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COHOMOLOGY OF FLAT VECTOR BUNDLES

HONG-JONG KIM

ABSTRACT. In this article, we calculate the cohomology groups of flat vector bundles on some manifolds.

1. Introduction

One of the most important invariants of a compact manifold is the Euler characteristic. This invariant can be obtained in many different interesting ways. For instance, if a smooth structure is given, H. Hopf's theory of vector fields, M. Morse's theory of the critical points of functions and Gauss-Bonnet-Chern's theory of curvatures all describe beautiful ways to understand the Euler characteristic. De Rham's cohomology theory also gives the invariant and its generalization to arbitrary flat vector bundles also computes the Euler characteristic.

When one considers a Morse function f on a compact (smooth connected) manifold (without boundary) M, it is well-known [Mil] that the number $c_k(f)$ of the critical points of f of index k is bounded below by the k-th Betti number $b_k(M)$ of M;

$$c_k(f) \geq b_k(M)$$

for every integer k. In fact one can replace f in the above inequality by an arbitrary Morse-Novikov 1-form ω on M;

$$c_k(\omega) \geq b_k(M).$$

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Note that a 1-form ω on M is called a *Morse-Novikov 1-form* if $\omega = df$ locally for some Morse function f [BF, DFN, N, NS, P]. The number $c_k(\omega)$ is the number of zeroes of ω whose "indices" are equal to k.

Let

$$\begin{aligned} c_k(M) &:= \min\{c_k(\omega) \mid \omega : \text{Morse-Novikov 1-form on } M\} \\ \beta_k(M) &:= \sup\{\beta_k(M,E) \mid E \to M \text{ is a flat vector bundle}\}, \end{aligned}$$

where

$$\beta_k(M,E) = \frac{h^k(M,E)}{\operatorname{rank} E}$$

and $h^k(M,E)$ is the dimension of the k-th cohomology space $H^k(M,E)$ of a flat vector bundle E over M.

In [K1], the above weak Morse inequality is generalized to the following inequality:

THEOREM 1. *For a compact smooth manifold M, the inequality*

$$c_k(M) \geq \beta_k(M)$$

holds for all integer k.

Note that when $E \to M$ is a trivial line bundle with the trivial connection, $h^k(M,E)$ is equal to the ordinary k-th Betti number $b_k(M)$ of M. Note also that the existence [DFN] of a Morse function with one local maximum point and one local minimum point on a compact n-manifold implies that

$$c_0(M) = 1 = c_n(M).$$

A Morse function $f : M \to \mathbb{R}$ is called an *exact* or a *minimal* function [F] if $c_k(f) = c_k(M)$ for all integer k. It is known [Sm, F] that on a simply connected compact manifold of dimension ≥ 6, there exists an exact Morse function and

$$c_k(M) = b_k(M) + q_k(M) + q_{k-1}(M)$$

where $q_k(M)$ denotes the minimal number of the generators of the torsion subgroup of $H_k(M,\mathbb{Z})$.

In [Bott], a Morse function f defined on M is said to be *perfect* if $c_k(f) = b_k(M)$ for all k. Perfect functions are clearly exact. Thus if M is *perfect*, i.e., if M admits a perfect Morse function, then

$$b_k(M) = \beta_k(M)$$

for all k. For instance, spheres $\mathbf{S}^n$, tori $\mathbf{T}^n$ and the complex projective spaces $\mathbf{P}^n(\mathbb{C})$ are all perfect.

Note that non orientable compact manifolds are not perfect, since any Morse function admits a maximum and the top Betti numbers are equal to zero for such manifolds.

We have the following observations.

COROLLARY. *Let E be a flat vector bundle over a perfect compact manifold M. Then*

$$\beta_k(M, E) \leq b_k(M)$$

for all integer k.

