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THE INDUCED AND INTRINSIC CONNECTIONS OF CARTAN TYPE IN A FINSLERIAN HYPERSURFACE

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ABSTRACT. The main purpose of the present paper is to derive the induced (Finsler) connections on the hypersurface from the Finsler connections of Cartan type (a Wagner, Miron, Cartan C- and Cartan Y - connection) of a Finsler space and to seek the necessary and sufficient conditions that the induced connections coincide with the intrinsic connections. And we show the differences of quantities with respect to the respective g connections and an induced Cartan connection. Finally we show some examples.

0. Introduction

Recently M.Matsumoto has obtained in his excellent paper [3] a synthesis of all results published in the field of hypersurfaces in a Finsler space during about a half century. He pointed out gaps and considered the theory of hypersurfaces in Finsler spaces endowed with the important connections (a Cartan, Berwald, Rund and Hashiguchi connection [5]).

In the present paper we are concerned with a hypersurface F^{n-1} in a Finsler spaces F^n endowed with Finsler connections of Cartan type (generalized Cartan connections [2] : a Wagner, Miron, Cartan C - and Cartan Y - connection). The main purpose of the present paper is to derive the induced (Finsler) connections on F^{n-1} from the Finsler connections of Cartan type of F^n and to seek the necessary and sufficient

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conditions that the induced connections coincide with the intrinsic connections. And we show the differences of the quantities with respect to the respective induced connections and an induced Cartan connection. Finally we get the quantities of the $(h)h$-torsion tensors for the respective induced connections of a hypersurface in a C-reducible Finsler space.

1. Preliminaries

Let $F^n = (M^n,\ L(x,y))$, be an n-dimensional Finsler space with fundamental function $L(x,y)$ which is assumed to be (1) p-homogeneous in $y = (y^i)$, $y^i = dx^i/dt$, where the equations $x^i = x^i(t)$ represent a curve of an n-dimensional differentiable mainfold M^n. The Finsler metric tensor $g_{ij}(x,\ y)$ is given by $g_{ij}(x,\ y) = (\dot{\partial}_i\dot{\partial}_j L^2)/2$, where $\dot{\partial}_i = \partial/\partial y^i$. (Throughout the present paper, Latin indices take values $1,\cdots,n$.) And *Cartan's C-tensor* $C_{ijk} = (\dot{\partial}_k g_{ij})/2$ is denoted by g_{ijk} to avoid confusion. Then a hypersurface $F^{n-1} = (M^{n-1}, \underline{L}(u,v))$ of the F^n is an (n-1)-dimensional Finsler space with the *induced metric function* $\underline{L}(u,v)$ [3], where $v = (v^\alpha)$ is a supporting element of M^{n-1} at a point $u = (u^\alpha)$ of M^{n-1} ($y^i = B^i_\alpha(u)v^\alpha$, *projection factors* $B^i_\alpha = \partial x^i/\partial u^\alpha$ and Greek indices run from 1 to $n-1$).

$l_\alpha = \dot{\partial}_\alpha \underline{L}$, the metric tensor $g_{\alpha\beta} = (\dot{\partial}_\alpha\dot{\partial}_\beta \underline{L}^2)/2$ and Cartan's C-tensor $g_{\alpha\beta\gamma} = (\dot{\partial}_\gamma g_{\alpha\beta})/2$ of F^{n-1} are given by

$$(1.1)\qquad l_\alpha = l_i B^i_\alpha,\ g_{\alpha\beta} = g_{ij}B^{ij}_{\alpha\beta}\ \textit{and}\ g_{\alpha\beta\gamma} = g_{ijk}B^{ijk}_{\alpha\beta\gamma}.$$

where $\dot{\partial}_\alpha = \partial/\partial v^\alpha$, $l_i = \dot{\partial}_i L$, $B^{ij\cdots}_{\alpha\beta\cdots} = B^i_\alpha B^j_\beta\cdots$.

A unit normal vector $B^i(u,v)$ at each point u of F^{n-1} is defined by

$$(1.2)\qquad g_{ij}(x(u),\ y(u,v))B^i_\alpha(u)B^j = 0,\ g_{ij}(x(u),\ y(u,v))B^iB^j = 1.$$

Then for the angular metric tensor $h_{ij} = g_{ij} - l_il_j$, we have

$$(1.3)\qquad h_{ij}B^{ij}_{\alpha\beta} = h_{\alpha\beta},\ h_{ij}B^i_\alpha B^j = 0,\ h_{ij}B^iB^j = 1.$$

We introduce important tensors from g_{ijk} :

$$(1.4)\qquad M_{\alpha\beta} = g_{ijk}B^{ij}_{\alpha\beta}B^k,\ M_\alpha = g_{ijk}B^i_\alpha B^jB^k,\ M = g_{ijk}B^iB^jB^k.$$

The h and v-covariant derivatives $X^i{}_{|k}, X^i|_k$ of a Finsler vector field $X^i(x,y)$ with respect to a Finsler connection $F\Gamma = (F_j{}^i{}_k,\ N^i{}_k,\ C_j{}^i{}_k)$ are defined by

$$X^i{}_{|k} = \delta_k X^i + X^h F_h{}^i{}_k,\ X^i|_k = \dot{\partial}_k X^i + X^h C_h{}^i{}_k, \tag{1.5}$$

where $\delta_k = \partial_k - N^h{}_k \dot{\partial}_h (\partial_k = \partial/\partial x^k)$.

In [3] the *induced (Finsler) connection* $IF\Gamma = (F_\beta{}^\alpha{}_\gamma,\ N^\alpha{}_\gamma,\ C_\beta{}^\alpha{}_\gamma)$ on the F^{n-1} from the $F\Gamma$ of F^n is given by

$$F_\beta{}^\alpha{}_\gamma = B_i^\alpha\{B_{\beta\gamma}^i + B_\beta^j(F_j{}^i{}_k B_\gamma^k + C_j{}^i{}_k B^k H_\gamma)\}, \tag{1.6}$$

$$N^\alpha{}_\beta = B_i^\alpha(B_{0\beta}^i + N^i{}_j B_\beta^j), \tag{1.7}$$

$$C_\beta{}^\alpha{}_\gamma = C_j{}^i{}_k B_i^\alpha B_{\beta\gamma}^{jk}. \tag{1.8}$$

where $B_{\alpha\beta}^i = \partial_\beta B_\alpha^i$ $(\partial_\beta = \partial/\partial u^\beta)$, $B_{0\beta}^i = v^\alpha B_{\alpha\beta}^i$ and a normal curvature vector $H_\beta := B_i(B_{0\beta}^i + N^i{}_j B_\beta^j)$. And the $(h)h$-torsion tensor $T_\beta{}^\alpha{}_\gamma$, $(v)hv$-torsion tensor $P^\alpha{}_{\gamma\beta}$, $(v)v$-torsion tensor $S^\alpha{}_{\beta\gamma}$ and deflection tensor $D^\alpha{}_\gamma$ of the $IF\Gamma$ and $\dot{\partial}_\beta H_\gamma - H_{\beta\gamma}$ are given by

$$T_\beta{}^\alpha{}_\gamma = B_i^\alpha\{T_j{}^i{}_k B_{\beta\gamma}^{jk} + C_j{}^i{}_k(B_\beta^j H_\gamma - B_\gamma^j H_\beta)B^k\}, \tag{1.9}$$

$$P^\alpha{}_{\gamma\beta} = 2H_\gamma M_\beta^\alpha + B_i^\alpha(P^i{}_{kj} B_{\gamma\beta}^{kj} - H_\gamma C_k{}^i{}_j B_\beta^k B^j), \tag{1.10}$$

$$S^\alpha{}_{\beta\gamma} = S^i{}_{jk} B_i^\alpha B_{\beta\gamma}^{jk}, \tag{1.11}$$

$$D^\alpha{}_\gamma = B^\alpha{}_i(D^i{}_k B_\gamma^k + C_0{}^i{}_k B^k H_\gamma), \tag{1.12}$$

$$\dot{\partial}_\beta H_\gamma - H_{\beta\gamma} = M_\beta H_\gamma + B_i(P_{kj}^i B_{\gamma\beta}^{kj} - H_\gamma C_k{}^i{}_j B_\beta^k B^j), \tag{1.13}$$

where $T_j{}^i{}_k, P^i{}_{kj}, S^i{}_{jk}$ and $D^i{}_k$ are the $(h)h$-torsion tensor, $(v)hv$-torsion tensor, $(v)v$-torsion tensor and deflection tensor of $F\Gamma$ respectively and a *second fundamemtal h-tensor* $H_{\beta\gamma} := B_i\{B_{\beta\gamma}^i + B_\beta^j(F_j{}^i{}_k B_\gamma^k + C_j{}^i{}_k B^k H_\gamma)\}$.

