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NOTES ON GROUPS WITH FINITE BASE

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ABSTRACT. We define a group property of finite base which is closely related to finite Prüfer rank, and then study the class of groups having such a property.

1. Introduction

Throughout this paper when we talk of a group having *finite rank*, we mean finite *Prüfer rank*. Thus G has finite rank if there is an integer n such that each finitely generated subgroup of G can be generated by n elements; its Prüfer rank is then the least integer n with this property.

Groups with finite rank, when restricted to soluble groups or residually finite groups, have been studied by several authors for many decades, recently by Dashkova, O. Yu, see [1]. A large number of results may be found in [7].

Here we shall define a group property, called *finite base* which appears similar to finite rank. Then we shall establish the corresponding results for groups having finite base. We first start with the notation before stating explicitly this property and its implications.

If H, K are subgroups of a group G and $H \leq K$, let $\sqrt[K]{H}$ denote the set

$$\{g \in K : g^n \in H \text{ for some integer } n\}.$$

$\sqrt[K]{H}$ is called the isolator of H in K. If $H = \sqrt[K]{H}$, we say H is isolated in K. We say G has the isolator property if the isolator of H in G is a subgroup for all $H \leq G$. It is well known that nilpotent groups

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have the isolator property, see [4], [6], and groups with this property were discussed in [2], [3]. However in general $\sqrt[K]{H}$ does not form a group from which leads us to consider following definition.

DEFINITION 1.1. Let H, K be subgroups of a group G and $H \leq K$. We denote by $R_K^1(H)$ the subgroup generated by the set $\sqrt[K]{H}$. Let $R_K^{i+1}(H) = R_K^1(R_K^i(H))$ and $R_K(H) = \bigcup_{i=1}^{\infty} R_K^i(H)$. We call $R_K(H)$ the *root* of H in K.

Note $R_K(H)$ is isolated for since $x^r \in R_K(H)$ then $x^r \in R_K^i(H)$ for some i and $x \in R_K^{i+1}(H)$. Hence $x \in R_K(H)$. So $R_K(H)$ is exactly the smallest isolated subgroup of K containing H.

EXAMPLE 1.2. If $G = R_G(H)$ for some subgroup H of a nilpotent group G. Then by the isolator property of nilpotent groups, a power of $x \in R_G^2(H)$ lies in $R_G^1(H)$ and again its power lies in H. So $R_G^2(H) \subset R_G^1(H)$ and hence $G = R_G^1(H)$.

As we see in the above example, the root does not grow in nilpotent groups. The following example indicates that a class of groups having this property is quite limited, by showing that even in the polycyclic group the root grows. Here its growth has still an upper bound. In the example 1.4 we give a group and its subgroup in which the growth has no upper bound.

EXAMPLE 1.3. Let $N = \langle\ a, b\ ; [a, b] = c, [c, a] = [c, b] = e \rangle$ be a free nilpotent group of class 2 and let t be the automorphism of N defined by $a^t = b, b^t = (ba)^{-1}$. Then $G = \langle\ a, b, t\ ; [a, b] = c, [c, a] = [c, b] = e, a^t = b, b^t = (ba)^{-1}, t^3 = c\ \rangle$ is a polycyclic group. Now we consider the root $R_G(\langle e \rangle)$ of $\langle e \rangle$ in G. We claim that $R_G^1(\langle e \rangle) \neq R_G^2(\langle e \rangle)$ and $R_G(\langle e \rangle) = R_G^2(\langle e \rangle) = G$. Let P be $R_G^1(\langle e \rangle)$ which is a subgroup generated by all periodic elements of G. We show that G/P is isomorphic to a cyclic group of order 3. Let a, b and c be as above. Then first we have that $a^n b^m = b^m a^n c^{nm}$ for all integers, n, m. Moreover $(a^{-1}b^{-1})^n = b^{-n}a^{-n}c^{n(n+1)/2}$ for all integers, n. This can be easily checked by induction on n. Here we have that a^3, b^3 and t^3 lie in P. To see this, Note that $(at^{-1})^3 = aa^t a^{t^2} t^{-3} = a \cdot b \cdot (ba)^{-1} c^{-1} = e$. And so $at^{-1} \in P$. Hence $(at^{-1})^t = bt^{-1}$, $(bt^{-1})^t = a^{-1}b^{-1}t^{-1}$, and $(at^{-1})^{a^{-1}} =$

$a^2b^{-1}t^{-1}$ lie in P. So $a^2b^{-1}t^{-1}\cdot tba = a^3 \in P$ and so does $b^3 = (a^3)^t$. Moreover $at^{-1}\cdot(bt^{-1})^{-1} = ab^{-1}$, $(bt^{-1})^t\cdot(bt^{-1})^{-1} = a^{-1}b^{-2} \in P$ and $a^{-3}\cdot ab^{-1}\cdot(b^{-3})^t\cdot a^{-1}b^{-2} = a^{-1}bab^{-1} = c^{-1} \in P$. So $t^3 \in P$.