PROPOSITION. *The product of two perfect compact manifolds is perfect.*

PROOF. Let $f_1 : M_1 \to \mathbb{R}$ and $f_2 : M_2 \to \mathbb{R}$ be Morse functions on compact manifolds. Since f_i's are bounded, we may assume that they are positive, by adding some constants, if necessary. Let $\mathrm{Crit}(f_i) \subset M_i$ be the set of critical points of f_i. Let

$$f(m_1, m_2) := f_1(m_1) f_2(m_2), \qquad (m_1, m_2) \in M_1 \times M_2.$$

Then

$$df = f_1 df_2 + f_2 df_1$$

and hence the critical set of the function f is

$$\mathrm{Crit}(f) = \mathrm{Crit}(f_1) \times \mathrm{Crit}(f_2).$$

Moreover, the Hessian of f at each critical point (m_1, m_2) is

$$\mathrm{Hess}\, f(m_1, m_2) = f_1(m_1)\, \mathrm{Hess}\, f_2(m_2) + f_2(m_2)\, \mathrm{Hess}\, f_1(m_1)$$

Thus Hess $f(m_1, m_2)$ is nonsingular, and the Morse *index*, i.e., the maximal dimension of the tangential subspace on which Hess $f(m_1, m_2)$ is negative definite, is given by

$$\operatorname{ind} f(m_1, m_2) = \operatorname{ind} f_1(m_1) + \operatorname{ind} f_2(m_2).$$

Thus the number of critical points are related by

$$c_k(f) = \sum_{k_1+k_2=k} c_{k_1}(f_1) \cdot c_{k_2}(f_2)$$

for all integer k. In particular, when f_1 and f_2 are perfect, we have

$$c_k(f) = \sum_{k_1+k_2=k} b_{k_1}(M_1) \cdot b_{k_2}(M_2) = b_k(M_1 \times M_2).$$

This completes the proof.

In this article we compute the cohomology spaces of flat vector bundles over some manifolds. Note that a parallel section on a flat vector bundle is determined by its value at a single point and hence

$$\beta_0(M) = 1$$

for every compact manifold M. We also prove the following theorem.

THEOREM 2. *Let $\tilde{M}$ be a Galois covering space of a compact manifold M with the finite Galois group $\Gamma := \pi_1(M)/\pi_1(\tilde{M})$. Let ρ be a representation of Γ on some finite dimensional complex vector space V. Then*

$$\beta_k(M, \tilde{M} \times_\rho V) \le b_k(\tilde{M})$$

for all k.

As a corollary we have,

COROLLARY. *Let $\tilde{M}$ be a universal covering space of a compact manifold M with the finite fundamental group. Then*

$$\beta_k(M) \le b_k(\tilde{M})$$

for all integer k.

We like to thank KyungHo Oh [Oh] for explaining algebraic aspect of flat connections as in [Borel].

2. Review of Flat Vector Bundles

Let E be a (complex) vector bundle over a smooth connected n dimensional manifold M. Then a connection ∇ for the bundle $E \to M$ induces a sequence of exterior derivatives:

$$(1) \qquad 0 \to A^0(M,E) \xrightarrow{d_\nabla} A^1(M,E) \xrightarrow{d_\nabla} \cdots \xrightarrow{d_\nabla} A^n(M,E) \to 0,$$

where $A^k(M,E)$ denotes the space of differential k-forms on M with values in E. We say that the connection ∇ is *flat* if the above sequence is a chain complex, i.e.,

$$d_\nabla \circ d_\nabla = 0.$$

A vector bundle $E \to M$ together with a flat connection ∇ on it is called a ***flat vector bundle***. We will often say that E is a flat vector bundle, when it really means that the pair (E, ∇) is flat. For instance, the tangent bundles of the "space forms" [W] and the affine manifolds [BP] are all flat.

A flat bundle $E \to M$ gives rise to cohomology spaces $H^k(M,E)$ associated to the above chain complex (1), and the dimension of $H^k(M,E)$ will be denoted by $h^k(M,E)$;

$$h^k(M,E) := \dim_{\mathbb{C}} H^k(M,E).$$

This cohomology space can be used to compute one of the most important topological invariants of the base manifold, the Euler characteristic;

$$\chi(M) = \sum_k (-1)^k \beta_k(M,E)$$

where $\beta_k(M,E) := \frac{h^k(M,E)}{\operatorname{rank} E}$. The above identity can be proved by the Atiyah-Singer index formula [AS], or by the strong Morse inequality [K1].

