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ON A WEIGHTED HARDY-SOBOLEV SPACE FUNCTIONS (I)

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Abstract. Using a special property of Bloch functions with Hardmard gaps and using the geometric properties of the self maps of the unit disc, we give a way of constructing explicit examples of Bloch functions f whose derivative is in H^p $(0 < p < 1)$ but $f \notin BMOA$.

1. Introduction

Let U be the open unit disc in the complex plane and let T be the boundary of U identified with $[-\pi, \pi]$. H^p, $0 < p \leq \infty$, denotes the classical Hardy space of holomorphic functions defined on U. $BMOA$ denotes the space of analytic functions of bounded mean oscillation. It consists of those $f \in H^1$ for which

$$|f'(z)|^2(1-|z|^2)dxdy$$

is a Carleson measure, that is to say,

$$\int\int_{S_{\delta,\theta}} |f'(z)|^2(1-|z|^2)dxdy = O(\delta),$$

where $0 < \delta \leq 1$ and

$$S_{\delta,\theta} = \{re^{it} : |\theta - t| \leq \delta, 1-\delta \leq r < 1\}, \quad \theta \in T.$$

$\mathcal{B}$ denotes the Bloch space. It consists of those f holomorphic in U for which

$$\|f\|_{\mathcal{B}} = \sup_{z \in U} |f'(z)|(1-|z|^2) < \infty.$$

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$\mathcal{B}$ is quite similar to $BMOA$ in a certain sense (see, for example, [G, pp 281-283]). But, in comparison with the fact that functions of H^p or $BMOA$ should have radial limits almost everywhere, Bloch functions need not. The following is the motivation of this note.

THEOREM A ([K, Remarks 2.6 (2)]). *If $f \in \mathcal{B}$ and $f' \in H^p$ for some p, $0 < p < 1$, then $f \in H^q$ for all $q < \infty$.*

Concerning the sharpness of this result, the author mentioned that $q = \infty$ can not be allowed in the conclusion of Theorem A by giving the example $f(z) = log(1 - z)$. Noting that $log|x|$, x : real, is a typical of unbounded BMO function, natural question in this direction is whether the f in the conclusion of Theorem A belongs to $BMOA$. We prove

THEOREM 1. *For each $p : 0 < p < 1$, there is a Bloch function f with $f' \in H^p$ such that $f \notin BMOA$.*

This result is sharp because if $f' \in H^1$ then, by a well known result of Privalov(see [D, pp 42-52]), f is continuous on $\bar{U}$ and absolutely continuous on T so that $f \in H^\infty \subset BMOA$.

It seems that there are many ways of showing the existence of functions requisted in Theorem 1. Our point is that we can transfer, by use of composition operators, the problem to a parallel problem of holomorphic self maps on U with hyperbolic geometry as in Theorem 2 of next section.

We construct explicit and easy examples of such fuctions satisfying Theorem 1. By Theorem A, the f of Theorem 1 belongs to

$$\mathcal{B} \bigcap H^* \setminus BMOA, \tag{1.1}$$

where $H^* = \bigcap_{0<p<\infty} H^p$, and explicit example of functions belonging to the set given by (1.1) was called for in [CCS] and [HT].

2. Proof of Theorem 1

The hyperbolic version of Theorem 1 can be stated as the following.

THEOREM 2. *For each $p : 0 < p < 1$, ther is a holomorphic self map h of U satisfying the following (2.1) and (2.2).*

$$\sup_r \int_T \left(\frac{|h'(re^{i\theta})|}{1 - |h(re^{i\theta})|^2} \right)^p d\theta < \infty. \tag{2.1}$$

$$\limsup_{\delta \to 0} \frac{1}{\delta} \int\int_{S_{\delta,0}} \left(\frac{|h'(z)|}{1 - |h(z)|^2} \right)^2 (1 - |z|^2)\, dxdy = \infty. \tag{2.2}$$

We will prove Theorem 2 in Section 4. The following theorem on Bloch functions plays the role of reducing Theorem 1 to Theorem 2.

THEOREM B (see [RU, Proposition 5.4] and [P]). *There exist a large natural number q and Bloch functions g_1, g_2 satisfying the following (2.3) and (2.4) respectively.*

$$|g_1'(z)| \geq \frac{1}{1 - |z|^2}, \quad 1 - \frac{1}{q^k} \leq |z| \leq 1 - \frac{1}{q^{k+\frac{1}{2}}}, \quad k = 1, 2, \ldots \tag{2.3}$$

$$|g_2'(z)| \geq \frac{1}{1 - |z|^2}, \quad 1 - \frac{1}{q^{k+\frac{1}{2}}} \leq |z| \leq 1 - \frac{1}{q^{k+1}}, \quad k = 1, 2, \ldots \tag{2.4}$$

In fact, for large enough q, such functions may be given by constant multiples of the functions

$$g_1(z) = \sum_{j=0}^{\infty} z^{q^j}, \quad z \in U$$

and

$$g_2(z) = \sum_{k=0}^{\infty} z^{n_k}, \quad z \in U,$$

where n_k is the integer closest to $q^{k+1/2}$.