The terminologies and notations are referred to M.Matsumoto's monographs [3], [5].

2. The induced Wagner connection

As a typical Finsler connection of Cartan type (generalized Cartan connection) we have a *Wagner connection* [1] $W\Gamma(s) = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ of F^n ($W1$: h-metrical, $W2$: semi-symmetric, $W3$: deflection $D^1{}_k = 0$, $W4$: v-metrical, $W5$: $(v)v$-torsion $S^i_{jk} = 0$).

We shall denote by $IW\Gamma$ the connection of a hypersurface F^{n-1} induced from the Wagner connection $W\Gamma(s)$ and indicate the quantities with respect to $IW\Gamma$ by putting "w" on them. Then (1.1), (1.8) and (W4,5) show $\overset{w}{C}_\beta{}^\alpha{}_\gamma = g_\beta{}^\alpha{}_\gamma$, so $\overset{w}{S}_\beta{}^\alpha{}_\gamma = 0$. Thus $IW\Gamma = (\overset{w}{F}_\beta{}^\alpha{}_\gamma, \overset{w}{N}{}^\alpha{}_\gamma, g_\beta{}^\alpha{}_\gamma)$.

What sort of Finsler connection is the induced connection $IW\Gamma$? ($W1$) and ($W4$) show that it is metrical ($g_{\alpha\beta|\gamma} = 0 = g_{\alpha\beta}|_\gamma$). Next ($W3$) and (W4,5) give $\overset{w}{D}{}^\alpha{}_\gamma = 0$ from (1.12). However (1.9) does not lead to $\overset{w}{T}_\beta{}^\alpha{}_\gamma = \delta^\alpha_\beta s_\gamma - \delta^\alpha_\gamma s_\beta$, but from (1.4) we get

$$\overset{w}{T}_\beta{}^\alpha{}_\gamma = \delta^\alpha_\beta s_\gamma - \delta^\alpha_\gamma s_\beta + M^\alpha_\beta \overset{w}{H}_\gamma - M^\alpha_\gamma \overset{w}{H}_\beta. \tag{2.1}$$

Thus, according to the theory of generalized Cartanconnections due to M. Hashiguchi [1] and M. Matsumoto [5], we have

THEOREM 2.1. *The connection $IW\Gamma$ of a hypersurface F^{n-1} in a Finsler space F^n, induced from the Wagner connection $W\Gamma(s)$ of F^n, is a generalized Cartan connection which is uniquely determined from the induced metric $\underline{L}(u,v)$ by the following five axioms :*

(IW1) $g_{\alpha\beta|\gamma} = 0$,

(IW2) The $(h)h$-torsion tensor $\overset{w}{T}_\beta{}^\alpha{}_\gamma$ of $IW\Gamma$ is given by (2.1),

(IW3) The deflection tensor $\overset{w}{D}{}^\alpha_\gamma = 0$,

(IW4) $\mathrm{g}_{\alpha\beta}|_\gamma = 0$,

(IW5) The $(v)v$-torsion tensor $\overset{w}{S}{}^\alpha{}_{\beta\gamma} = 0$.

We shall apply the procedure to find a generalized Cartan connection to this $IW\Gamma$.

From ($IW1$: $IW\Gamma$ is h-metrical), making use of Christoffel process with respect to α, β, γ, we have

$$\overset{w}{F}_{\alpha\beta\gamma} = (\delta_\gamma \mathrm{g}_{\alpha\beta} + \delta_\alpha \mathrm{g}_{\beta\gamma} - \delta_\beta \mathrm{g}_{\gamma\alpha})/2 + A_{\alpha\beta\gamma}, \tag{2.2}$$

where we put

$$A_{\alpha\beta\gamma} := (\overset{w}{T}_{\alpha\beta\gamma} - \overset{w}{T}_{\beta\gamma\alpha} + \overset{w}{T}_{\gamma\alpha\beta})/2. \tag{2.3}$$

Hence, since $\delta_\gamma = \partial_\gamma - \overset{w}{N}{}^{\delta}_{\gamma}\dot{\partial}_\delta$, $(IW4)$ and $(IW5)$ give

$$F_{\alpha\beta\gamma} = \gamma_{\alpha\beta\gamma} - g_{\alpha\beta\delta}\overset{w}{N}{}^{\delta}{}_{\gamma} - g_{\beta\gamma\delta}\overset{w}{N}{}^{\delta}{}_{\alpha} + g_{\gamma\alpha\delta}\overset{w}{N}{}^{\delta}{}_{\beta} + A_{\alpha\beta\gamma}. \tag{2.4}$$

Thus from $(IW3)$ we get

$$\begin{aligned} \overset{w}{N}{}^{\alpha}{}_{\gamma} &= \overset{w}{F}_0{}^{\alpha}{}_{\gamma} = \gamma_0{}^{\alpha}{}_{\gamma} - g^{\alpha}{}_{\gamma\delta}\overset{w}{N}{}^{\delta}{}_0 + A_0{}^{\alpha}{}_{\gamma}, \\ \overset{w}{N}{}^{\delta}{}_0 &= \gamma_0{}^{\delta}{}_0 + A_0{}^{\delta}{}_0. \end{aligned} \tag{2.5}$$

Substituting (2.2) into (2.4), we have

$$A_{\alpha\beta\gamma} = M_{\alpha\gamma}\overset{w}{H}_\beta - M_{\beta\gamma}\overset{w}{H}_\alpha + g_{\alpha\gamma}s_\beta - g_{\beta\gamma}s_\alpha. \tag{2.6}$$

Consequently, denoting by $W\underline{\Gamma} = (F_\beta{}^\alpha{}_\gamma,\ N^\alpha{}_\gamma,\ C_\beta{}^\alpha{}_\gamma)$ the intrinsic Wagner connection of F^{n-1} determined from the $\underline{L}(u,v)$, we have

$$\begin{aligned} \overset{w}{F}_\beta{}^{\alpha}{}_{\gamma} &= F_\beta{}^{\alpha}{}_{\gamma} + (g_\beta{}^{\alpha}{}_{\delta}M^{\delta}_{\gamma} + g^{\alpha}{}_{\gamma\delta}M^{\delta}_{\beta} - g_{\beta\gamma}{}^{\delta}M^{\alpha}_{\delta})\overset{w}{H}_0 \\ &\quad + M_{\beta\gamma}\overset{w}{H}{}^{\alpha} - M^{\alpha}_{\gamma}\overset{w}{H}_\beta, \\ \overset{w}{N}{}^{\alpha}{}_{\gamma} &= N^{\alpha}{}_{\gamma} - M^{\alpha}_{\gamma}\overset{w}{H}_0, \\ \overset{w}{C}_\beta{}^{\alpha}{}_{\gamma} &= C_\beta{}^{\alpha}{}_{\gamma}. \end{aligned} \tag{2.7}$$

Therefore we have

COROLLARY 2.1. *The induced Wagner connection* $IW\Gamma = (\overset{w}{F}_\beta{}^\alpha{}_\gamma,\ \overset{w}{N}{}^\alpha{}_\gamma,\ \overset{w}{C}_\beta{}^\alpha{}_\gamma)$ *on a hypersurface* F^{n-1} *of the Finsler space* $(F^n, W\Gamma)$ *is given by* (2.7).

REMARK. (2.7) shows the difference between $IW\Gamma$ and $W\underline{\Gamma}$.