A flat bundle and its cohomology spaces can be interpreted as a *system of local coefficients* [St, Die] or a *locally constant sheaf* [Sp]. This point of view will not be considered in this article.

2.1. Holonomy Representation. Let $E \to M$ be a flat vector bundle. We fix a point $m_0 \in M$, then the fundamental group $\pi_1(M, m_0)$ of M at m_0 acts on the fiber E_{m_0} of $E \to M$ as follows. Let

$$\gamma : [0,1] \to M$$

be a piecewise smooth path with $\gamma(0) = m_0$ and $\gamma(1) =: m_1$. Then for each element $e \in E_{m_0}$ there exists a unique parallel section $\tilde{\gamma}_e$ of E along γ with $\tilde{\gamma}_e(0) = e$. Then the linear map

$$E_{m_0} \ni e \mapsto \tilde{\gamma}_e(1) =: e \cdot \gamma \in E_{m_1}$$

is called the *parallel transport* along γ. Since our connection is flat, the parallel transport depends only on the homotopy class of paths. In particular, when the base manifold M is simply connected, every flat vector bundle is trivial and hence $\beta_k(M, E) = b_k(M)$ for all k.

If we consider the loops on M based at m_0, then we obtain the action of the fundamental group $\pi_1(M, m_0)$ on the fiber E_{m_0}. This representation is called the *holonomy representation.*

2.2. Universal covering space. Let $p : \tilde{M} \to M$ be a universal covering space of M, then the fundamental group $\pi_1(M, m_0)$ of M at a point $m_0 \in M$ acts on the right of $\tilde{M}$, when we fix a point $\tilde{m}_0 \in \tilde{M}$ with $p(\tilde{m}_0) = m_0$. In fact the action

$$R_g : \tilde{M} \to \tilde{M}, \qquad \forall g \in \pi_1(M, m_0)$$

is given as follows: Let $\tilde{m} \in \tilde{M}$. Take a smooth path $\tilde{\gamma} : [0,1] \to \tilde{M}$ such that $\tilde{\gamma}(0) = \tilde{m}_0$ and $\tilde{\gamma}(1) = \tilde{m}$. Then

$$\gamma := p \circ \tilde{\gamma}$$

is a path on M from m_0 to $m := p(\tilde{m})$. Now

$$\gamma^{-1} * g * \gamma$$

is an element of $\pi_1(M, m)$. Then the end point of the lifting of this homotopy class of loops is by definition $R_g(\tilde{m})$.

Then it is easy to see that a flat vector bundle $E \to M$ is isomorphic to the vector bundle

$$\tilde{M} \times_\rho E_{m_0}$$

associated to the "principal bundle" $\tilde{M} \to M$ and the holonomy representation ρ.

Conversely, let V be a finite dimensional complex vector space and let $\rho : \pi_1(M, m_0) \to \mathrm{GL}(V)$ be a representation. Then

$$E := \tilde{M} \times_\rho V$$

becomes a flat vector bundle over M. Note that each $\tilde{m} \in \tilde{M}$ induces an isomorphism

$$j_{\tilde{m}} : E_{p(\tilde{m})} \ni [(\tilde{m}, v)] \mapsto v \in V.$$

Let

$$A^k_\rho(\tilde{M}, V)$$

be the space of differential k-forms ξ on $\tilde{M}$ with values in the vector space V satisfying

$$R^*_g \xi = \rho(g)^{-1}\xi, \qquad \forall\, g \in \pi_1(M, m_0).$$

Then the map

$$p^* : A^k(M, E) \to A^k_\rho(\tilde{M}, V)$$

defined by

$$(p^*\xi)_{\tilde{m}}(w_1, \ldots, w_k) := j_{\tilde{m}}\xi_{p(\tilde{m})}(p_*w_1, \ldots, p_*w_k), \quad \forall w_1, \ldots, w_k \in T_{\tilde{m}}\tilde{M}$$

is an isomorphism [C, Rag]. In fact this isomorphism commutes with the exterior differentiations and hence the chain complex (1) is isomorphic to

$$0 \to A^0_\rho(\tilde{M}, V) \xrightarrow{d} A^1_\rho(\tilde{M}, V) \to \cdots \to A^n_\rho(\tilde{M}, V) \to 0.$$

This observation gives an easy way to compute the cohomology groups of many flat bundles. We will often denote by

$$H^k(M, \rho)$$

the cohomology spaces of the flat bundle associated to a representation ρ of the fundamental group of M.

