Comm. Korean Math. Soc. **11** (1996), No. 3, pp. 539–546

REDUCTIONS OF IDEALS IN COMMUTATIVE NOETHERIAN SEMI-LOCAL RINGS

YEONG MOO SONG AND SE GYEONG KIM

ABSTRACT. The purpose of this paper is to show that the Noetherian semi-local property of the underlying ring enables us to develope a satisfactory concept of the theory of reduction of ideals in a commutative Noetherian ring.

1. Introduction

The important ideas of reduction and integral closure of an ideal in a commutative Noetherian ring R (with identity) were introduced by Northcott and Rees [4] ; a brief and direct approach to their theory is given in [5, (1.1)] , and it is appropriate for us to begin by briefly summarizing some of the main aspects.

Let $\mathfrak{a}$ be an ideal of R. We say that $\mathfrak{a}$ is a *reduction* of the ideal $\mathfrak{b}$ of R if $\mathfrak{a} \subset \mathfrak{b}$ and there is a positive integer s such that $\mathfrak{a}\mathfrak{b}^s = \mathfrak{b}^{s+1}$. An element x of R is said to be *integrally dependent on* $\mathfrak{a}$ if there exists a positive integer s and elements$c_1, c_2, \cdots, c_s$ of R with $c_i \in \mathfrak{a}^i$ for $i = 1, 2, \cdots, s$ such that

$$x^s + c_1x^{s-1} + \cdots + c_{s-1}x + c_s = 0.$$

It turns out that this is the case if and only if $\mathfrak{a}$ is a reduction of $\mathfrak{a} + Rx$; moreover,

$$\overline{\mathfrak{a}} = \{y \in R \mid y \text{ is integrally dependent on } \mathfrak{a}\}$$

Received September 7, 1995. Revised June 8, 1996.

1991 AMS Subject Classification: 13A15, 13E05, 13H99.

Key words and phrases: reduction, analytic spread, minimal reduction.

This work was supported by NONDIRECTED RESEARCH FUND, Korea Research Foundation, 1994.

is an ideal of R, called the *integral closure of* $\mathfrak{a}$, and is the largest ideal of R which has $\mathfrak{a}$ as a reduction in the sense that $\mathfrak{a}$ is a reduction of $\overline{\mathfrak{a}}$ and any ideal of R which has $\mathfrak{a}$ as a reduction must be contained in $\overline{\mathfrak{a}}$.

In the case when R is, in addition, local with maximal ideal $\mathfrak{m}$, and has infinite residue field k, the theory of reductions of the ideal $\mathfrak{a}$ of R is intimately related to the *analytic spread* $\ell(\mathfrak{a})$ *of* $\mathfrak{a}$: see [4, §§3.4]. There is a rational polynomial ϕ such that, for all large positive integers n, the vector space dimension $dim_k(\mathfrak{a}^n/\mathfrak{a}^n\mathfrak{m})$ is equal to $\phi(n)$. Let d denote the degree of ϕ (with the convention that the degree of the zero polynomial is taken to be -1) ; then $\ell(\mathfrak{a})$ is defined to be $d+1$. It turns out [4, p.151] that $\ell(\mathfrak{a})$ is equal to the smallest number of elements which will generate a reduction of $\mathfrak{a}$.

The elegant theory of Northcott and Rees [4] uses the Noetherian property of the underlying ring. In [7], Sharp and Taherizadeh introduced the concept of reduction and integral closure of ideals relative to Artinian modules, and established many properties of these concepts which reflect results of Northcott and Rees in the classical situation. Moverover, Ansari and Sharp [2] introduced concept of reduction and integral closure of ideals relative to injective modules, and showed that these concepts have poperties which reflect some of those of the classical concepts of reduction and integral closure introduced by Northcott and Rees.

Now, an interesting question arises: whether there are, in the semi-local situation, some companion results similar to those discussed for instance in [4] .

The purpose of this paper is to show that the Noetherian semi-local property of the underlying ring enables us to develope a satisfactory concept of the theory of reduction of ideals in R; and the authors hope that this note presents topics for further research.

We would like to express our thanks to professor MeeKyoung Kim, Sung Kyun Kwan University,for helpful discussions.

2. Reductions of ideals

Throughout this paper, R will denote a commutative Noetherin semi-local ring with all of its distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_n$, $\mathcal{J}$ will denote a Jacobson radical of R and $\mathfrak{a}$ will denote an ideal of R.