Let $K = \langle a^3, b^3, t^3\rangle^G \leq N$ and $Q = \langle K, at^{-1}\rangle^G \leq P$. Now we claim $P = Q$ by showing that every periodic element of G is in Q. For a periodic element $x = c^\ell a^m b^n t^\epsilon$ in G where $\epsilon = 1, -1$, we denote (ℓ, m, n, ϵ) by x. We do for the case of $\epsilon = -1$. Similar argument can be applied for the case of $\epsilon = 1$. Then $x^2 = (\ell, m, n, -1)\cdot(\ell, m, n, -1) = (2\ell + mn + n(n+1)/2, m-n, m, -2)$ and $x^3 = (3\ell + mn + n(n+1)/2 + m(m+1)/2 - 1, 0, 0, 0)$. Since x is periodic, $3\ell + mn + n(n+1)/2 + m(m+1)/2 - 1 = 0$. We consider this relation modulo 3. This relation gives $(m+n)(m+n+1) \equiv 2 \; mod \; 3$. So $m + n \equiv 1$. On the other hand, since $bt^{-1} = (at^{-1})^t \in Q$, G/Q is a cycle group of order at most 3, and so $x = c^\ell a^m b^n t^\epsilon \equiv a^{m+n-1} \equiv e \; mod \; Q$. Therefore $x \in Q$ and so $P = Q$. Recall that if a group G is a 2-Engel group then $\langle x\rangle^G$ is abelian for all $x \in G$ (see, for example, Theorem 7.13 of [7]). Since G/K has an exponent 3 and $|G/K| = 27$, it is a 2-Engel group (see, Theorem 7.14 of [7]). Note $P/K = \langle(at^{-1}K)\rangle^{G/K}$ and so P/K is abelian. If $G = P$, then G/K is abelian. This is a contradiction. From the defining relations, it follows that G/P is an abelian group of order 3. In fact $G/K \cong (Z_3 \times Z_3) \rtimes Z_3$ and $G/P = \langle(aP)\rangle \cong Z_3$.

EXAMPLE 1.4. In general case, we may have a group G and a subgroup H such that $R^i_G(H)$ is strictly increasing for $i \geq 1$. For an infinite cyclic group T_i generated by t_i, set $G_1 = T_1$ and G_i the free product of $G_{(i-1)}$ and T_i with amalgamation subgroup $\langle(t_1t_2\cdots t_{(i-1)})\rangle = \langle t_i^2\rangle$. Then $G = \bigcup_{i=1}^{\infty} G_i$ and T_1 are required groups.

DEFINITION 1.5. We will say that a group G has *finite base* n if there is an integer n such that every subgroup H of G has an n-generated subgroup K of H with $R_H(K) = H$ and if n is the least integer with the property. In particular, if $H = R_H(\langle e\rangle)$ for each subgroup H of G then we shall say that G has base 0.

Clearly the additive group of rational numbers has finite base 1. All periodic groups have finite base 0. One can easily find a locally finite infinite group which doesn't have finite rank, for example a direct product of infinite many copies of a finite group. Hence the group property of finite base doesn't imply finite rank. It is not known that the converse implication holds. One of our results is that both properties are equivalent for torsion-free nilpotent groups.

2. Finite base

Here we consider the closure properties of groups with finite base.

LEMMA 2.1. *The class of groups with finite base is S(Subgroup), H(Homomorphic image) and P(Extension)-closed.*