2.3. Indecomposable Flat bundles. A flat vector bundle $E \to M$ is said to be *decomposable*, if there exist nontrivial subbundles E_1 and E_2 such that $E = E_1 \oplus E_2$ and E_i's are invariant under the parallel transports. Note that a flat vector bundle is decomposable if and only if it corresponds to a *decomposable* representation of the fundamental group of the base manifold. In this case each direct summand of E is also flat and

$$H^k(M,E) = H^k(M,E_1) \oplus H^k(M,E_2).$$

Note that

$$\frac{h^k(M,E)}{\operatorname{rank} E} \le \max\left\{\frac{h^k(M,E_1)}{\operatorname{rank} E_1}, \frac{h^k(M,E_2)}{\operatorname{rank} E_2}\right\}.$$

Thus the invariant $\beta_k(M)$ of a compact manifold M is equal to the supremum of $\{\beta_k(M,E)\}$, where E runs through *indecomposable* flat bundles over M.

PROPOSITION. *Let M be a compact manifold whose fundamental group is abelian. Then*

$$\beta_k(M) = \max\{h^k(M,L) : L \to M \textit{ is a flat line bundle }\}$$

for all integer k.

PROOF. Note that any complex representation of an abelian group is one dimensional. Thus if $E \to M$ is indecomposable, then it is a line bundle.

Note that a complex line bundle $L \to M$ admits a flat connection if and only if the real first Chern class $c_1(L)_{\mathbb{R}} \in H^2(M,\mathbb{R})$ vanishes.

3. Hodge Theory

Let $\tilde{M}$ be a smooth n-manifold and let Γ be a *properly discontinuous* group of diffeomorphisms acting on $\tilde{M}$ on the right. Thus [Die] each $\tilde{m} \in \tilde{M}$ has a neighborhood U such that

$$U \cap U\gamma = \emptyset$$

for any nontrivial $\gamma \in \Gamma$. Then the quotient map $p : \tilde{M} \to M := M/\Gamma$ is a covering map. We will assume that M is compact. Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation of Γ on some complex vector space V.

We will assume that $\tilde{M}$ admits a Riemannian metric for which Γ acts as isometries so that the quotient M inherits a Riemannian metric. We also assume that V admits a Hermitian inner product for which Γ acts as isometries. Then the vector bundle

$$E := \tilde{M} \times_\rho V \to M$$

is equipped with a natural Hermitian metric and a compatible flat connection ∇. Let d_∇^* be the formal adjoint of the chain complex

$$0 \to A^0(M,E) \xrightarrow{d_\nabla} A^1(M,E) \xrightarrow{d_\nabla} \cdots \to A^n(M,E) \to 0$$

and let

$$\Delta := d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$$

be the associated Laplacian. Then the cohomology spaces $H^k(M,E)$ are isomorphic to the finite dimensional space

$$H^k_\Delta(M,E) = \{\xi \in A^k(M,E) : \Delta\xi = 0\}$$

of *harmonic* sections.

Note that if $d^* : A^{k+1}(\tilde{M},V) \to A^k(\tilde{M},V)$ is the formal adjoint of $d : A^k(\tilde{M},V) \to A^{k+1}(\tilde{M},V)$, then

$$d^*(A_\rho^{k+1}(\tilde{M},V)) \subset A_\rho^k(\tilde{M},V), \qquad d^*p^* = p^*d_\nabla^*$$

and

$$\tilde{\Delta}(A_\rho^k(\tilde{M},V)) \subset A_\rho^k(\tilde{M},V), \qquad \tilde{\Delta} := dd^* + d^*d.$$

Thus

$$H^k(M,E) \simeq H^k_\Delta(M,E) \simeq \ker(\tilde{\Delta}|_{A_\rho^k(\tilde{M},V)}).$$

In particular we have

THEOREM 3. *Let $\tilde{M}$ be a universal covering space of a compact manifold M. If the fundamental group of M is finite, then*

$$\beta_k(M) \leq b_k(\tilde{M})$$

for all integer k.