LEMMA 2.1. *Let* $\mathfrak{b}$ *be an ideal of* R *such that* $\mathfrak{a} \subset \mathfrak{b}$ *and let* $\mathfrak{a}$ *be a reduction of* $\mathfrak{b}$. *Then* $\mathfrak{a} \subset \mathfrak{m}_i$ *if and only if* $\mathfrak{b} \subset \mathfrak{m}_i$.

PROOF. This is an immediate consequence of the definition of reduction. □

THEOREM 2.2. *Let* $\mathfrak{b}$ *be an ideal of* R *such that* $\mathfrak{a} \subset \mathfrak{b}$. *Then* $\mathfrak{a}$ *is a reduction of* $\mathfrak{b}$ *if and only if* $\mathfrak{a}R_{\mathfrak{m}_i}$ *is a reduction of* $\mathfrak{b}R_{\mathfrak{m}_i}$ *if* $\mathfrak{b} \subset \mathfrak{m}_i$ *and* $\mathfrak{a}R_{\mathfrak{m}_i} = R_{\mathfrak{m}_i}$ *if* $\mathfrak{b} \not\subset \mathfrak{m}_i$.

PROOF. By definition, there exist positive integers s_i such that

$$(\mathfrak{a}R_{\mathfrak{m}_i})(\mathfrak{b}R_{\mathfrak{m}_i})^{s_i} = (\mathfrak{b}R_{\mathfrak{m}_i})^{s_i+1}$$

if $\mathfrak{b} \subset \mathfrak{m}_i$. Let $s = \sum_{\mathfrak{b}\subset\mathfrak{m}_i} s_i$. Then, by (2.1), for all $i = 1, 2, \cdots, n$,

$$(\mathfrak{a}R_{\mathfrak{m}_i})(\mathfrak{b}R_{\mathfrak{m}_i})^{s} = (\mathfrak{b}R_{\mathfrak{m}_i})^{s+1}$$

So, for all $i = 1, 2, \cdots, n$, $(\mathfrak{a}\mathfrak{b}^s)R_{\mathfrak{m}_i} = (\mathfrak{b}^{s+1})R_{\mathfrak{m}_i}$. Hence $\mathfrak{a}\mathfrak{b}^s = \mathfrak{b}^{s+1}$. It follows that $\mathfrak{a}$ is a reduction of $\mathfrak{b}$. The converse is obvious. □

LEMMA 2.3. *Let* $\mathfrak{b}$ *be an ideal of* R *such that* $\mathfrak{a} \subset \mathfrak{b} + \mathfrak{a}\mathcal{J}$. *Then* $\mathfrak{a} \subset \mathfrak{b}$. *In particular, if* $\mathfrak{b} \subset \mathfrak{a}$ *and* $\mathfrak{a} \subset \mathfrak{b} + \mathfrak{a}\mathcal{J}$, *then* $\mathfrak{a} = \mathfrak{b}$.

PROOF. From $\mathfrak{a} \subset \mathfrak{b} + \mathfrak{a}\mathcal{J}$ it follows, by an easy induction, that $\mathfrak{a} \subset \mathfrak{b} + \mathfrak{a}\mathcal{J}^s$ for every positive integer s. Consequently,

$$\mathfrak{a} \subset \bigcap_{s=1}^{\infty} (\mathfrak{b} + \mathfrak{a}\mathcal{J}^s) \subset \bigcap_{s=1}^{\infty} (\mathfrak{b} + \mathcal{J}^s).$$

But $\cap_{s=1}^{\infty}(\mathfrak{b} + \mathcal{J}^s) = \mathfrak{b}$ by [3, p.208, corollary] . This completes the proof. □

LEMMA 2.4. *Let* $\mathfrak{c}$ *be an ideal of* R *contained in* $\mathfrak{a}$. *Then* $\mathfrak{c}$ *is a reduction of* $\mathfrak{a}$ *if and only if* $\mathfrak{c} + \mathfrak{a}\mathcal{J}$ *is a reduction of* $\mathfrak{a}$.

PROOF. If $\mathfrak{c}$ is a reduction of $\mathfrak{a}$ then, since $\mathfrak{c} \subset \mathfrak{c} + \mathfrak{a}\mathcal{J} \subset \mathfrak{a}$, it follows that $\mathfrak{c} + \mathfrak{a}\mathcal{J}$ is also a reduction of $\mathfrak{a}$.

