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ON THE MCSHANE INTEGRABILITY

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ABSTRACT. For a given separable space X which contains no copy of C_0 and a weakly compact T, we show that a Dunford integrable function f : $[a,b] \to X$ is intrinsically-separable valued *if and only if* f is McShane integrable. Also, for a given separable space X which contains no copy of C_0, a weakly compact T and a Dunford integrable function f we show that if there exists a sequence (f_n) of McShane integrable functions from $[a,b]$ to X such that for each $x^* \in X^*$, $x^*f_n \to x^*f$ a.e., then f is McShane integrable. Finally, let X contain no copy of C_0. If $f : [a,b] \to X$ is McShane integrable, then F is a countably additive on Σ.

1. Introduction

In [8], we introduce the intrinsically separable valued function. In this paper, we generalize the idea put forth in [4] and [6] using the intrinsically separable valued function and investigate some properties of the McShane integrability.

Throughout this paper X is a real Banach space with continuous dual X^* and the closed unit ball of X^* will be denoted by B_{X^*}.

Let (Ω, Σ, μ) be a finite measure space. A weakly measurable function $f : \Omega \to X$ is *separable-like* if there exists a separable subspace D of X such that for every $x^* \in X^*$,

$$x^*|_D f = x^* f \quad \mu\text{-a.e.}.$$

A function $f : \Omega \to X$ is *Pettis integrable* if
(a) f is weakly μ-measurable and weakly μ-integrable, and

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(b) for every $E \in \Sigma$, there exists an $x_E \in X$ such that

$$x^*(x_E) = \int_E x^* f \, d\mu$$

for every $x^* \in X^*$.

Let the operator $T : X^* \to L^1(\mu)$ be defined by $T(x^*) = x^* f$. Then by the closed graph theorem, T is bounded linear operator. Let T^* denote the adjoint of T. Then $T^* : L^\infty(\mu) \to X^{**}$ is given by

$$T_g^*(x^*) = \int_\Omega g T(x^*) \, d\mu = \int_\Omega g x^* f d\mu$$

for every $g \in L^\infty(\mu)$, $x^* \in X^*$.

In particular, define $F : \Sigma \to X^{**}$ by

$$F(E) = (D) - \int_E f \, d\mu$$

for each $E \in \Sigma$, and $F(E) = T^*_{\chi_E}$ is called the *Dunford integral* of f over E. In the case that $F(E) \in X$ for each $E \in \Sigma$, we write $F(E) = (P) - \int_E f \, d\mu$ and it is called the Pettis integral of f over E.

The function F is not necessarily countably additive. It can be shown that F is countably additive *if and only if* T is a weakly compact operator *if and only if* $\{x^* f : x^* \in B_{X^*}\}$ is uniformly integrable in $L^1(\mu)$.

Here we recall that a subset K of $L^1(\mu)$ is called *uniformly integrable* if

$$\lim_{\mu(E) \to 0} \int_E |f| \, d\mu = 0$$

uniformly in $f \in K$.

2. McShane Integrability

Let $\Omega = [a, b]$ be a closed interval in $\mathbb{R}$. In [6], Russell A. Gordon introduces the concept of the McShane integrability on the interval $[a, b]$.

We recall the following definitions.

(a) Let $\varphi : [a, b] \to (0, \infty)$ be a function. A tagged interval $([a_i, b_i], t_i)$ consists of an interval $[a_i, b_i] \subset [a, b]$ and a point $t_i \in [a, b]$, $1 \leq i \leq N$. The tagged interval $([a_i, b_i], t_i)$ is ***subordinate*** to φ, say *sub*φ if $[a_i, b_i] \subset (t_i - \varphi(t_i), t_i + \varphi(t_i))$, $1 \leq i \leq N$.

(b) A McShane partition of $[a, b]$ is a finite sequence $(([a_i, b_i], t_i))_{1 \leq i \leq N}$ such that $([a_i, b_i])_{1 \leq i \leq N}$ is a non-overlapping family of intervals covering $[a, b]$ and $t_i \in [a, b]$ for each i.