Now we prove Theorem 1. Let h be a holomorphic self map satisfying (2.1) and (2.2). Let q and g_1, g_2 be an integer and Bloch functions satisfying (2.3) and (2.4). Let

$$A_1 = A_1(h) = \bigcup_{k=1}^{\infty} \left\{ z : 1 - \frac{1}{q^k} \leq |h(z)| \leq 1 - \frac{1}{q^{k+1/2}} \right\}$$

and

$$A_2 = A_2(h) = \bigcup_{k=1}^{\infty} \left\{ z : 1 - \frac{1}{q^{k+\frac{1}{2}}} \le |h(z)| \le 1 - \frac{1}{q^{k+1}} \right\}.$$

Then, A_j, $j = 1, 2$ are measurable and by (2.2)

$$I_j(h) := \limsup_{\delta \to 0} \frac{1}{\delta} \int\int_{S_\delta \cap A_j} \left(h^{\#}(z)\right)^2 (1 - |z|^2) dx dy = \infty$$

for at least one j, $j = 1, 2$, where $h^{\#}(z) = \frac{|h'|}{1-|h|^2}(z)$, and $S_\delta = S_{\delta,0}$. Assume $I_1(h) = \infty$. Let

$$f(z) = g_1 \circ h(z), \quad z \in U.$$

Then, by the Schwarz-Pick Lemma ([G]), f is a Bloch since g_1 is. Since

$$\int_T |f'(re^{i\theta})|^p \, d\theta \le \|g_1\|_{\mathcal{B}}^p \int_T \left(h^{\#}(e^{i\theta})\right)^p d\theta, \tag{2.5}$$

by (2.1), $f' \in H^p$. But, by (2.3) and the assumption on $I_1(h)$,

$$\begin{aligned}
&\limsup_{\delta \to 0} \frac{1}{\delta} \int\int_{S_\delta} |f'(z)|^2 (1 - |z|^2) dx dy \\
\ge &\limsup_{\delta \to 0} \frac{1}{\delta} \int\int_{S_\delta \cap A_1} |f'(z)|^2 (1 - |z|^2) dx dy \\
\ge &\limsup_{\delta \to 0} \frac{1}{\delta} \int\int_{S_\delta \cap A_1} \left(h^{\#}(z)\right)^2 (1 - |z|^2) dx dy = \infty,
\end{aligned}$$

so that $f \notin BMOA$. If $I_2(h) = \infty$, by considering $f = g_2 \circ h$, a similar argument shows that $f' \in H^p$ and that $f \notin BMOA$. The proof is complete. □

3. An example.

By a direct calculation, we show in this section that functions of the form $\frac{(1+z)}{2}$ satisfy both (2.1) with $p < \frac{1}{2}$ and (2.2).

Fix $s > 0$, $e^{it} \in T$, and take $h(z) = e^{it}\frac{1+sz}{s+1}$, $z \in U$. Then h maps U onto

$$\Delta(e^{it}, s) = \left\{ z \in U : \left| z - \frac{e^{it}}{s+1} \right| < \frac{s}{s+1} \right\},$$

the boundary of which is internally tangent to the unit circle. Let $h^{\#} = \frac{|h'|}{1-|h|^2}$. Then $(h^{\#})^p$ is subharmonic so that

$$\begin{aligned} &\sup_r \int_T \left(h^{\#}(re^{i\theta})\right)^p d\theta \\ &= \int_T \left(h^{\#}(e^{i\theta})\right)^p d\theta \\ &= \left(\frac{s+1}{2}\right)^p \int_{-\pi}^{\pi} (1-cos\theta)^{-p}\, d\theta \\ &< \infty \end{aligned} \tag{3.1}$$

for all $p : 0 < p < \frac{1}{2}$.

On the other hand, if $0 < \delta < 1$ then $1 - \delta < cos\theta$ for $|\theta| < \delta$, so that

$$\begin{aligned} &\iint_{S_{\delta,0}} \left(h^{\#}(z)\right)^2 (1-|z|^2)dxdy \\ &= \iint_{S_{\delta,0}} \left(\frac{1+s}{s(1-r^2)+2-2rcos\theta}\right)^2 (1-r^2)rdrd\theta \\ &\geq \left(\frac{1+s}{2+s}\right)^2 \int_{-\delta}^{\delta} d\theta \int_{1-\delta}^{cos\theta} \frac{1-r^2}{(1-r^2)^2} rdr \\ &= \left(\frac{1+s}{2+s}\right)^2 \int_{-\delta}^{\delta} log\frac{1-(1-\delta)^2}{sin^2\theta}\frac{d\theta}{2}. \end{aligned} \tag{3.2}$$

Since it is easy to see that $1-(1-\delta)^2 \geq \frac{1}{\delta} sin^2\theta$ for all $\theta : |\theta| < \delta$ provided $\delta < \frac{1}{3}$, we have, by (3.2),

$$\iint_{S_{\delta,0}} \left(h^{\#}(z)\right)^2 (1-|z|^2)dxdy \geq \left(\frac{1+s}{2+s}\right)^2 \delta\, log\frac{1}{\delta}, \quad 0 < \delta < \frac{1}{3}.$$