Coversely, assume that $\mathfrak{c} + \mathfrak{a}\mathcal{J}$ is a reduction of $\mathfrak{a}$. Then there exists a positive integer t such that $(\mathfrak{c} + \mathfrak{a}\mathcal{J})\,\mathfrak{a}^t = \mathfrak{a}^{t+1}$, that is to say $\mathfrak{a}^{t+1} = \mathfrak{c}\mathfrak{a}^t + \mathfrak{a}^{t+1}\mathcal{J}$. It now follows immediately from (2.3) that $\mathfrak{a}^{t+1} = \mathfrak{c}\mathfrak{a}^t$. □

LEMMA 2.5. *Let M be an R–module. Then there exists an R–isomo rphism*

$$\eta : M/\mathcal{J}M \longrightarrow \bigoplus_{i=1}^{n} M_{\mathfrak{m}_i}/\,(\mathfrak{m}_i R_{\mathfrak{m}_i})\,M_{\mathfrak{m}_i}$$

such that $\eta(x+\mathcal{J}M) = (x/1 + (\mathfrak{m}_1 R_{\mathfrak{m}_1} M_{\mathfrak{m}_1}), \cdots, x/1 + (\mathfrak{m}_n R_{\mathfrak{m}_n} M_{\mathfrak{m}_n}))$, *where* $x \in M$.

PROOF. Let $\mathcal{U} = \{\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_n\}$ and $\mathcal{U}' = \phi$. Then $\mathcal{U} - \mathcal{U}'$ is low with respect to $\mathcal{U}$ in the sense of [6,(2.1)] . Also it is easy to see that *Supp*$(M/\mathcal{J}M) \subset \{\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_n\}$. Hence, by [6 (2.3)] there exists an R–homomorphism

$$\xi : M/\mathcal{J}M \longrightarrow \bigoplus_{i=1}^{n} (M/\mathcal{J}M)_{\mathfrak{n}_i}$$

such that $\xi(x+\mathcal{J}M) = ((x+\mathcal{J}M)/1, \cdots, (x+\mathcal{J}M)/1)$, where $x \in M$. Further, by [6, (2.5)] , *Supp*$(Ker\xi) \subset \mathcal{U}' = \phi$ and *Supp*$(Coker\xi) \subset \phi$. Therefore, $Ker\xi = 0$ and $Coker\xi = 0$ which implies that ξ is an R–isomorphism. So there exists an R–isomorphism

$$\begin{aligned}\eta : M/\mathcal{J}M \longrightarrow \bigoplus_{i=1}^{n} (M/\mathcal{J}M)_{\mathfrak{m}_i} &\longrightarrow \bigoplus_{i=1}^{n} M_{\mathfrak{m}_i}/(\mathcal{J}M)_{\mathfrak{m}_i} \\ &= \bigoplus_{i=1}^{n} M_{\mathfrak{m}_i}/(\mathfrak{m}_i R_{\mathfrak{m}_i})M_{\mathfrak{m}_i}\end{aligned}$$

such that, for $x \in M$,

$$\begin{aligned} x+\mathcal{J}M &\mapsto (x+\mathcal{J}M/1, \cdots, x+\mathcal{J}M/1) \\ &\mapsto (x/1 + (\mathfrak{m}_1 R_{\mathfrak{m}_1} M_{\mathfrak{m}_1}), \cdots, x/1 + (\mathfrak{m}_n R_{\mathfrak{m}_n} M_{\mathfrak{m}_n})).\end{aligned}$$

This completes the proof. □

3. The Analytic Spread of an ideal

Now for a positive integer h, we will denote the length of $\mathfrak{a}^h/\mathfrak{a}^h\mathcal{J}$ by $L_R(\mathfrak{a}^h/\mathfrak{a}^h\mathcal{J})$. Since $\mathfrak{a}^h/\mathfrak{a}^h\mathcal{J}$ is a finitely generated $R/\mathcal{J}-$module and the ring $R/\mathcal{J}$ is both Noetherian and Artinian ring, $\mathfrak{a}^h/\mathfrak{a}^h\mathcal{J}$ is both Noetherian and Artinian as $R/\mathcal{J}-$module by [1, (6.5)]. Hence, for each positive integer h, $L_R(\mathfrak{a}^h/\mathfrak{a}^h\mathcal{J})$ is finite by [1,(6.8)].