PROOF. S and H-closure are immediate. For P-closure, let $N \lhd G$ and suppose that N has finite base r and G/N has finite base s. Let H be a subgroup of G and $H \cap N = L$. Then modulo L, $\overline{H} = R_{\overline{H}}(\langle \overline{x}_1, \ldots, \overline{x}_s\rangle)$ for some $x_i \in H$ and $L = R_L(\langle y_1, \ldots, y_r\rangle)$ for some $y_i \in L$. Clearly $L \subset R_H(\langle y_1, \ldots, y_r, x_1, \ldots, x_s\rangle)$. For $x \in H\backslash L$, $\overline{x} \in R_{\overline{H}}(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$. Let $\overline{x} \in R^1_{\overline{H}}(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$. Then $\overline{x} = \overline{g}_1 \cdots \overline{g}_r$ and $x = g_1 \cdots g_r \ell_1$, where $\overline{g}_i^m \in \langle \overline{x}_1, ..., \overline{x}_s\rangle$ and $\ell_1 \in L$. So $g_i^m = x_{i_1} \cdots x_{i_n} \ell_2 \in \langle x_1, ..., x_s, L\rangle$. Thus $x \in R^1_H(\langle x_1, ..., x_s, L\rangle) \subset R_H(\langle x_1, ..., x_s, L\rangle)$. Suppose that if $\overline{x} \in R^{i-1}_{\overline{H}}(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$, then $x \in R_H(\langle x_1, ..., x_s, L\rangle)$. Let $\overline{x} \in R^i_{\overline{H}}(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$. Then $\overline{x} = \overline{g}_1 \cdots \overline{g}_s$ and $x = g_1 \cdots g_s \ell_3$ where $\overline{g}_i^m \in R^{i-1}_{\overline{H}}(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$. By induction hypothesis, $g_i^m \in R_H(\langle x_1, ..., x_s, L\rangle)$. Thus $x \in R^1_H(R_H(\langle x_1, ..., x_s, L\rangle)) = R_H(\langle x_1, ..., x_s, L\rangle)$. So if $\overline{x} \in R_{\overline{H}}(\langle \overline{x}_1, ..., \overline{x}_s\rangle)$, then $x \in R_H(\langle x_1, ..., x_s, L\rangle)$.

Note that $R_H(\langle x_1, ..., x_s, y_1, ..., y_r\rangle)$ contains $x_1, ..., x_s, L$ and is isolated. Thus $R_H(\langle x_1, ..., x_s, L\rangle) = R_H(\langle x_1, ..., x_s, y_1, ..., y_r\rangle)$. □

LEMMA 2.2. *If a group G has finite base, then it satisfies maximal condition on isolated subgroups.*

PROOF. Suppose that there is a proper ascending series of isolated subgroups, $H_1 < H_2 < \cdots$. Let $H = \bigcup_{i=1}^{\infty} H_i$. Then there is an n-generated subgroup $K = \langle x_1, \ldots, x_n\rangle$ of H with $R_H(K) = H$. Take a subgroup H_ℓ containing $x_1, \ldots, x_n$ and then $H_\ell = R_H(H_\ell) = R_H(K) = H$. □

LEMMA 2.3. *Let G be a torsion-free abelian group with finite base 1. Then G is isomorphic to a subgroup of the additive group of rational numbers.*

PROOF. Since G has finite base 1, $G = R_G(\langle x \rangle)$ for some x in G. For $g \in G$, $g^\ell = x^n$ for some ℓ, n. Hence G is indecomposable. Let H be a finitely generated subgroup of G. Then H is also indecomposable, i.e., $H \cong \mathbb{Z}$. This means that G is locally cyclic and torsion-free. Hence G is isomorphic to a subgroup of $\mathbb{Q}$. $\square$

The result of lemma 2.3 can be extended to the class of torsion-free nilpotent groups.

LEMMA 2.4. *Let G be a torsion-free nilpotent group with base 1. Then it is isomorphic to a subgroup of the additive group of rational numbers.*

PROOF. Induction on the nilpotent class n. By Lemma 5.3, it is clear if $n = 1$. Suppose that it is true if $n < r$. For a group G of the nilpotent class r, $\langle G', g \rangle$ has the nilpotent class $< r$ for all g in G. In particular for $g \in Z(G)$, the center of G, $\langle G', g \rangle$ is isomorphic to a subgroup of $\mathbb{Q}$. Hence $x^\ell = g^n$ for some $x \in G'$. For all $y \in G$, $\langle G', y \rangle$ is isomorphic to a subgroup of $\mathbb{Q}$. Hence $y^\alpha = x^\beta$ and $y^{\alpha\ell} = x^{\beta\ell} = g^{n\beta}$. So $y^{\alpha\ell}$ lies in $Z(G)$. Since the center of a torsion-free nilpotent group is isolated, y lies in $Z(G)$. Hence G is abelian. $\square$

Let G be an abelian group with finite rank r. Then the factor group of G with respect to its torsion subgroup is isomorphic with a subgroup of a direct product of $r_0 (\leq r)$ copies of the additive group of rational numbers. Hence G has finite base. From this and the P-closure it follows that soluble groups with finite rank have finite base. For a locally soluble group G with finite rank r, there is an integer n depending only on r such that $G^{(n)}$ is periodic (See Lemma 10.39 in [7]). Hence

THEOREM 2.5. *A locally soluble group G with finite rank has finite base.*

Moreover a residually finite group with finite rank is almost locally soluble, see [5] and hence has finite base.

