every submodule of R is isomorphic to R then R is correct.

(2) We know that (2) is true from the fact that aR is equal to R if and only if R is a division ring

· Usually a quasi-Frobenius ring is defined as a ring which is self injective and left- right noetherian. Thus we can get the following corollary easily from Proposition 1 and Theorem 2.

COROLLARY. *If a quasi-Frobenius ring R has no nonzero zero divisors, then R is a division ring.*

PROOF. From Proposition 1, R_R is strongly correct so R is a division ring from Theorem 2.

LEMMA 3. *If every injective R-module is correct, then every R-module is injective.*

PROOF. Let A be not injective R-module. Then there exists an injective hull $E(A)$ of A. And let M be the direct product of infinite copies of $E(A)$. Then M is injective since direct produc: of injective modules is injective. Let $N = M \oplus A$. Clearly N is not injective since A is not injective. Thus N is not isomorphic to M. But M is isomorphic to M itself where M is a submodule of N and there exists a submodule K of M such that K is isomorphic to N sicne A is a submodule of $E(A)$. This contradicts to the fact that M is correct (M is injective). Thus every module A is injective.

From the above lemma, we can get the following theorem.

THEOREM 4. *The following statements are equivalent.*

(1) *Every R-module is correct*
(2) *Every injective R-module is correct*
(3) *R is semisimple right artinian ring.*

PROOF. Clearly (1) implies (2) and (2) implies (3) since every R-module is injective if and only if R is semisimple artinian. And (3) implies (1) since every semisimple R-module is correct.

THEOREM 5. *If every projective R-module is correct, then R is right hereditary.*

PROOF. Let I be arbitrary right ideal of R. And let M be the direct sum of infinite copies of R. Then M is isomorphic to M a submodule of $M \oplus I$ and $M \oplus I$ is isomorphic to a submodule of M since I is a submodule of R. Thus M is isomorphic to $M \oplus I$ since M is correct(M is projective). Thus I is projective.

The following example shows that the converse statement is not true.

EXAMPLE 6. Let $R = Z[\sqrt{-5}]$ Since R is a Dedekind domain, it is hereditary. But R is not correct since $I = 3R + (1 + \sqrt{-5})R$ is not isomorphic to R.

Recall that a ring R is called a graded ring of type a group G if $R = \oplus R_g$ and $R_g R_h \subset R_{gh}$ for every $g, h \in G$. To get more detail informations about graded ring theory one can read "Graded Ring Theory" [3]. We will define graded correct modules and graded strongly correct modules.

DEFINITION. (1) Let R be a graded ring of type G and M be a graded R-module. Then M is called graded correct if the following conditions are satisfied. For every graded R-module N, if there exist a graded submodule N' of N and a graded submodule M' of M such that N' is isomorphic to M and M' is isomorphic to N, then N is isomorphic to M.

(2) M is called graded strongly correct if the following condition is satisfied. For every graded R-module N, if there exist monomorphisms f and g of degree e from N to M and from M to N respectively, then f is isomorphism.

Instantly we know that if a graded R-module M is correct or strongly correct then M is graded correct or graded strongly correct. But the converse is not true. The following example shows that there exists a module which is graded correct but not correct.

EXAMPLE 7. Let $R = Z[x, x^{-1}]$ the set of all Laurent series over Z. Then R is a graded ring of type Z, the integer group. Then we can know that all graded ideals of R are generated by one element. Thus R is a graded corret R-module. But $I = 2R + (3 + 5x)R$ is an ideal of R (not graded ideal) which is not generated by one element so R is not correct R-module. Clearly $2R$ is isomorphic to R and I is not isomorphic to R.

Let R be a graded ring of type G and M be a graded R module. Then R_e is always a subring of R and M_e is an R_e-module. We can get the following theorem for strongly graded ring.

THEOREM 8. *Let R be a strongly graded ring of type a group G(i.e. $R_gR_h = R_{gh}$ for every $g, h \in G$) . Then M is a graded (strongly) correct R-module if and only if M_e is a (strongly) correct R_e-module.*

PROOF. Suppose that there exists an R_e-modue N such that there are an R_e submodule N' of N and an R_e-submodule M_e' of M_e where M_e' is isomorphic to N and N' is isomorphic to M_e. Let $NR = \oplus NR_g$ and $N'R = \oplus N'R_g$ where NR_g is the set of all formal product nr for $n \in N$ and $r \in R_g$. We know that NR is graded right R-module and $N'R$ is a graded submodule of NR via $(n_1 r_{g_1} + \cdots + n_k r_{g_k})r_h = n_1 r_{g_1} r_h + \cdots + n_k r_{g_k} r_h$. Then $N'R$ is graded isomorphic to M via the graded isomorphism ψ which is defined by $\psi(a_1 r_{g_1} + \cdots + a_k r_{g_k}) = \phi(a_1)r_{g_1} + \cdots + \phi(a_k)r_{g_k}$ where ϕ is the isomorphism from N' into M_e and $a_i \in N'$. In fact ψ is onto since M is the direct sums of R_eM_g (R is strongly graded ring). And similarly NR is isomorphic to $\oplus M_e'R_g$. So M is graded isomorphic to NR since M is graded correct. Thus an R_e-module M_e is isomorphic to an R_e-module N. Conversely suppose that M_e is a correct R_e-module. For a graded R-module N there exist a graded submodule N' of N and a graded submodule M' of M such that N is isomorphic to M' and M is isomrophic to N'. Then M_e is isomorphic to $N'e$ and N_e is isomorphic to $M'e$. Thus M_e is isomorphic to N_e. Thus M is graded isomorphic to N. Similarly we can prove that if M is a graded strongly correct R-module if and only if M_e is a strongly correct R_e-module.

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