THEOREM 3.1. *Let $\phi(h,\mathfrak{a}) = L_R(\mathfrak{a}^h/\mathfrak{a}^h\mathcal{J})$, and let for $i = 1, 2, \cdots, n$*

$$\phi(h, \mathfrak{a}R_{\mathfrak{m}_i}) = dim_{k(\mathfrak{m}_i)}(\mathfrak{a}R_{\mathfrak{m}_i})^h/(\mathfrak{a}R_{\mathfrak{m}_i})^h\mathfrak{m}_iR_{\mathfrak{m}_i},$$

where $k(\mathfrak{m}_i) = R_{\mathfrak{m}_i}/\mathfrak{m}_iR_{\mathfrak{m}_i}$. Then $\phi(h,\mathfrak{a})$, for large h, is a polynomial function in h such that

$$1 + deg\phi(h,\mathfrak{a}) = max_{1\le i\le n}\ell(\mathfrak{a}R_{\mathfrak{m}_i})$$

(here, as we reviewed in the introduction, $\ell(\mathfrak{a}R_{\mathfrak{m}_i})$ denote the analytic spread of $\mathfrak{a}R_{\mathfrak{m}_i}$ in $R_{\mathfrak{m}_i}$ if $\mathfrak{a} \subset \mathfrak{m}_i$ and is 0 otherwise).

PROOF. By (2.5), there exists an $R-$isomorphism

$$\eta : \mathfrak{a}^h/\mathfrak{a}^h\mathcal{J} \longrightarrow \bigoplus_{i=1}^{n}(\mathfrak{a}R_{\mathfrak{m}_i})^h/(\mathfrak{a}R_{\mathfrak{m}_i})^h\mathfrak{m}_iR_{\mathfrak{m}_i}$$

such that $\eta(x+\mathfrak{a}^h\mathcal{J}) = \left(x/1 + (\mathfrak{a}R_{\mathfrak{m}_1})^h\mathfrak{m}_1R_{\mathfrak{m}_1}, \cdots, x/1 + (\mathfrak{a}R_{\mathfrak{m}_n})^h\mathfrak{m}_nR_{\mathfrak{m}_n}\right)$, where $x \in \mathfrak{a}^h$. So

$$\phi(h,\mathfrak{a}) = \sum_{i=1}^{n} L_R\left((\mathfrak{a}R_{\mathfrak{m}_i})^h/(\mathfrak{a}R_{\mathfrak{m}_i})^h\mathfrak{m}_iR_{\mathfrak{m}_i}\right).$$

Let $M_i = (\mathfrak{a}R_{\mathfrak{m}_i})^h/(\mathfrak{a}R_{\mathfrak{m}_i})^h\mathfrak{m}_iR_{\mathfrak{m}_i}$. Then we can easily see that for a subset N_i of M_i, N_i is an $R-$sumodule if and only if N_i is an $R_{\mathfrak{m}_i}-$submodule. Hence

$$\phi(h,\mathfrak{a}) = \sum_{i=1}^{n} L_{R_{\mathfrak{m}_i}}(M_i) = \sum_{i=1}^{n} dim_{k(\mathfrak{m}_i)}M_i = \sum_{i=1}^{n}\phi(h,\mathfrak{a}R_{\mathfrak{m}_i})$$

Now, for each i, $\phi(h, \mathfrak{a}R_{\mathfrak{m}_i})$ is, for large h, equal to a polynomial in h, with $1 + deg\phi(h, \mathfrak{a}R_{\mathfrak{m}_i}) = \ell(\mathfrak{a}R_{\mathfrak{m}_i})$. Also, for all large h, the leading coefficient of the polynomial to which $\phi(h, \mathfrak{a}R_{\mathfrak{m}_i})$ becomes equal is positive for $i = 1, 2, \cdots, n$. So for large h, $\phi(h, \mathfrak{a})$ is a polynomial in h with $deg\phi(h, \mathfrak{a}) = max_{1 \leq i \leq n} deg\phi(h, \mathfrak{a}R_{\mathfrak{m}_i})$. Therefore,

$$\begin{aligned} 1 + deg\phi(h, \mathfrak{a}) &= 1 + max_{1 \leq i \leq n} deg\phi(h, \mathfrak{a}R_{\mathfrak{m}_i}) \\ &= 1 + max_{1 \leq i \leq n} (-1 + \ell(\mathfrak{a}R_{\mathfrak{m}_i})) \\ &= max_{1 \leq i \leq n} \ell(\mathfrak{a}R_{\mathfrak{m}_i}) \qquad \square \end{aligned}$$