We shall find the quantites and relations with respect to $IW\Gamma$ induced from $W\Gamma = (F_j{}^i{}_k, N^i{}_k, g_j{}^i{}_k)$. Indicating the quantities with respect to the induced Cartan connection $IC\Gamma$ by putting "c" on them, from (1.7) and (1.6) we have

$$\overset{w}{N}{}^\alpha{}_\gamma = \overset{c}{N}{}^\alpha{}_\gamma - \underline{L}^2(g_\gamma{}^\alpha{}_\delta s^\delta + M^\alpha_\gamma B_i s^i) + s^\alpha v_\gamma - \delta^\alpha_\gamma s_0, \tag{2.8}$$

$$\begin{aligned} \overset{w}{H}_\gamma &= \overset{c}{H}_\gamma - \underline{L}^2(M_{\gamma\delta} s^\delta + M_\gamma B_i s^i) + v_\gamma B_i s^i, \\ \overset{w}{H}_0 &= \overset{c}{H}_0 + \underline{L}^2 B_i s^i, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \overset{w}{F}_\beta{}^\alpha{}_\gamma = \overset{c}{F}_\beta{}^\alpha{}_\gamma &+ \underline{L}^2\{(S_\beta{}^\alpha{}_{\gamma\delta} + g_\beta{}^\alpha{}_\epsilon g^\epsilon{}_{\gamma\delta})s^\delta \\ &+ (g_\beta{}^\alpha{}_\delta M^\delta_\gamma + g^\alpha{}_{\gamma\delta} M^\delta_\beta - {}_{\beta\gamma}{}^\delta M^\alpha_\delta) B_i s^i \\ &- M_{\beta\gamma}(M^\alpha_\delta s^\delta + M^\alpha B_i s^i) + M^\alpha_\gamma(M_{\beta\delta} s^\delta \\ &+ M_\beta B_i s^i)\} + (v^\alpha g_{\beta\gamma\delta} - v_\beta g^\alpha{}_{\gamma\delta} - v_\gamma g_\beta{}^\alpha{}_\delta)s^\delta \\ &+ g_\beta{}^\alpha{}_\gamma s_0 + g_{\beta\gamma} s^\alpha - \delta^\alpha_\gamma s_\beta + (M_{\beta\gamma} v^\alpha - M^\alpha_\gamma v_\beta) B_i s^i, \end{aligned} \tag{2.10}$$

where $S_\beta{}^\alpha{}_{\gamma\delta}$ are the components of the v-curvature tensor of the intrinsic Cartan connection $C\underline{\Gamma} = (\Gamma^{*\alpha}_\beta{}_\gamma, \Gamma^{*\alpha}_0{}_\gamma, g_\beta{}^\alpha{}_\gamma)$.

The second fundamental h-tensor $\overset{w}{H}_{\beta\gamma}$ is given by

$$\overset{w}{H}_{\beta\gamma} = B_i(B_\beta{}^i{}_\gamma + F_j{}^i{}_k B^{jk}_{\beta\gamma}) + M_\beta \overset{w}{H}_\gamma, \tag{2.11}$$

from which (2.9) give

$$\begin{aligned} \overset{w}{H}_{\beta\gamma} = \overset{c}{H}_{\beta\gamma} &+ \underline{L}^2\{(g_\beta{}^\delta{}_\epsilon M_{\delta\gamma} + g_\gamma{}^\delta{}_\epsilon M_{\delta\beta} - g_\beta{}^\delta{}_\gamma M_{\delta\epsilon} \\ &+ M_{\beta\epsilon} M_\gamma - M_{\beta\gamma} M_\epsilon)s^\epsilon + (2M_{\beta\delta} M^\delta_\gamma + M_\beta M_\gamma \\ &- M M_{\beta\gamma} - g_\beta{}^\delta{}_\gamma M_\delta) B_i s^i\} - (v_\beta M_{\gamma\delta} + v_\gamma M_{\beta\delta})s^\delta \\ &+ (g_{\beta\gamma} - v_\beta M_\gamma) B_i s^i + M_{\beta\gamma} s_0. \end{aligned} \tag{2.12}$$

Hence $\overset{w}{H}_{\beta\gamma}$ is generally not symmetric :

$$\begin{aligned}\overset{w}{H}_{\beta\gamma} - \overset{w}{H}_{\gamma\beta} &= M_\beta \overset{w}{H}_\gamma - M_\gamma \overset{w}{H}_\beta \\ &= \underline{L}^2(M_{\beta\delta}M_\gamma - M_{\gamma\delta}M_\beta)s^\delta \\ &\quad - (v_\beta M_\gamma - v_\gamma M_\beta)B_i s^i + M_\beta \overset{c}{H}_\gamma - M_\gamma \overset{c}{H}_\beta.\end{aligned} \tag{2.13}$$

Further $(W3)$ yields

$$\overset{w}{H}_{0\gamma} = \overset{w}{H}_\gamma, \tag{2.14}$$

and from the fact that $F_j{}^i{}_0 = T_j{}^i{}_0 + F_0{}^i{}_j$ we get

$$\overset{w}{H}_{\beta 0} = \overset{w}{H}_\beta + M_\beta \overset{w}{H}_0. \tag{2.15}$$

Making use of the Bianchi identity (11.4′) of [5] and the Christoffel process, we can prove that the $(v) - hv$ torsion $P^i{}_{jk}$ of $W\Gamma$ is equal to $\mathrm{g}^i{}_{jk|0}$. If we put

$$\begin{aligned}Q_{\alpha\beta\gamma} &= \mathrm{g}_{ijk|0}B^{ijk}_{\alpha\beta\gamma}, \\ Q_{\alpha\beta} &= \mathrm{g}_{ijk|0}B^{ij}_{\alpha\beta}B^k, \\ Q_\alpha &= \mathrm{g}_{ijk|0}B^i_\alpha B^j B^k,\end{aligned} \tag{2.16}$$

then (1.10) and (1.13) respectively give

$$\overset{w}{P}{}^\alpha{}_{\gamma\beta} = \overset{w}{H}_\gamma M^\alpha_\beta + Q^\alpha{}_{\gamma\beta}, \tag{2.17}$$

$$\dot{\partial}_\beta \overset{w}{H}_\gamma - \overset{w}{H}_{\beta\gamma} = Q_{\beta\gamma}. \tag{2.18}$$

From (2.18) and (2.15) we get

$$\dot{\partial}_\beta \overset{w}{H}_0 = 2\overset{w}{H}_\beta + M_\beta \overset{w}{H}_0. \tag{2.19}$$

Consequently we have

PROPOSITION 2.3. *The normal curvature* $\overset{w}{H}_0 = \overset{w}{H}_\beta v^\beta$ *vanishes if and only if the normal curvature vector* $\overset{w}{H}_\beta$

And we also have

THEOREM 2.2. *The induced Wagner connection* $IW\Gamma$ *of* F^{n-1} *coincides with the intrinsic Wagner connection* $W\underline{\Gamma}$ *of* F^{n-1} *if and only if (1) a Brown tensor* $M_{\alpha\beta} = 0$ *or (2) a normal curvature vector* $\overset{w}{H}_\beta = 0$.

PROOF. It is obvious from Theorem 2.1 that $IW\Gamma$ coincides with $W\underline{\Gamma}$ if and only if $\overset{w}{T}_\beta{}^\alpha{}_\gamma = T_\beta{}^\alpha{}_\gamma : M^\alpha_\beta \overset{w}{H}_\gamma = M^\alpha_\gamma \overset{w}{H}_\beta$.
If $\overset{w}{H}_\beta \neq 0$, we have quantities $h^\alpha (= \overset{w}{H}_\beta M^\alpha_\beta / |\overset{w}{H}_\beta|^2)$ satisfying $M^\alpha_\gamma = h^\alpha \overset{w}{H}_\gamma$. Since $M_{\alpha\gamma}$ is symmetric, $h_\alpha \overset{w}{H}_\gamma = h_\gamma \overset{w}{H}_\alpha$. Hence we get a quantity $h (= \overset{w}{H}{}^\gamma h_\gamma / |\overset{w}{H}_\gamma|^2)$ satisfying $h_\alpha = h\overset{w}{H}_\alpha$, and so $M_{\alpha\gamma} = h\overset{w}{H}_\alpha \overset{w}{H}_\gamma$. Then $M_{\alpha 0} = 0$ leads to $h\overset{w}{H}_0 = 0$. Since $\overset{w}{H}_\beta \neq 0$ implies $\overset{w}{H}_0 \neq 0$ from Proposition 2.3, we get $h = 0$, and so $M_{\alpha\gamma} = 0$.

REMARK. (2.8), (2.9), (2.10) and (2.12) show the differences of quantities of the induced Wagner connection $IW\Gamma$ and the typical induced Cartan connection $IC\Gamma$.

3. The induced Miron connection

As an example of a generalized Cartan connection with surviving deflection we have a Miron connection [2] $M\Gamma(s) = (F_j{}^i{}_k,\ N^i{}_k,\ C_j{}^i{}_k)$ of F^n (M1 : h-metrical, M2 : semi-symmetric, M3 : the non linear connection is the Cartan one $\Gamma^*{}_0{}^i{}_k$, M4 : v-metrical, M5 : (v)v-torsion $S^i{}_{jk} = 0$).