One can compare this theorem with that of Leray-Hirsch [BT, Die]. The above theorem can be easily generalized to the following theorem.

THEOREM 4. *Let $\tilde{M}$ be a Galois covering space of a compact manifold M with the finite Galois group $\Gamma := \pi_1(M)/\pi_1(\tilde{M})$. Let ρ be a representation of Γ on some finite dimensional complex vector space V. Then*

$$\beta_k(M, \tilde{M} \times_\rho V) \leq b_k(\tilde{M})$$

for all k.

This is a generalization of the following well-known fact [BP, p.305].

COROLLARY. *Let $\tilde{M}$ be a finite Galois covering space of a compact manifold M. Then*

$$b_k(M) \leq b_k(\tilde{M})$$

for all k.

COROLLARY. *If a manifold M is covered by a sphere $\mathbf{S}^n$, $n \geq 2$, then*

$$\beta_k(M) = 0, \qquad 0 < k < n.$$

We now compute the top dimension $H^n(M, \rho)$ for a compact manifold M with a finite Galois covering $\tilde{M}$ and a representation $\rho : \pi_1(M) \to \pi_1(M)/\pi_1(\tilde{M}) \to \mathrm{GL}(V)$. We may assume that $\tilde{M}$ is oriented and there exists a Riemannian metric on $\tilde{M}$ invariant under the action of the Galois group $\Gamma := \pi_1(M)/\pi_1(\tilde{M})$. Then the volume form vol $\in A^n(\tilde{M})$ is harmonic and

$$R_\gamma^* \operatorname{vol} = \operatorname{sgn}(\gamma) \operatorname{vol}, \qquad \forall \gamma \in \gamma,$$

where $\operatorname{sgn}(\gamma)$ is equal to 1 if $R_\gamma : \tilde{M} \to \tilde{M}$ preserves the orientation, and is equal to -1 if R_γ reverses the orientation. Thus we have

$$H^n(M, \rho) \simeq H_\rho^n(\tilde{M}, V) \simeq \{f \in V : \operatorname{sgn}(\gamma) f = \rho(\gamma) f, \quad \forall \gamma \in \pi_1(M)\}.$$

For instance, if ρ is the nontrivial one-dimensional representation of the fundamental group of the real projective space $\mathbb{RP}^n$, then

$$h^k(\mathbb{RP}^n, \rho) = \begin{cases} 1, & k = n,\ n \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

We have the following observation.

PROPOSITION. *For any compact n-manifold M, $\beta_n(M) = 1$.*

PROOF. Note that these exists a Morse function on M with a unique local maximum point [DFN]. This implies that $\beta_n(M) \leq 1$. Thus it suffices to show the existence of a flat vector bundle E over M with $\beta_n(M, E) = 1$. When M is orientable, one can take the trivial flat line bundle for E. Now suppose that M is not orientable. Then there exists an orientable double covering space $\tilde{M}$ of M. The flat line bundle L over M associated to this double covering space has the desired property; for, if $\gamma : \tilde{M} \to \tilde{M}$ is the nontrivial deck transformation, then γ reverses the orientation of $\tilde{M}$ and hence if vol denotes the volume form of $\tilde{M}$ with respect to a metric for which γ is an isometry, then

$$\gamma^* \operatorname{vol} = -\operatorname{vol}.$$

Thus a harmonic n-form $\xi = c \operatorname{vol}$ on $\tilde{M}$ satisfies

$$\gamma^* \xi = -\xi.$$

Thus $\beta_n(M, L) \geq 1$. This completes the proof.

As an example we consider flat bundles over the Lens spaces. Let $g \in \mathrm{U}(n)$ acts on the odd dimensional sphere

$$\mathbf{S}^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

We will assume that $g^p = 1$ for some positive integer p and g^k has no fixed points for $0 < k < p$. Then we have a free action of the cyclic group $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ of order p on the sphere $\mathbf{S}^{2n-1}$.