Now the above theorem provides motivation for the following definition. We define the *analytic spread of* $\mathfrak{a}$ to be $max_{1 \leq i \leq n} \ell(\mathfrak{a}R_{\mathfrak{m}_i})$ and it is denoted by $\ell(\mathfrak{a})$.

LEMMA 3.2. *Let $\mathfrak{b}$ be an ideal of R, and let $\mathfrak{b}$ be a reduction of $\mathfrak{a}$. Then at least $\ell(\mathfrak{a})$ elements are required to generate $\mathfrak{b}$.*

PROOF. Let $\mathfrak{b} = (b_1, b_2, \cdots, b_r)$ for some positive integer r. Then $\mathfrak{b}R_{\mathfrak{m}_i} = (b_1/1)R_{\mathfrak{m}_i} + \cdots + (b_r/1)R_{\mathfrak{m}_i}$ for $i = 1, 2, \cdots, n$. Since $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, by (2.2), $\mathfrak{b}R_{\mathfrak{m}_i} = (b_1/1)R_{\mathfrak{m}_i} + \cdots + (b_r/1)R_{\mathfrak{m}_i}$ is a reduction of $\mathfrak{a}R_{\mathfrak{m}_i}$ in $R_{\mathfrak{m}_i}$ for $\mathfrak{a} \subset \mathfrak{m}_i$. So, by [4, §4. Theorem 2] and (2.1), $r \geq \ell(\mathfrak{a}R_{\mathfrak{m}_i})$ for $i = 1, 2, \cdots, n$. It follows that $r \geq \ell(\mathfrak{a})$. $\square$

DEFINITION 3.3. Let $\mathfrak{c}$ be an ideal of R, and let $\mathfrak{c}$ be a reduction of $\mathfrak{a}$. Then $\mathfrak{c}$ is called a *minimal reduction of* $\mathfrak{a}$ if it can be generated by $\ell(\mathfrak{a})$ elements.

The reader should note that in the special case in which R is local, by [4, §4. Theorem 2], the above definition is consistent to that of [4, §1. Definition 2].

COROLLARY 3.4. *Let $t = \ell(\mathfrak{a})$, and let $\mathfrak{c} = (c_1, c_2, \cdots, c_t)$ be a minimal reduction of $\mathfrak{a}$. Then $c_1, c_2, \cdots, c_t$ form a minimal generating set for $\mathfrak{c}$.*

PROOF. This is an immediate consequence of (3.2). $\square$

Let $\mathfrak{a} = (u_1, u_2, \cdots, u_t)$ and let $\phi(X_1, X_2, \cdots, X_t)$ be a s−form of $R[X_1, X_2, \cdots, X_t]$ such that not all these coefficients are in $\cup_{\mathfrak{a} \subset \mathfrak{m}_i} \mathfrak{m}_i$.

Let ϕ_i be the image of $\phi(X_1, X_2, \cdots, X_t)$ under the cannonical map $R[X_1, X_2, \cdots, X_t] \longrightarrow R_{\mathfrak{m}_i}[X_1, X_2, \cdots, X_t]$ and $\bar{\phi}_i$ be the image of ϕ_i under the cannonical map

$$R_{\mathfrak{m}_i}[X_1, X_2, \cdots, X_t] \longrightarrow (R_{\mathfrak{m}_i}/\mathfrak{m}_i R_{\mathfrak{m}_i})[X_1, X_2, \cdots, X_t] = k(\mathfrak{m}_i)[X_1, X_2, \cdots, X_t].$$

DEFINITION 3.5 ([4, §3, Definition 1-2]). Under the above assumption, the form $\bar{\phi}_i$ is said to be a *null form of* $\mathfrak{a}R_{\mathfrak{m}_i}$ if $\phi_i(u_1, u_2, \cdots, u_t) \in \mathfrak{a}^s \mathfrak{m}_i R_{\mathfrak{m}_i}$. And the homogeneous ideal $\mathfrak{n}_i$ of $k(\mathfrak{m}_i)[X_1, X_2, \cdots, X_t]$ which is generated by the null forms of $\mathfrak{a}R_{\mathfrak{m}_i}$ is said to be the *null form ideal of* $\mathfrak{a}R_{\mathfrak{m}_i}$.