We shall denote by $IM\Gamma$ the connection of a hypersurface F^{n-1} induced from the Miron connection $M\Gamma(s)$ and indicate the quantities with respect to $IM\Gamma$ by putting "m" on them. Then (1.1) and (1.8) show $\overset{m}{C}_\beta{}^\alpha{}_\gamma = g_\beta{}^\alpha{}_\gamma$. Thus $IM\Gamma = (\overset{m}{F}_\beta{}^\alpha{}_\gamma,\ \overset{m}{N}{}^\alpha{}_\gamma,\ g_\beta{}^\alpha{}_\gamma)$.

We shall find the axioms which determine $IM\Gamma$. ($M1$) and ($M4$) show that $IM\Gamma$ is metrical. (1.7), ($M3$) and ((5.2) of [3]) give

$$\overset{m}{N}{}^\alpha{}_\gamma = \overset{c}{N}{}^\alpha{}_\gamma = \Gamma^*{}_0{}^\alpha{}_\gamma - M^\alpha_\gamma \overset{c}{H}_0. \tag{3.1}$$

From (1.9), $(M2)$ and (1.4) we get

$$\overset{m}{T}_{\beta}{}^{\alpha}{}_{\gamma} = \delta^{\alpha}_{\beta} s_{\gamma} - \delta^{\alpha}_{\gamma} s_{\beta} + M^{\alpha}_{\beta} \overset{c}{H}_{\gamma} - M^{\alpha}_{\gamma} \overset{c}{H}_{\beta}. \tag{3.2}$$

From (1.11) and $(M5)$ we obtain $\overset{m}{S}{}^{\alpha}{}_{\beta\gamma} = 0$. Consequently we have

THEOREM 3.1. *The connection $IM\Gamma$, of a hypersurface of F^n induced from the Miron connection $M\Gamma(s)$ of F^n, is a generalized Cartan connection with deflection which is uniquely determined from the induced metric $\underline{L}(u,v)$ by the following five axioms :*

(IM1) $g_{\alpha\beta|\gamma} = 0$,

(IM2) The (h)h-torsion tensor $\overset{m}{T}_{\beta}{}^{\alpha}{}_{\gamma}$ is given by (3.2),

(IM3) The non-linear connection of $IM\Gamma$ is the one of $IC\Gamma$: $\overset{m}{N}{}^{\alpha}{}_{\gamma}$ is given by (3.1),

(IM4) $g_{\alpha\beta}|_{\gamma} = 0$,

(IM5) The (v)v-torsion tensor $\overset{m}{S}{}^{\alpha}{}_{\beta\gamma} = 0$.

We shall apply the procedure to find a generalized Cartan connection to this $IM\Gamma$. Since $A_{\alpha\beta\gamma} = g_{\alpha\gamma} s_{\beta} - g_{\beta\gamma} s_{\alpha} + M_{\alpha\gamma} \overset{c}{H}_{\beta} - M_{\beta\gamma} \overset{c}{H}_{\alpha}$, we have

$$\begin{aligned} \overset{m}{F}_{\alpha\beta\gamma} = \gamma_{\alpha\beta\gamma} - g_{\alpha\beta\delta} \overset{m}{N}{}^{\delta}{}_{\gamma} - g_{\beta\gamma\delta} \overset{m}{N}{}^{\delta}{}_{\alpha} + g_{\gamma\alpha\delta} \overset{m}{N}{}^{\delta}{}_{\beta} \\ + g_{\alpha\gamma} s_{\beta} - g_{\beta\gamma} s_{\alpha} + M_{\alpha\gamma} \overset{c}{H}_{\beta} - M_{\beta\gamma} \overset{c}{H}_{\alpha}. \end{aligned} \tag{3.3}$$

From (3.1) we get

$$\begin{aligned} \overset{m}{F}_{\beta}{}^{\alpha}{}_{\gamma} = \Gamma^{*}{}_{\beta}{}^{\alpha}{}_{\gamma} + g_{\beta\gamma} s^{\alpha} - \delta^{\alpha}_{\gamma} s_{\beta} \\ + (g_{\beta}{}^{\alpha}{}_{\delta} M^{\delta}_{\gamma} + g^{\alpha}{}_{\gamma\delta} M^{\delta}_{\beta} - g_{\beta\gamma}{}^{\delta} M^{\alpha}_{\delta}) \overset{c}{H}_{0} \\ + M_{\beta\gamma} \overset{c}{H}{}^{\alpha} - M^{\alpha}_{\gamma} \overset{c}{H}_{\beta}. \end{aligned} \tag{3.4}$$

Consequently, denoting by $M\underline{\Gamma} = (F_{\beta}{}^{\alpha}{}_{\gamma},\ N^{\alpha}{}_{\gamma},\ C_{\beta}{}^{\alpha}{}_{\gamma})$ the intrinsic Miron connection of F^{n-1} determined from the induced metric $\underline{L}(u,v)$, we have

$$(3.5)\quad \begin{aligned} \overset{m}{F}_{\beta}{}^{\alpha}{}_{\gamma} &= F_{\beta}{}^{\alpha}{}_{\gamma} + (g_{\beta}{}^{\alpha}{}_{\delta} M^{\delta}_{\gamma} + g^{\alpha}{}_{\gamma\delta} M^{\delta}_{\beta} - g_{\beta\gamma}{}^{\delta} M^{\alpha}_{\delta}) \overset{c}{H}_0 \\ &\quad + M_{\beta\gamma} \overset{c}{H}{}^{\alpha} - M^{\alpha}_{\gamma} \overset{c}{H}_{\beta}, \\ \overset{m}{N}{}^{\alpha}{}_{\gamma} &= N^{\alpha}{}_{\gamma} - M^{\alpha}_{\gamma} \overset{c}{H}_0, \\ \overset{m}{C}_{\beta}{}^{\alpha}{}_{\gamma} &= C_{\beta}{}^{\alpha}{}_{\gamma}. \end{aligned}$$

Therefore we have

COROLLARY 3.1. *The induced Miron connection* $IM\Gamma = (\overset{m}{F}_{\beta}{}^{\alpha}{}_{\gamma}, \overset{m}{N}{}^{\alpha}{}_{\gamma}, \overset{m}{C}_{\beta}{}^{\alpha}{}_{\gamma}))$ *on a hypersurface* F^{n-1} *of the Finsler space* $(F^n, M\Gamma)$ *is given by* (3.5).

REMARK. (3.5) shows the difference between $IM\Gamma$ and $M\underline{\Gamma}$. The deflection tensor $\overset{m}{D}{}^{\alpha}{}_{\gamma}$ of $IM\Gamma$ is given by $\overset{m}{D}{}^{\alpha}{}_{\gamma} = D^{\alpha}{}_{\gamma} = s^{\alpha} v_{\gamma} - \delta^{\alpha}_{\gamma} s_0$.

THEOREM 3.2. *The indeced Miron connection* $IM\Gamma$ *of* F^{n-1} *coincides with the intrinsic Miron connection* $M\underline{\Gamma}$ *of* F^{n-1} *if and only if* $M^{\alpha}_{\beta} \overset{c}{H}_{\gamma}$ *is symmetric with respect to* β, γ.

PROOF. It is obvious from Theorem 3.1 that $IM\Gamma$ coincides with $M\underline{\Gamma}$ if and only if $M^{\alpha}_{\beta} \overset{c}{H}_{\gamma} = M^{\alpha}_{\gamma} \overset{c}{H}_{\beta}$ and $M^{\alpha}_{\gamma} \overset{c}{H}_0 = 0$. Since $M^{\alpha}_{\beta} \overset{c}{H}_{\gamma} = M^{\alpha}_{\gamma} \overset{c}{H}_{\beta}$ implies $M^{\alpha}_{\gamma} \overset{c}{H}_0 = 0$, the converse is proved.