A *gauge* on $[a, b]$ is a function $\varphi : [a, b] \to (0, \infty)$. A McShane partition $(([a_i, b_i], t_i))_{1 \leq i \leq N}$ is subordinate to a gauge φ if $t_i - \varphi(t_i) \leq a_i \leq b_i \leq t_i + \varphi(t_i)$ for every $t_i \in [a, b]$ and $1 \leq i \leq N$.

We say that a function $f : [a, b] \to X$ is *McShane integrable,* with McShane integral m if for every $\varepsilon > 0$, there is a gauge $\varphi : [a, b] \to (0, \infty)$ such that

$$\| m - \sum_{i=1}^{N} (b_i - a_i) f(t_i) \| \leq \varepsilon$$

for every McShane partition $(([a_i, b_i], t_i))_{1 \leq i \leq N}$ of $[a, b]$ subordinate to φ.

By [6,Theorem 19], we can consider the following proposition.

PROPOSITION 2.1. *Let X contain no copy of C_0. If $f : [a, b] \to X$ is McShane integrable, then f is Pettis integrable.*

The following definition is taken from [8].

DEFINITION 2.2. A function $f : \Omega \to X$ is said to be ***intrinsically-separable valued*** if there exists $E \in \Sigma$ with $\mu(E) = 0$ such that $f(\Omega - E)$ is a separable subset in X.

In particular, if f is a weakly measurable, then it is separable-like.

EXAMPLE 2.3. (a) Let $f : [0,1] \to X$ be a simple function. Then f is McShane integrable and f is intrinsically-separable valued since f is strongly measurable function.

(b) Let $f : [0,1] \to L^{\infty}[0,1]$ be a function defined by $f(t) = \chi_{[0,t]}$. Then f is McShane integrable, but f is not intrinsically separable valued.

PROPOSITION 2.4.[8, THEOREM 2.4]. *Let $f : \Omega \to X$ be Dunford integrable and T be a weakly compact. If f is intrinsically-separable valued, then f is Pettis integrable.*

By [2, Theorem 8, p.55] and [5, Theorem 3], we obtain the following proposition.

PROPOSITION 2.5. *Let $f : \Omega \to X$. If there is a sequence (f_n) of Pettis integrable functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e., then f is Pettis integrable.*

PROPOSITION 2.6.[8, THEOREM 2.6]. *Let $f : \Omega \to X$ be Dunford integrable and T be a weakly compact. If there exists a sequence (f_n) of Dunford integrable and intrinsically-separable valued functions from Ω into X such that for each $x^* \in X^*$, $x^* f_n \to x^* f$ a.e., then f is Pettis integrable.*

PROOF. By the hypothesis and Proposition 2.4, (f_n) is a sequence of Pettis integrable functions from Ω into X such that for each $x^* \in X^*$, $x^* f_n \to x^* f$ a.e.. Then, by Proposition 2.5, f is Pettis integrable.

By [6, Theorem 20] and [3, 2C], we give the following proposition.

PROPOSITION 2.7. (a) *Let X be separable which contains no copy of C_0. A function $f : [a,b] \to X$ is McShane integrable if and only if f is Pettis integrable.*

(b) *Let X be a separable. A function $f : [0,1] \to X$ is McShane integrable if and only if f is Pettis integrable.*

THEOREM 2.8. (a) *Let X be separble which contains no copy of C_0 and T be a weakly compact. Then a Dunford integrable function $f : [a,b] \to X$ is intrinsically-separable valued if and only if f is McShane integrable.*

(b) *Let X be a separable and T be a weakly compact. Then a Dunford integrable function $f : [0,1] \to X$ is intrinsically-separable valued if and only if f is McShane integrable.*

PROOF. (a) ($\Rightarrow$). Let $\Omega = [a,b]$. By Proposition 2.4, f is Pettis integrable. Then f is McShane integrable by (a) of Proposition 2.7.

($\Leftarrow$). Note that every McShane integrable function is Dunford integrable. Let $x^* \in X^*$. Since x^*f is real valued McShane integrable function, x^*f is Lebesgue integrable and so, x^*f is measurable. It means that f is weakly measurable. Since X is separable, f is separable valued and hence f is strongly measurable. Therefore f is intrinsically-separable valued.