PROPOSITION 3.6. *Under the above assumption,* $\phi_i(u_1, u_2, \cdots, u_t) \in \mathfrak{a}^s(\sum_{\mathfrak{a}\subset\mathfrak{m}_i} \mathfrak{m}_i)$ *if and only if* $\bar{\phi}_i$ *is the null form of* $\mathfrak{a}R_{\mathfrak{m}_i}$ *for all* $\mathfrak{m}_i$ *containing* $\mathfrak{a}$.

PROOF. The proposition follows from ;

$$\left[\mathfrak{a}^s\left(\sum_{\mathfrak{a}\subset\mathfrak{m}_i} \mathfrak{m}_i\right)\right]_{\mathfrak{m}} = \begin{cases} \mathfrak{a}^s\mathfrak{m}R_{\mathfrak{m}} & \text{if } \mathfrak{a}\subset\mathfrak{m} \\ R_{\mathfrak{m}} & \text{otherwise.} \end{cases}$$

□

Let $\lambda_j(X_1, X_2, \cdots, X_t) = \sum_{h=1}^{t} \alpha_{jh}X_h \quad (1 \leq j \leq m, \quad \alpha_{jh} \in R)$ and let λ_{ij} be the image of $\lambda_j(X_1, X_2, \cdots, X_t)$ under the cannonical map $R[X_1, X_2, \cdots, X_t] \longrightarrow R_{\mathfrak{m}_i}[X_1, X_2, \cdots, X_t]$ and $\bar{\lambda}_{ij}$ be the image of λ_{ij} under the cannonical map

$$R_{\mathfrak{m}_i}[X_1, X_2, \cdots, X_t] \longrightarrow (R_{\mathfrak{m}_i}/\mathfrak{m}_i R_{\mathfrak{m}_i})[X_1, X_2, \cdots, X_t] = k(\mathfrak{m}_i)[X_1, X_2, \cdots, X_t].$$

For the null form ideal $\mathfrak{n}_i$ of $\mathfrak{a}R_{\mathfrak{m}_i}$, by [4, §4. lemma1] and (2.2), we have the following theorem;

THEOREM 3.7. *Under the above assumptions,* $(\lambda_1(u_1, u_2, \cdots, u_t), \cdots, \lambda_m(u_1, u_2, \cdots, u_t))$ *is a reduction of* $\mathfrak{a}$ *if and only if* $(\bar{\lambda}_{i1}(u_1, u_2, \cdots, u_t), \cdots, \bar{\lambda}_{im}(u_1, u_2, \cdots, u_t), \mathfrak{n}_i)$ *is a* $(X_1, X_2, \cdots, X_t)$*−primary ideal of* $k(\mathfrak{m}_i)[X_1, X_2, \cdots, X_t]$ *if* $\mathfrak{a} \subset \mathfrak{m}_i$ *and it is* $k(\mathfrak{m}_i)[X_1, X_2, \cdots, X_t]$ *if* $\mathfrak{a} \not\subset \mathfrak{m}_i$.

References

1. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.
2. H. Ansari Toroghy and R. Y. Sharp, *Integral closures of ideals relative to injective modules over commutative Noetherian rings*, Oxford Quart. J. Math. **42** (1991), 393–402.
3. D. G. Northcott, *Lessons on rings, modules and multiplicities*, Cambridge University Press, 1968.
4. D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145–158.
5. D. Rees and R. Y. Sharp, *On a theorem of B. Teissier on multiplicities of ideals in local rings*, J. London Math. Soc. **18** (1978), 449–463.
6. R. Y. Sharp, *The cousin complex for a module over a commutative Noetherian ring*, Math. Zeit **112** (1969), 340–356.
7. R. Y. Sharp and A. -J. Taherizadeh, *Reductions and integral closures of ideals relative to an Artinian module*, J. London Math. Soc. **37** (1988), 203–218.

Department of Mathematics Education
Sunchon National University
Sunchon 540-742, Korea

Department of Mathematics
Chonnam National University
Kwangju 500–757, Korea