We shall find the quantities and relations with respect to $IM\Gamma$ induced from $M\Gamma = (F_j{}^i{}_k,\ \Gamma^*{}_0{}^i{}_k,\ g_j{}^i{}_k)$. The normal curvature vector $\overset{m}{H}_{\beta}$ is given by

$$(3.6)\qquad \overset{m}{H}_{\beta} = \overset{c}{H}_{\beta} = B_i(B^i_{0\beta} + \Gamma^*{}_0{}^i{}_j B^j_{\beta}),\ \overset{m}{H}_0 = \overset{c}{H}_0.$$

The second fundamental h-tensor $\overset{m}{H}_{\beta\gamma}$ is given by

$$(3.7)\qquad \overset{m}{H}_{\beta\gamma} = \overset{c}{H}_{\beta\gamma} + g_{\beta\gamma} B_i s^i,$$

from which we get

$$\overset{m}{H}_{\beta\gamma} - \overset{m}{H}_{\gamma\beta} = M_\beta \overset{c}{H}_\gamma - M_\gamma \overset{c}{H}_\beta, \tag{3.8}$$

that is, $\overset{m}{H}_{\beta\gamma}$ is generally not symmetric.

Furthermore (3.7) yields

$$\begin{aligned} \overset{m}{H}_{0\gamma} &= \overset{c}{H}_\gamma + v_\gamma B_i s^i, \\ \overset{m}{H}_{\beta 0} &= \overset{c}{H}_\beta + M_\beta \overset{c}{H}_0 + v_\beta B_i s^i. \end{aligned} \tag{3.9}$$

(1.10) and (1.13) respectively give

$$\overset{m}{P}{}^\alpha{}_{\gamma\beta} = \overset{c}{H}_\gamma M^\alpha_\beta + P^i{}_{jk} B^\alpha_i B^{jk}_{\gamma\beta}, \tag{3.10}$$

$$\dot\partial_\beta \overset{m}{H}_\gamma - \overset{m}{H}_{\beta\gamma} = P^i{}_{jk} B_i B^{jk}_{\gamma\beta}, \tag{3.11}$$

where $P^i{}_{jk}$ is the $(v)hv$-torsion tensor of $M\Gamma$. From (3.11) and (3.9) we get

$$\dot\partial_\beta \overset{m}{H}_0 = 2\overset{m}{H}_\beta + M_\beta \overset{m}{H}_0 + v_\beta B_i s^i + P^i_{0k} B_i B^k_\beta. \tag{3.12}$$

4. The induced Cartan C-connection

In 1986 M. Matsumoto improved W. Barthel's connection $C\Gamma(B)$ in the theory of minimal hypersurfaces as a Cartan C-connection [4] $C\Gamma(T_c)$ of F^n (T_c1 : h-metrical, T_c2 : $T_j{}^i{}_k = -C(\delta^i_j l_k - \delta^i_k l_j)$, where $C = LC^i{}_{|i}/(n-1)$, $C^i = C_j{}^i{}_k g^{jk}$, T_c3: deflection $D^i{}_k = 0$, T_c4 : $v-metrical$, T_c5 :(v)v-torsion $S^i{}_{jk} = 0$.)

Denoting by $IC\Gamma(T_c)$ the connection of a hypersurface F^{n-1} induced from the Cartan C-connection $C\Gamma(T_c)$ and indicating the quantities with respect to $IC\Gamma(T_c)$ by putting "t_c" on them, $IC\Gamma(T_c) = (\overset{t_c}{F}_\beta{}^\alpha{}_\gamma, \overset{t_c}{N}{}^\alpha{}_\gamma, \overset{t_c}{C}_\beta{}^\alpha{}_\gamma)$.

We shall find the axioms which determine $IC\Gamma(T_c)$. From(1.1) and (1.8) we have $\overset{t_c}{C}_\beta{}^\alpha{}_\gamma = g_\beta{}^\alpha{}_\gamma$. (T_c1) and (T_c4) show that $IC\Gamma(T_c)$ is metrical. Next (1.12) and (T_c3) give $\overset{t_c}{D}{}^\alpha{}_\gamma = 0$. Further (1.9), (T_c2) and (1.4) lead to

$$\overset{t_c}{T}_\beta{}^\alpha{}_\gamma = -(\underline{C} - \frac{L}{n-1}C^*)(\delta^\alpha_\beta l_\gamma - \delta^\alpha_\gamma l_\beta) + M^\alpha_\beta \overset{t_c}{H}_\gamma - M^\alpha_\gamma \overset{t_c}{H}_\beta, \tag{4.1}$$

where

$$\begin{aligned} &\underline{C} = \underline{L}g^\alpha{}_{|\alpha}/(n-1), \\ &C^* = B_i g^i \overset{t_c}{H}{}^\alpha{}_\alpha - B_i g^i{}_{|j} B^j + \overset{t_c}{H}_\alpha B^\alpha_i g^i|_j B^j - M^\alpha{}_{|\alpha}. \end{aligned} \tag{4.2}$$

From (1.11) and (T_c5) we get $\overset{t_c}{S}{}^\alpha{}_{\beta\gamma} = 0$. Consequently we have

THEOREM 4.1. *The connection $IC\Gamma(T_c)$ of a hypersurface of F^n, induced from the Cartan C-connection $C\Gamma(T_c)$ of F^n, is a generalized Cartan connection which is uniquely determined from the induced metric $\underline{L}(u,v)$ by the following five axioms :*

(IT_c1) $g_{\alpha\beta|\gamma} = 0$,

(IT_c2) *The (h)h-torsion tensor* $\overset{t_c}{T}_\beta{}^\alpha{}_\gamma$ *is given by* (4.1) *and* (4.2),

(IT_c3) *The deflection tensor* $\overset{t_c}{D}{}^\alpha{}_\gamma = 0$,

(IT_c4) $g_{\alpha\beta}|_\gamma = 0$,

(IT_c5) $\overset{t_c}{S}{}^\alpha{}_{\beta\gamma} = 0$.

We shall apply the procedure to find a generalized Cartan connection to this $IC\Gamma(T_c)$. Since $A_{\alpha\beta\gamma} = (\underline{C} - \frac{L}{n-1}C^*)(g_{\beta\gamma}l_\alpha - g_{\alpha\gamma}l_\beta) + M_{\alpha\gamma}\overset{t_c}{H}_\beta - M_{\beta\gamma}\overset{t_c}{H}_\alpha$, from $(IT_c1), (IT_c4)$ and (IT_c5) we get

$$\begin{aligned} \overset{t_c}{F}_{\alpha\beta\gamma} &= \gamma_{\alpha\beta\gamma} - g_{\alpha\beta\delta}\overset{t_c}{N}{}^\delta{}_\gamma - g_{\beta\gamma\delta}\overset{t_c}{N}{}^\delta{}_\alpha + g_{\gamma\alpha\delta}\overset{t_c}{N}{}^\delta{}_\beta \\ &\quad + (\underline{C} - \frac{L}{n-1}C^*)(g_{\beta\gamma}l_\alpha - g_{\alpha\gamma}l_\beta) + M_{\alpha\gamma}\overset{t_c}{H}_\beta - M_{\beta\gamma}\overset{t_c}{H}_\alpha. \end{aligned} \tag{4.3}$$

Hence (IT_c3) gives

$$\overset{t_c}{N}{}^{\alpha}{}_{\gamma} = \Gamma^*{}_0{}^{\alpha}{}_{\gamma} + (\underline{C} - \frac{\underline{L}}{n-1}C^*)\underline{L}h^{\alpha}_{\gamma} - M^{\alpha}_{\gamma}\overset{t_c}{H}_0. \tag{4.4}$$

Then (4.3) leads to

$$\begin{aligned}\overset{t_c}{F}_{\beta}{}^{\alpha}{}_{\gamma} = {} & \Gamma^*{}_{\beta}{}^{\alpha}\gamma + (\underline{C} - \frac{\underline{L}}{n-1}C^*)(l_{\beta}\delta^{\alpha}_{\gamma} - l^{\alpha}g_{\beta\gamma} - \underline{L}g_{\beta}{}^{\alpha}{}_{\gamma} \\ & + (g_{\beta}{}^{\alpha}{}_{\delta}M^{\delta}_{\gamma} + g^{\alpha}{}_{\gamma\delta}M^{\delta}_{\beta} - g^{\delta}_{\beta\gamma}M^{\alpha}_{\delta})\overset{t_c}{H}_0 \\ & + M_{\beta\gamma}\overset{t_c}{H}{}^{\alpha} - M^{\alpha}_{\gamma}\overset{t_c}{H}_{\beta}.\end{aligned} \tag{4.5}$$