After the diagonalization, we may assume that

$$g(z_1, \ldots, z_n) = (\omega_1 z_1, \ldots, \omega_n z_n)$$

for some p-th roots $\omega_1, \ldots, \omega_n$ of unity. Since the action is free, the roots

$$\omega_j = e^{2\pi i q_j/p}, \quad q_j \in \{1, \ldots, p-1\}, \qquad j = 1 \ldots, n$$

are primitive. Recall that a p-th root ω of unity is said to be *primitive* if $\omega^k \neq 1$ for $0 < k < p$. Thus p and each q_j are relatively prime. Now the quotient space of $\mathbf{S}^{2n-1}$ under this action is called the *Lens space* [Ray] and will be denoted by

$$L_p^{2n-1}(q_1 : \cdots : q_n).$$

Let

$$\rho_{p,q} : \mathbb{Z}_p \to \mathbb{C}^\times, \qquad q \in \{0, 1, \ldots, p-1\}$$

be the representation with

$$\rho_{p,q}(1) = e^{2\pi i q/p}.$$

Then the dimensions of the cohomology spaces of the flat line bundle over $L_p^{2n-1}(q_1 : \cdots : q_n)$ associated to the representation $\rho_{p,q}$, $q \neq 0$, are

$$h^k(L_p^{2n-1}, \rho_{p,q}) = 0$$

for all k, by the Hodge theory.

The Hodge theory can be applied to compute the cohomology spaces of flat vector bundles over tori [K2].

4. Poincaré duality

Let E be a vector bundle over a compact Riemannian n-manifold M. We will assume a Hermitian structure on E and a compatible connection ∇ on E. Then the adjoint d_∇^* of the exterior covariant differentiation

$$d_\nabla : A^k(M, E) \to A^{k+1}(M, E)$$

is given by

$$d_\nabla^* = -(-1)^{nk} \star d_\nabla \star : A^{k+1}(M, E) \to A^k(M, E),$$

where $\star$ denotes the "local" Hodge star. Then the kernel of the Laplacian

$$\Delta := d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$$

on $A^k(M,E)$ will be denoted by

$$H_\Delta^k(M,E).$$

Note that when ∇ is flat, this space of harmonic forms is isomorphic to the cohomology space $H^k(M,(E,\nabla))$. When M is orientable, there exists a global Hodge star $\star$, and $\star$ commutes with the Laplacian. Thus we have

PROPOSITION. *Let E be a Hermitian vector bundle over a compact orientable Riemannian n-manifold M, and let ∇ be a connection on E compatible with the Hermitian structure. Then*

$$H_\Delta^k(M,E) \simeq H_\Delta^{n-k}(M,E)$$

for all integer k.

COROLLARY. *Let E be a flat vector bundle over a compact n-manifold M associated to a unitary representation of the fundamental group of M. If M is orientable,*

$$\beta_k(M,E) = \beta_{n-k}(M,E)$$

for all integer k.

We now consider the dual vector bundle $E^\vee$ of a Hermitian vector bundle E over a compact Riemannian n-manifold M. Then a Hermitian connection ∇ on E induces a connection $\nabla^\vee$ on $E^\vee$, which is again compatible with the canonical Hermitian structure on $E^\vee$. Now let

$$\sharp : A^k(M,E) \to A^k(M,E^\vee)$$

be the canonical conjugate-linear map induced from the Hermitian structure. Then this map commutes with the local Hodge star and

$$\sharp \circ d_\nabla = d_{\nabla^\vee} \circ \sharp.$$

This implies that

$$\sharp \circ d_\nabla = d_{\nabla^\vee}^* \circ \sharp.$$

Thus $H_\Delta^k(M,E)$ and $H_\Delta^k(M,E^\vee)$ are conjugate-linear isomorphic.

COROLLARY. *Let E be a flat vector bundle over a compact n-manifold M associated to a unitary representation of the fundamental group of M. Then*

$$\beta_k(M,E) = \beta_k(M,E^\vee)$$

for all integer k.

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Department of Mathematics
Seoul National University
Seoul 151–742, Korea