This gives (2.2). □

A close look of the arguments used in (3.1) and (3.2) shows that the properties (2.1) and (2.2) are connected with the order of contact and the angular derivative. See [S], for example, for these two concepts. This motivates us to consider a function h, in the next section, whose image is quite similar to that of h in this section, and which has a contact (with T) of order worse than that of h in this section. For a theoretical background, we invoke two results: A theorem of Tsuji-Warschawski(see [T, p 366], [S, p 72] or [ST, p 479]) stating a necessary and sufficient contact condition for a univalent map to have an angular derivative; A rescent result of [CRU, Theorem 1] stating that if a holomorphic self map of U have an angular derivative then it should satisfy (2.2). Hence, to satisfy conditions of Theorem 2, it is recommended to find a holomorphic self map h having worse order of contact (to fullfill (2.1)) as far as it does not disturb the smooth contact condition of [T] (to have an angular derivative so that (2.2) holds), which also has p-integrable derivative on subarcs of T away from the contacting point.

4. Proof of Theorem 2

We borrow an example in [S]. Given $p : 0 < p < 1$, take $\alpha : 1 < \alpha < 2$ such that $\alpha p < 1$. Let f be the univalent map from $\Pi = \{w : Re(w) > 0\}$ into U defined by $f = \tau \circ \psi$, where

$$\tau(w) = \frac{w-1}{w+1}, \quad w \in \Pi$$

and

$$\psi(w) = 1 + w + w^{(2-\alpha)}, \quad w \in \Pi,$$

defined via principal branch. Let h be the univalent self map of U defined by

$$h(z) = f \circ \tau^{-1}(z), \quad z \in U.$$

Then h has an angular derivative at $z = 1$ which is the unique point of $h(\bar{U}) \cap T$. See [S, p 47]. Thus, by [CRU, Theorem 1], h satisfies (2.2).

Next we claim (2.1). It follows by a change of variable that

$$\begin{aligned}
&\int_T \left(\frac{|h'|}{1-|h|^2}\right)^p (e^{i\theta})\, d\theta \\
(4.1) \qquad &= \int_{-\infty}^{\infty} \frac{2}{1+y^2}\left(\frac{1+y^2}{2}\right)^p \left(\frac{|f'|}{1-|f|^2}\right)^p (iy)\, dy \\
&= 2\int_{-\infty}^{\infty} (1+y^2)^{p-1}\left(\frac{|\psi'|}{|1+\psi|^2-|1-\psi|^2}\right)^p (iy)\, dy.
\end{aligned}$$

Now a little more calculation shows that

$$\left(|1+\psi|^2-|1-\psi|^2\right)(iy) = 4\left(1+|y|^{2-\alpha} cos\, \frac{2-\alpha}{2}\pi\right)$$

and

$$|\psi'|(iy) = \sqrt{1+2(2-\alpha)|y|^{1-\alpha} cos\, \frac{1-\alpha}{2}\pi + (2-\alpha)^2|y|^{2(1-\alpha)}}$$

for all real y. Hence h satisfies (2.1) because the integrand of the last integral of (4.1) has the order of growth $O\left(|y|^{\alpha p-2}\right)$ as $|y| \to \infty$ and has $O\left(|y|^{(1-\alpha)p}\right)$ as $y \to 0$. □

5. Remarks

(1). Let h be the function of Section 4 and let g_j, $j = 1, 2$ be as in Theorem A. Then, as for explicit examples of the functions required by Theorem 1, our proof of Theorem 1 says that at least one of $g_j \circ h$ gives the example required. But, in fact, both of them are the required examples. This follows from the intuitively clear fact that, for this special h, $I_1(h) = \infty$ and $I_2(h) = \infty$ hold simultaneously once (2.2) is satisfied. By a similar reason, $g_j \circ h$, with h that in Section 3 or that in Section 4, gives an explicit example of function in the class given by (1.1).

(2). Note that in the language of hyperbolic geometry, (2.1) corresponds to "$h' \in H^p$" while (2.2) corresponds to "$h \in BMOA$". See, for example, [Y1] and [Y2]. Theorem 2 shows [Kw1, Theorem 2] is sharp in the sense that the hyperbolic H^p result can not be extensible to the hyperbolic $BMOA$ result.

(3). In view of (2.3), (2.4) and (2.5), we have the following characterization on the boundedness of the composition operator C_h which is defined on Blochs by

$$C_h(g) = g \circ h, \quad g \in \mathcal{B}.$$

See [Kw2] for the same vein.

THEOREM C ([Kw3]). *Let* $0 < p < \infty$ *and let* $H_1^p = \{f : f' \in H^p\}$. *Then the composition operator*

$$C_h : \mathcal{B} \to H_1^p$$

is bounded if and only if h satisfies (2.1).

(4). It is found out, by a careful handling of a self map which has contact worse than that of h in this paper, that one can have h with (2.1) for all $p : 0 < p < 1$ and simultaneously (2.2). Also, this can be done with h having no angular derivative. Holomorphic self maps of U of these types will be considered, in the context of hyperbolic geometry, and in connection with more general geometric properties of composition operators, at the coming paper of the author.

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