Consequently, denoting by $C\underline{\Gamma}(T_c) = (F_{\beta}{}^{\alpha}{}_{\gamma},\ N^{\alpha}{}_{\gamma},\ C_{\beta}{}^{\alpha}{}_{\gamma})$ the intrinic Cartan C-connection of F^{n-1} determined from the induced metric $\underline{L}(u,v)$, we have

$$\begin{aligned}\overset{t_c}{F}_{\beta}{}^{\alpha}{}_{\gamma} = {} & F_{\beta}{}^{\alpha}{}_{\gamma} - \frac{\underline{L}}{n-1}C^*(l_{\beta}\delta^{\alpha}_{\gamma} - l^{\alpha}g_{\beta\gamma} - \underline{L}g_{\beta}{}^{\alpha}{}_{\gamma}) \\ & + (g_{\beta}{}^{\alpha}{}_{\delta}M^{\delta}_{\gamma} + g^{\alpha}{}_{\gamma\delta}M^{\delta}_{\beta} - g_{\beta\gamma}{}^{\delta}M^{\alpha}_{\delta})\overset{t_c}{H}_0 \\ & + M_{\beta\gamma}\overset{t_c}{H}{}^{\alpha} - M^{\alpha}_{\gamma}\overset{t_c}{H}_{\beta}, \\ \overset{t_c}{N}{}^{\alpha}{}_{\gamma} = {} & N^{\alpha}{}_{\gamma} - \frac{\underline{L}^2}{n-1}C^*h^{\alpha}_{\gamma} - M^{\alpha}_{\gamma}\overset{t_c}{H}_0, \\ \overset{t_c}{C}_{\beta}{}^{\alpha}{}_{\gamma} = {} & C_{\beta}{}^{\alpha}{}_{\gamma}.\end{aligned} \tag{4.6}$$

Therefore we have

COROLLARY 4.1. *The induced Cartan C-connection $IC\Gamma(T_c)$ $(\overset{t_c}{F}_{\beta}{}^{\alpha}{}_{\gamma},\ \overset{t_c}{N}{}^{\alpha}{}_{\gamma},\ \overset{t_c}{C}_{\beta}{}^{\alpha}{}_{\gamma})$ on a hypersurface F^{n-1} of the Finsler space $(F^n, C\Gamma(T_c))$ is given by (4.6).*

REMARK. (4.6) shows the difference between $IC\Gamma(T_c)$ and $C\underline{\Gamma}(T_c)$.

THEOREM 4.2. *The induced Cartan C-connection $IC\Gamma(T_c)$ of F^{n-1} coincides with the intrinsic Cartan C-connection $C\underline{\Gamma}(T_c)$ of F^{n-1} if and only if $M^\alpha_\beta \overset{t_c}{H}_\gamma$ is symmetric with respect to β, γ and C^* vanishes.*

We shall find the quantities and relations with respect to $IC\Gamma(T_c)$. From ((5.4) of [3]) and (1.3) we have

$$(4.7)\qquad \overset{t_c}{H}_\beta = \overset{c}{H}_\beta,\ \overset{t_c}{H}_0 = \overset{c}{H}_0.$$

Then (4.4), (4.5) and ((5.2) of [3]) yield

$$(4.8)\qquad \begin{aligned} \overset{t_c}{N}{}^\alpha{}_\gamma &= \overset{c}{N}{}^\alpha{}_\gamma + C\underline{L}h^\alpha_\gamma, \\ \overset{t_c}{F}_\beta{}^\alpha{}_\gamma &= \overset{c}{F}_\beta{}^\alpha{}_\gamma + C(l_\beta\delta^\alpha{}_\gamma - l^\alpha g_{\beta\gamma} - \underline{L}g_\beta{}^\alpha{}_\gamma). \end{aligned}$$

From ((5.5) of [3]) we get

$$(4.9)\qquad \overset{t_c}{H}_{\beta\gamma} = \overset{c}{H}_{\beta\gamma} - C\underline{L}M_{\beta\gamma}.$$

Hence $\overset{t_c}{H}_{\beta\gamma}$ is generally not symmetric. From (4.9) and ((5.7) of [3]) we get

$$(4.10)\qquad \overset{t_c}{H}_{0\gamma} = \overset{c}{H}_\gamma,\ \overset{t_c}{H}_{\beta 0} = \overset{c}{H}_\beta + M_\beta \overset{c}{H}_0.$$

(1.10) and (1.13) respectively give

$$(4.11)\qquad \overset{t_c}{P}{}^\alpha{}_{\gamma\beta} = \overset{c}{H}_\gamma M^\alpha_\beta + P^i{}_{kj}B^\alpha_i B^{kj}_{\gamma\beta},$$

$$(4.12)\qquad \dot{\partial}_\beta \overset{t_c}{H}_\gamma - \overset{t_c}{H}_{\beta\gamma} = P^i{}_{kj}B_i B^{kj}_{\gamma\beta},$$

where $P^i{}_{jk}$ is the $(v)hv$-torsion tensor of $C\Gamma(T_c)$.

5. The induced Cartan Y-connection

In the theory of Y-extremal hyperfaces, M. Matsumoto defined a Cartan Y-connection [4,6] $CY\Gamma = (F_j{}^i{}_k,\ N^i{}_k,\ C_j{}^i{}_k)$ of F^n (Y1 : h-metrical, Y2 : $T_j{}^i{}_k = -L^*(C_j{}^i{}_r Y^r_k - C_k{}^i{}_r Y^r_j)$, where $L^*(x,y) = L(x,y)/L(x,Y)$ and $Y^i_j = \partial_j Y^i(x) + N^i{}_j(x,Y)$, Y3 : deflection $D^i{}_k = 0$, Y4 : v-metrical, Y5 : (v)v-torsion $S^i{}_{jk} = 0$).

Denoting by $ICY\Gamma$ the connection of a hypersurface F^{n-1} induced from the Cartan Y-connection $CY\Gamma$ of F^n and indicating the quantities with respect to $ICY\Gamma$ by putting "y" on them, $ICY\Gamma = (\overset{y}{F}_\beta{}^\alpha{}_\gamma, \overset{y}{N}{}^\alpha{}_\gamma\ \overset{y}{C}_\beta{}^\alpha{}_\gamma)$. We shall find the axioms which determine $ICY\Gamma$. From (1.1) and (1.8) we have $\overset{y}{C}_\beta{}^\alpha{}_\gamma = g_\beta{}^\alpha{}_\gamma$. It is obvious from $(Y1)$ and $(Y4)$ that $ICY\Gamma$ is metrical. Next (1.12) and $(Y3)$ give $\overset{y}{D}{}^\alpha{}_\gamma = 0$. Putting $V^\alpha = B^\alpha_i Y^i$, we have

$$
(5.1)\qquad
\begin{aligned}
Y^m_k B^\delta_m B^k_\gamma &= \partial_\gamma V^\delta + N^\delta{}_\gamma(u,V),\\
Y^m_k B_m B^k_\gamma &= \overset{y}{H}_\gamma,
\end{aligned}
$$

from which(1.9), $(Y2)$ and (1.4) yield

$$
(5.2)\qquad \overset{y}{T}_\beta{}^\alpha{}_\gamma = -\underline{L}^*(g_\beta{}^\alpha{}_\delta V^\delta_\gamma - g_\gamma{}^\alpha{}_\delta V^\delta_\beta + (1-\underline{L}^*)(M^\alpha_\beta \overset{y}{H}_\gamma - M^\alpha_\gamma \overset{y}{H}_\beta),
$$

where

$$
(5.3)\qquad
\begin{aligned}
\underline{L}^*(u,v) &= \underline{L}(u,v)/\underline{L}(u,V),\\
V^\delta_\gamma(u) &= \partial_\gamma V^\delta(u) + N^\delta{}_\gamma(u,V).
\end{aligned}
$$

From (1.11) and (Y5) we get $\overset{y}{S}{}^\alpha{}_{\beta\gamma} = 0$. Consequently we have

THEOREM 5.1. *The connection $ICY\Gamma$ of a hypersurface of F^n, induced from the Cartan Y-connection $CY\Gamma$ of F^n, is a generalized Cartan connection which is uniquely determined from the induced metric $\underline{L}(u,v)$ by the following five axioms:*

(IY1) $g_{\alpha\beta|\gamma} = 0$,

(IY2) The $(h)h$-torsion tensor $\overset{y}{T}_\beta{}^\alpha{}_\gamma$ is given by (5.2) and (5.3),

(IY3) The deflection tensor $\overset{y}{D}^{\alpha}{}_{\gamma} = 0$,
(IY4) $g_{\alpha\beta}|_{\gamma} = 0$,
(IY5) The $(v)v$-torsion tensor $\overset{y}{S}^{\alpha}{}_{\beta\gamma} = 0$.