THEOREM 2.6. *A torsion-free nilpotent group with finite base has finite rank.*

PROOF. Let G be a torsion-free abelian group with finite base and A a subgroup such that $G/A = \langle \bar{t} \rangle$ is infinite cyclic. If s is the smallest integer such that $A = R_A(\langle x_1, \ldots, x_s \rangle)$, $x_i \in A$, then G has finite base $\geq s$. Now suppose G has finite base s. Then $\langle A, t \rangle = R_{\langle A,t \rangle}(\langle t^{\alpha_1} a_1, \ldots, t^{\alpha_p} a_p, a_{p+1}, \ldots, a_s \rangle)$ where $a_i \in A$. Here we have an expression for t in the right hand side. Since t is torsion-free, we have $1 = a_1^{b_1} \cdots a_p^{b_p} a_{p+1}^{b_{p+1}} \cdots a_s^{b_s}$ where $b_i \in \mathbb{Z}$ and $b_i \neq 0$ for some i. Hence $A = R_A(\langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_s \rangle)$, a contradiction. Thus G cannot have infinitely many independent elements. This means that G has finite rank. $\square$

Recall that for an abelian group A, the 0-*rank* of A is the cardinal of a maximal independent set consisting of elements of A with infinite order.

COROLLARY 2.7. *If G is a locally nilpotent group, then the following properties of G are equivalent.*

i) G has finite base.

ii) Each abelian subgroup of G has finite 0-rank.

iii) The factor group G by its torsion subgroup is a torsion-free nilpotent group of finite base.

COROLLARY 2.8. *A torsion-free locally nilpotent group with finite base has finite rank.*

Let G be a soluble group, and $1 = G^{(n)} \leq G^{(n-1)} \leq \cdots \leq G^{(1)} \leq G^{(0)} = G$ be the derived series of G. If G has finite base, then by Lemma 2.1 and Corollary 2.7 each factor has finite base and hence finite 0-rank.

Conversely suppose each factor $G^{(i)}/G^{(i+1)}$ has finite 0-rank. Let T_i be the preimage of the torsion subgroup of a factor group $G^{(i)}/G^{(i+1)}$. Then $1 = G^{(n)} \leq T_{n-1} \leq G^{(n-1)} \leq \ldots \leq G^{(1)} \leq T_0 \leq G^{(0)} = G$. The torsion subgroup T_{n-1} of abelian group $G^{(n-1)}$ is just $R^1_{G^{(n-1)}}(\langle e \rangle)$. In general, we have

$$T_i = R^1_{G^{(i)}}(G^{(i+1)}), \quad n-1 \geq i \geq 0.$$

Let $a_{i1}, a_{i2}, \ldots a_{im(i)}$ be maximal independent elements of $G^{(i)}$ mod T_i. Then

$$G^{(n-1)} = R^1_{G^{(n-1)}}(\langle a_{(n-1)1}, a_{(n-1)2}, \cdots, a_{(n-1)m(n-1)}\rangle) \text{ and}$$

$$T_{n-2} \subseteq R^2_{G^{(n-2)}}(\langle a_{(n-1)1}, \cdots, a_{(n-1)m(n-1)}\rangle).$$

For $g \in G^{(n-2)}$, let a power g^ℓ be in $R^1_{G^{(n-2)}}(\langle a_{(n-1)1}, \cdots, a_{(n-1)m(n-1)}\rangle)$. Then $g^\ell \in R^1_{G^{(n-2)}}(G^{(n-1)}) = T_{n-2}$ and hence $g \in T_{n-2}$, which implies

$$T_{n-2} = R^2_{G^{(n-2)}}(\langle a_{(n-1)1}, \cdots, a_{(n-1)m(n-1)}\rangle).$$

By the similar argument, we induce that

$$T_i = R^2_{G^{(i)}}(\langle a_{jm(j)}; n-1 \geq j \geq i+1\rangle).$$

Hence $G = T_0 \cdot R^1_G(\langle a_{01}, \cdots, a_{0m(0)}\rangle) = R^2_G(H)$ where H is a subgroup generated by all maximal independent elements $a_{i1}, a_{i2}, \ldots a_{im(i)}$ of each factor $G^{(i)}/G^{(i+1)}$. Thus we have

THEOREM 2.9. *A soluble group G has finite base if and only if each factor $G^{(i)}/G^{(i+1)}$ has finite 0-rank.*

It can be checked that if a soluble group has finite base s, then s is at most the sum of finite 0-ranks of each factor, that is $\sum_{i=0}^{n-1} m(i)$.

Suppose G is a soluble group with finite base. If $G = R_G(K)$ for some subgroup K, then each maximal independent element a_{ij} of each factor $G^{(i)}/G^{(i+1)}$ lies in $R_G^{m(ij)}(K)$ for some integer $m(ij)$. Hence $G = R^\ell_G(K)$ for some positive integer ℓ.

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