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A PROPERTY OF SURFACE GROUPS

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ABSTRACT. We prove that if G is the fundamental group of a closed surface or a Seifert fibered space and K is a finitely generated subgroup of G, and if for any element g in G there exists an integer n_g such that g^{n_g} belongs to K, then K is of finite index in G.

An easy consequence of K being a finite index subgroup of a group G is that for all $g \in G$, there exists a non-zero integer n_g such that g^{n_g} belongs to K. In the paper [1], R. Canary pointed out by giving J. Anderson's proof that the converse is true for the case where G is the fundamental group of an infinite volume geometrically finite hyperbolic 3-manifold and K is finitely generated. In this paper, we will prove the converse for fundamental groups of closed surfaces in the case where K is finitely generated. If F is an orientable closed surface of genus$\geqslant 2$, then $F = \mathbb{H}^2/G$, where G is a Fuchsian group of the first kind. We will use some properties of Fuchsian groups of the first kind in the course of the proof. Using the result for closed surface groups we will also show that the converse is true for the fundamental groups of Seifert fibered spaces. In section 1, we state some of the properties of Fuchsian groups out of [3]. In section 2, we provide a proof of the converse for closed surfaces and Seifert fibered spaces.

1. Preliminaries

A Fuchsian group is a discrete subgroup of $PSL(2;\mathbb{R})$, which is considered as a subgroup of the group of isometries of the hyperbolic plane

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$\mathbb{H}^2$. There are three types of elements in $PSL(2;\mathbb{R})$, elliptic, parabolic and hyperbolic depending on the types of fixed points. If a Fuchsian group G acts freely on $\mathbb{H}^2$, then $\mathbb{H}^2/G$ is a hyperbolic surface.

DEFINITION 1.1. Let G be a Fuchsian group. The limit set L_G of G is defined to be the set of all possible limit points of G-orbits G_z, $z \in \mathbb{H}^2$.

For any Fuchsian group G, $L_G \subset S^1_\infty$, where S^1_∞ is the circle at infinity.

DEFINITION 1.2. A Fuchsian group G is called of the first kind if $L_G = S^1_\infty$.

PROPOSITION 1.3. *Let G be a Fuchsian group. If L_G contains more than one point, then the set of fixed points of the hyperbolic elements in G is dense in L_G.*

It is known that there exists a convex fundamental region for a Fuchsian group called the Dirichlet region of the given Fuchsian group. A Dirichlet region F of a Fuchsian group G is bounded by a number of geodesic segments and possibly segments of S^1_∞. The boundary geodesic segments are called sides of F. A Fuchsian group G is called cocompact if $\mathbb{H}^2/G$ is compact.

PROPOSITION 1.4. *A Fuchsian group G is cocompact if and only if the area of a fundamental Dirichlet region is finite and G has no parabolic elements.*

An immediate consequence of Proposition 1.4 is that every element of $G \setminus \{1\}$ is hyperbolic if $\mathbb{H}^2/G$ is a closed hyperbolic surface.

DEFINITION 1.5. A Fuchsian group G is called geometrically finite if there exists a convex fundamental region for G with finitely many sides.

The following 3 propositions characterize geometric finiteness of Fuchsian groups.

PROPOSITION 1.6. *If G is a geometrically finite Fuchsian group of the first kind, then G has a fundamental region of finite area.*

PROPOSITION 1.7. *If a Fuchsian group G has a fundamental region of finite area, then G is of the first kind.*

PROPOSITION 1.8. *If a Fuchsian group is finitely generated, then it is geometrically finite.*

2. Main Theorem

We first provide a Lemma on limit sets.

LEMMA 2.1. *Let G be a Fuchsian group having at least one hyperbolic element and K be a subgroup of G. If for any $g \in G$ there is a non-zero integer n_g such that $g^{n_g} \in K$, then $L_G = L_K$.*

PROOF. If g is a hyperbolic element of G, with fixed points x and y, then g^{n_g} is a hyperbolic element of K with the same fixed points x and y. Hence the set of fixed points of hyperbolic elements of G is the same as the set of fixed points of hyperbolic elements of K. By Proposition 1.3, $L_K = L_G$.

Next we establish the result for orientable closed surfaces.