We shall find the procedure to find a genearlized Cartan connection to this $ICY\Gamma$. Since $A_{\alpha\beta\gamma} = \underline{L}^*(g_{\beta\gamma\delta}V^{\delta}_{\alpha} - g_{\gamma\alpha\delta}V^{\delta}_{\beta}) +$ $(1-\underline{L}^*)(M_{\gamma\alpha}\overset{y}{H}_{\beta} - M_{\beta\gamma}\overset{y}{H}_{\alpha})$, from $(IY1)$, $(IY4)$ and $(IY5)$ we get

$$\begin{aligned}\overset{y}{F}_{\alpha\beta\gamma} &= \gamma_{\alpha\beta\gamma} - g_{\alpha\beta\delta}\overset{y}{N}^{\delta}{}_{\gamma} - g_{\beta\gamma\delta}\overset{y}{N}^{\delta}{}_{\alpha} \\ &+ g_{\gamma\alpha\delta}\overset{y}{N}^{\delta}{}_{\beta} + \underline{L}^*(g_{\beta\gamma\delta}V^{\delta}_{\alpha} - g_{\gamma\alpha\delta}V^{\delta}_{\beta}) \\ &+ (1-\underline{L}^*)(M_{\gamma\alpha}\overset{y}{H}_{\beta} - M_{\beta\gamma}\overset{y}{H}_{\alpha}).\end{aligned} \tag{5.4}$$

Hence (IY3) gives

$$\overset{y}{N}^{\alpha}{}_{\gamma} = \Gamma^*{}_0{}^{\alpha}{}_{\gamma} + \underline{L}^* textg^{\alpha}{}_{\gamma\delta}V^{\delta}_0 - (1-\underline{L}^*)M^{\alpha}_{\gamma}\overset{y}{H}_0. \tag{5.5}$$

Then (5.4) leads to

$$\begin{aligned}\overset{y}{F}_{\beta}{}^{\alpha}{}_{\gamma} &= \Gamma^*{}_{\beta}{}^{\alpha}{}_{\gamma} - \underline{L}^*\{(g_{\beta}{}^{\alpha}{}_{\delta}g^{\delta}{}_{\gamma\epsilon} + g^{\alpha}{}_{\gamma\delta}g^{\delta}{}_{\beta\epsilon} - g^{\delta}_{\beta\gamma}g_{\delta}{}^{\alpha}{}_{\epsilon})V^{\epsilon}_{\delta} \\ &+ g^{\delta}_{\beta\gamma}V^{\alpha}_{\delta} - g^{\alpha}_{\gamma\delta}V^{\delta}_{\beta}\} + (1-\underline{L}^*)(g_{\beta}{}^{\alpha}{}_{\delta}M^{\delta}_{\gamma} + g^{\alpha}{}_{\gamma\delta}M^{\delta}_{\beta} \\ &- g_{\beta\gamma}{}^{\delta}M^{\alpha}_{\delta})\overset{y}{H}_0 + M_{\beta\gamma}\overset{y}{H}^{\alpha} - M^{\alpha}_{\gamma}\overset{y}{H}_{\beta}.\end{aligned} \tag{5.6}$$

Consequently we have

COROLLARY 5.1. *The induced Cartan Y-connection* $ICY\Gamma = (\overset{y}{F}_{\beta}{}^{\alpha}{}_{\gamma}, \overset{y}{N}^{\alpha}{}_{\gamma}, \overset{y}{C}_{\beta}{}^{\alpha}{}_{\gamma})$ *on a hypersurface* F^{n-1} *of the Finsler space* $(F^n,$ $CY\Gamma)$ *is given by* (5.6), (5.5)*and* $\overset{y}{C}_{\beta}{}^{\alpha}{}_{\gamma} = g_{\beta}{}^{\alpha}{}_{\gamma}$.

REMARK. (5.6) and (5.5) show the difference between $ICY\Gamma$ and $CY\underline{\Gamma}$ (intrinsic Cartan Y-connection).

THEOREM 5.2. *The induced Cartan Y-connection $ICY\Gamma$ of F^{n-1} coincides with the intrinsic Cartan Y-connection $CY\underline{\Gamma}$ of F^{n-1} if and only if(1) $\underline{L}^*(u,v)=1$ or (2) $M^\alpha_\beta \overset{y}{H}_\gamma$ is symmetric with respect to β,γ.*

We shall find the quantities and relations with respect to $ICY\Gamma$ and compare these with them with respect to $IC\Gamma$. Since $Y_0^m B^\alpha_m = V_0^\alpha$ and $Y_0^m B_m = \overset{y}{H}_0$, the normal curvature vector $\overset{y}{H}_\beta$ is given by

$$\overset{y}{H}_\beta = \overset{c}{H}_\beta + \underline{L}^*(M_{\alpha\beta}V_0^\alpha + M_\beta \overset{y}{H}_0), \quad \overset{y}{H}_0 = \overset{c}{H}_0. \tag{5.7}$$

And we have the second fundamental h-tensor

$$\begin{aligned}\overset{y}{H}_{\beta\gamma} = \overset{c}{H}_{\beta\gamma} - \underline{L}^*\{&(M_{\beta\epsilon}g^\epsilon{}_{\gamma\delta} + M_{\gamma\epsilon}g^\epsilon{}_{\beta\delta} - M_{\epsilon\delta}g_{\beta\gamma}{}^\epsilon \\ &+ M_\gamma M_{\beta\delta} - M_\delta M_{\beta\gamma})V_0^\delta + (M_{\beta\epsilon}M^\epsilon_\gamma + M_{\gamma\epsilon}M^\epsilon_\beta \\ &+ M_\beta M_\gamma - g_{\beta\gamma}{}^\epsilon M_\epsilon - MM_{\beta\gamma})\overset{y}{H}_0 + g_\beta{}^\epsilon{}_\gamma \overset{y}{H}_\epsilon \\ &+ B_i Y^i_s B^s M_{\beta\gamma} - M_{\gamma\delta}V^\delta_\beta - M_\gamma \overset{y}{H}_\beta\},\end{aligned} \tag{5.8}$$

from which we get

$$\begin{aligned}\overset{y}{H}_{\beta\gamma} - \overset{y}{H}_{\gamma\beta} = M_\beta \overset{c}{H}_\gamma - M_\gamma \overset{c}{H}_\beta - \underline{L}^*\mathcal{U}_{(\beta\gamma)}[&M_\gamma M_{\beta\delta}V_0^\delta \\ &- M_{\gamma\delta}V^\delta_\beta - M_\gamma\{\overset{c}{H}_\beta + \underline{L}^*(M_{\beta\delta}V_0^\delta + M_\beta \overset{c}{H}_0\}],\end{aligned} \tag{5.9}$$

where the notation $\mathcal{U}_{(\beta\gamma)}$ denotes the interchange of indices (β,γ) and subtraction,

$$\overset{y}{H}_{0\gamma} = \overset{c}{H}_\gamma - M_{\gamma\delta}V_0^\delta - M_\gamma \overset{c}{H}_0, \quad \overset{y}{H}_{\beta 0} = \overset{c}{H}_\beta + M_\beta \overset{c}{H}_0. \tag{5.10}$$

Next (5.6) and (5.7) respectively give

$$\overset{y}{N}^\alpha{}_\gamma = \overset{c}{N}^\alpha{}_\gamma + \underline{L}^*(g^\alpha{}_{\gamma\delta}V_0^\delta + M^\alpha_\gamma \overset{c}{H}_0), \tag{5.11}$$

$$
(5.12)\quad \begin{aligned}
\overset{y}{F}_{\beta}{}^{\alpha}{}_{\gamma} = \overset{c}{F}_{\beta}{}^{\alpha}{}_{\gamma} - \underline{L}^*[(\mathrm{g}_{\beta}{}^{\alpha}{}_{\delta}\mathrm{g}^{\delta}{}_{\gamma\epsilon} + \mathrm{g}^{\alpha}{}_{\gamma\delta}\mathrm{g}^{\delta}{}_{\beta\epsilon} - \mathrm{g}_{\beta\gamma}{}^{\delta}\mathrm{g}_{\delta}{}^{\alpha}{}_{\epsilon})V_0^{\epsilon} \\
+ (\mathrm{g}_{\beta}{}^{\alpha}{}_{\delta}M^{\delta}_{\gamma} + \mathrm{g}^{\alpha}{}_{\gamma\delta}M^{\delta}_{\beta} - \mathrm{g}^{\delta}_{\beta\gamma}M^{\alpha}_{\delta})\overset{c}{H}_0 \\
+ M_{\beta\gamma}\overset{c}{H}{}^{\alpha} - M^{\alpha}_{\gamma}\overset{c}{H}_{\beta} + \underline{L}^*\{(M_{\beta\gamma}M^{\alpha}_{\delta} - M^{\alpha}_{\gamma}M_{\beta\delta})V_0^{\delta} \\
+ (M_{\beta\gamma}M^{\alpha} - M^{\alpha}_{\gamma}M_{\beta})\overset{c}{H}_0\}].
\end{aligned}
$$