(b) ($\Rightarrow$). In (a), we can let $\Omega = [0,1]$. By Proposition 2.4 and (b) of Proposition 2.7, we obtain the result.

($\Leftarrow$). It is trivial.

THEOREM 2.9. (a) *Let X be a separable which contains no copy C_0, T be a weakly compact and $f : [a,b] \to X$ be a Dunford integrable. If there exists a sequence (f_n) of McShane integrable functions from $[a,b]$ to X such that for each $x^* \in X^*$, $x^*f_n \to x^*f$ a.e., then f is McShane integrable.*

(b) *Let X be a seperable, T be a weakly compact and $f : [0,1] \to X$ be a Dunford integrable. If there exists a sequence (f_n) of McShane integrable functions from $[0,1]$ to X such that for each $x^* \in X^*$, $x^*f_n \to x^*f$ a.e., then f is McShane integrable.*

PROOF. (a) Since (f_n) is a sequence of McShane integrable functions from $[a,b]$ to X, (f_n) is a sequence of Dunford integrable and intrinsically separable valued functions from $[a,b]$ to X by (a) of Theorem 2.8. By Proposition 2.6, f is Pettis integrable. Since X is separable which contains no copy of C_0, f is McShane integrable by (a) of Proposition 2.7.

(b) In (a), we take $[a,b] = [0,1]$. Then f is McShane integrable by proposition 2.6 and (b) of Proposition 2.7.

In a real Banach space, if X is reflexive, then $X^{**} = X$. From the above Theorems, we obtain the following corollaries.

COROLLARY 2.10. (a) *Let X be a reflexive separable space which contains no copy of C_0 and T be a weakly compact. Then a Dunford integrable function $f : [a,b] \to X^{**}$ is intrinsically-separable valued if and only if f is McShane integrable.*

(b) *Let X be a reflexive separable space and T be a weakly compact. Then a Dunford integrable function $f : [0,1] \to X^{**}$ is intrinsically-separable valued if and only if f is McShane integrable.*

COROLLARY 2.11. (a) *Let X be a reflexive separable space which contains no copy of C_0, T be a weakly compact and $f : [a,b] \to X^{**}$ be a Dunford integrable. If there exists a sequence (f_n) of McShane integrable functions from $[a,b]$ to X^{**} such that for each $x^* \in X^*$, $x^* f_n \to x^* f$ a.e., then f is McShane integrable.*

(b) *Let X be a reflexive separable space, T be a weakly compact and $f : [0,1] \to X^{**}$ be a Dunford integrable. If there exists a sequence (f_n) of McShane integrable functions from $[0,1]$ to X^{**} such that for each $x^* \in X^*$, $x^* f_n \to x^* f$ a.e., then f is McShane integrable.*

3. Properties Of The McShane Integrability

THEOREM 3.1. *Let X contain no copy of C_0. If $f : [a,b] \to X$ is McShane integrable, then F is a countably additive on Σ.*

PROOF. Let $\{E_n\}$ be a sequence of disjoint members of Σ. By Proposition 2.1, f is Pettis integrable. Then we can define $F : \Sigma \to X^{**}$ by $F(E) = (P)-\int_E f \, d\mu$ and $F(E)$ is in X. We will show that F is weakly countably additive. For each $x^* \in X^*$,

$$
\begin{aligned}
x^*(F(\cup_{n=1}^{\infty} E_n)) &= \int_{\cup_{n=1}^{\infty} E_n} x^* f \, d\mu \\
&= \sum_{n=1}^{\infty} \int_{E_n} x^* f \, d\mu \\
&= \sum_{n=1}^{\infty} x^*(F(E_n)).
\end{aligned}
$$

The next corollary is trivial.

COROLLARY 3.2. *Let X contain no copy of C_0. If $f : [a,b] \to X$ is McShane integrable, then $\{ x^*f \mid x^* \in B_{X^*} \}$ is uniformly integrable in $L^1(\mu)$.*

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