THEOREM 2.2. *Let G be the fundamental group of an orientable closed surface F, and let K be a finitely generated subgroup of G. If for all $g \in G$, there exists a non-zero integer n_g such that $g^{n_g} \in K$, then K is of finite index in G.*

PROOF. If F ia a torus or sphere, then $G = \mathbb{Z} \times \mathbb{Z}$ or $G = 1$, in which case the conclusion is trivial. Suppose F is a closed surface of genus$\geqslant$ 2. G may be considered as a Fuchsian group consisting of hyperbolic elements only. Since G is cocompact, G has a fundamental region of finite hyperbolic area, so $L_G = S^1_\infty$. By Lemma 2.1, $L_K = L_G = S^1_\infty$. Proposition 1.8 assures that K is geometrically finite. It follows that K has a fundamental region of finite area. Hence $\mathbb{H}^2/K$ is a closed surface by Proposition 1.4, as all the elements of K are hyperbolic. Now the natural map $\mathbb{H}^2/K \to \mathbb{H}^2/G$ is a covering map between closed surfaces. Thus the covering is finite to one, which implies that K has finite index in G.

THEOREM 2.3. *Let G be the fundamental group of a closed surface F and let K be a finitely generated subgroup of G. If for all $g \in G$ there is a non-zero integer n_g such that $g^{n_g} \in K$, then K has finite index in G.*

PROOF. If F is orientable, we are done in Theorem 2.2. If F is non-orientable, then there is a orientable double cover $\tilde{F}$ of F. Note that $\pi_1(\tilde{F}) \cap K$ is finitely generated and for all $g \in \pi_1(\tilde{F})$ there is a non-zero integer n_g such that $g^{n_g} \in \pi_1(\tilde{F}) \cap K$. By Theorem 2.2, $\pi_1(\tilde{F}) \cap K$ is of finite index in $\pi_1(\tilde{F})$. Since $\pi_1(\tilde{F})$ is of index 2 in G, $\pi_1(\tilde{F}) \cap K$ is of finite index in G. It follows that K is of finite index in G, as $\pi_1(\tilde{F}) \cap K \subset K$.

Recall that a Seifert fibered space is a S^1-bundle over a closed 2-dimensional orbifold. It is known that if M is a closed Seifert fibered space with base orbifold X, then there is an exact sequence:

$$1 \longrightarrow N \longrightarrow \pi_1(M) \longrightarrow \pi_1(X),$$

where N denotes the cyclic subgroup of $\pi_1(M)$ generated by a regular fiber. The group N is infinite except in the case where M is covered by S^3. Furthermore $\pi_1(X)$ has the fundamental group of an surface as a finite index subgroup.(See [2] or [4])

COROLLARY 2.4. *Let M be a closed Seifert fibered space, and let K be a subgroup of $\pi_1(M)$. If for all $g \in \pi_1(M)$ there is a nonzero integer n_g such that $g^{n_g} \in K$, then K has finite index in $\pi_1(M)$.*

PROOF. If M is covered by S^3, $\pi_1(M)$ is finite, in which case the conclusion follows easily. Suppose M is not covered by S^3. Then we have the following exact sequence:

$$1 \longrightarrow N \longrightarrow \pi_1(M) \xrightarrow{\phi} \pi_1(X),$$

where $N \cong \mathbb{Z}$. Let G be the fundamental group of a closed surface whose index in $\pi_1(X)$ is finite. Then for any $g \in G$, there exists a nonzero integer n_g with $g^{n_g} \in \phi(K) \cap G$. By Theorem 2.3, $\phi(K) \cap G$ is of finite index in G, and so of finite index in $\pi_1(X)$. It follows that $\phi^{-1}(\phi(K))$ has finite index in $\pi_1(M)$. Note that $\phi^{-1}(\phi(K)) = < t, K >$, where N is generated by t and $< t, K >$ is the subgroup of $\pi_1(M)$ generated by t and K. Each element x in $< t, K >$ can be written as $t^m \cdot k$ for some integer m and $k \in K$. Since there is an integer n_t such that $t^{n_t} \in K$, there are less than $n_t + 1$ left cosets of K in $< t, K >$. Hence K has finite index in $\phi^{-1}(\phi(K))$, and so K is of finite index in $\pi_1(M)$.

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