From (1.10) and (1.13) we have

$$
(5.13)\quad \overset{y}{P}{}^{\alpha}{}_{\gamma\beta} = \overset{y}{H}_{\gamma}M^{\alpha}_{\beta} + P^{i}{}_{kj}B^{\alpha}_{i}B^{kj}_{\gamma\beta}.
$$

$$
(5.14)\quad \dot{\partial}_{\beta}\overset{y}{H}_{\gamma} - \overset{y}{H}_{\beta\gamma} = P^{i}{}_{kj}B_{i}B^{kj}_{\gamma\beta}.
$$

6. Some examples

It is well-known ([8], [9]) that the (h)hv-torsion tensor C_{ijk} of a Randers space and a Kropina space is respectively given by

$$
(6.1)\quad \overset{R}{C}_{ijk} = (h_{ij}L_k + h_{jk}L_i + h_{ki}L_j)/2L,
$$

where

$$
(6.2)\quad L_i = b_i - \mu l_i,\ \mu = \beta/\alpha,
$$

and

$$
(6.3)\quad \overset{K}{C}_{ijk} = (h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)/2L,
$$

where

$$
(6.4)\quad m_i = l_i - \tau b_i,\ \tau = \alpha^2/\beta^2.
$$

From (6.1) and (6.3) respectively we have

$$(6.5)\qquad \overset{R}{C}_{\alpha\beta\gamma} = (h_{\alpha\beta}L_\gamma + h_{\beta\gamma}L_\alpha + h_{\gamma\alpha}L_\beta)/2\underline{L}, \quad \overset{R}{M}_{\alpha\beta} = b_i B^i h_{\alpha\beta}/2\underline{L}$$

and

$$(6.6)\qquad \overset{K}{C}_{\alpha\beta\gamma} = (h_{\alpha\beta}m_\gamma + h_{\beta\gamma}m_\alpha + h_{\gamma\alpha}m_\beta)/2\underline{L}, \quad \overset{K}{M}_{\alpha\beta} = -\tau b_i B^i h_{\alpha\beta}/2\underline{L}.$$

Let $F^n = (M^n,\ L(x,y))$ be a C-reducible Randers space. Then the $(h)h$-torsion tensors $\overset{w}{T}_\beta{}^\alpha{}_\gamma$, $\overset{m}{T}_\beta{}^\alpha{}_\gamma$, $\overset{t_c}{T}_\beta{}^\alpha{}_\gamma$ and$\overset{y}{T}_\beta{}^\alpha{}_\gamma$ of a hypersurface F^{n-1} in F^n from(6.5) are respectively given by

$$(6.7)\qquad \overset{w}{T}_\beta{}^\alpha{}_\gamma = \delta^\alpha_\beta s_\gamma - \delta^\alpha_\gamma s_\beta + h^\alpha_\beta \overset{w}{R}_\gamma - h^\alpha_\gamma \overset{w}{R}_\beta, \quad \overset{w}{R}_\beta = b_i B^i \overset{w}{H}_\beta/2\underline{L}(u,v),$$

$$(6.8)\qquad \overset{m}{T}_\beta{}^\alpha{}_\gamma = \delta^\alpha_\beta s_\gamma - \delta^\alpha_\gamma s_\beta + h^\alpha_\beta \overset{c}{R}_\gamma - h^\alpha_\gamma \overset{c}{R}_\beta, \quad \overset{c}{R}_\beta = b_i B^i \overset{c}{H}_\beta/2\underline{L}(u,v),$$

$$(6.9)\qquad \overset{t_c}{T}_\beta{}^\alpha{}_\gamma = -(\underline{C} - \frac{\underline{L}}{n-1}C^*)(\delta^\alpha_\beta l_\gamma - \delta^\alpha_\gamma l_\beta) + h^\alpha_\beta \overset{c}{R}_\gamma - h^\alpha_\gamma \overset{c}{R}_\beta,$$

$$(6.10)\qquad \overset{y}{T}_\beta{}^\alpha{}_\gamma = -\underline{L}^*(\mathrm{g}_\beta{}^\alpha{}_\delta V^\delta_\gamma - \mathrm{g}_\gamma{}^\alpha{}_\delta V^\delta_\beta) + (1-\underline{L}^*)(h^\alpha_\beta \overset{y}{R}_\gamma - h^\alpha_\gamma \overset{y}{R}_\beta), \quad \overset{y}{R}_\beta = b_i B^i \overset{y}{H}_\beta/2\underline{L}(u,v),$$

Thus we have

THEOREM 6.1. *Any hypersurface F^{n-1} of a C-reducible Randers space is also C-reducible. The $IW\Gamma, IM\Gamma, IC\Gamma(Tc)$ and $ICY\Gamma$ of F^{n-1} satisfy* (6.7) – (6.10).

Let $F^n = (M^n,\ L(x,y))$ be a C-reducible Kropina space. Then the $(h)h$-torsion tensors $\overset{w}{T}_\beta{}^\alpha{}_\gamma$, $\overset{m}{T}_\beta{}^\alpha{}_\gamma$, $\overset{t_c}{T}_\beta{}^\alpha{}_\gamma$ and $\overset{y}{T}_\beta{}^\alpha{}_\gamma$ of a hypersurface F^{n-1} in F^n, from (6.6), are respectively given by

$$\begin{aligned} \overset{w}{T}_\beta{}^\alpha{}_\gamma &= \delta^\alpha_\beta s_\gamma - \delta^\alpha_\gamma s_\beta + h^\alpha_\beta \overset{w}{K}_\gamma - h^\alpha_\gamma \overset{w}{K}_\beta, \\ \overset{w}{K}_\beta &= -\tau b_i B^i \overset{w}{H}_\beta / 2\underline{L}, \end{aligned} \tag{6.11}$$

$$\begin{aligned} \overset{m}{T}_\beta{}^\alpha{}_\gamma &= \delta^\alpha_\beta s_\gamma - \delta^\alpha_\gamma s_\beta + h^\alpha_\beta \overset{c}{K}_\gamma - h^\alpha_\gamma \overset{c}{K}_\beta, \\ \overset{c}{K}_\beta &= -\tau b_i B^i \overset{c}{H}_\beta / 2\underline{L}, \end{aligned} \tag{6.12}$$

$$\overset{t_c}{T}_\beta{}^\alpha{}_\gamma = -(\underline{C} - \frac{\underline{L}}{n-1} C^*)(\delta^\alpha_\beta l_\gamma - \delta^\alpha_\gamma l_\beta) + h^\alpha_\beta \overset{c}{K}_\gamma - h^\alpha_\gamma \overset{c}{K}_\beta, \tag{6.13}$$

$$\begin{aligned} \overset{y}{T}_\beta{}^\alpha{}_\gamma &= -\underline{L}^*(\mathrm{g}_\beta{}^\alpha{}_\delta V^\delta_\gamma - \mathrm{g}_\gamma{}^\alpha{}_\delta V^\delta_\beta) + (1 - \underline{L}^*)(h^\alpha_\beta \overset{y}{K}_\gamma - h^\alpha_\gamma \overset{y}{K}_\beta), \\ \overset{y}{K}_\beta &= -\tau b_i B^i \overset{y}{H}_\beta / 2\underline{L}. \end{aligned} \tag{6.14}$$

Thus we have

THEOREM 6.2. *Any hypersurface F^{n-1} of a C-reducible Kropina space is also C-reducible. The $IW\Gamma$, $IM\Gamma$, $IC\Gamma(T_c)$ and $ICY\Gamma$ of F^{n-1} satisfy* (6.11) – (6.14).

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