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ON CERTAIN GRADED RINGS WITH MINIMAL MULTIPLICITY

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ABSTRACT. Let (R, m) be a Cohen-Macaulay local ring with an infinite residue field and let $J = (a_1, \cdots, a_l)$ be a minimal reduction of an equimultiple ideal I of R. In this paper we shall prove that the following conditions are equivalent:

(1) $I^2 = JI$.

(2) $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal N.

(3) $N^2 = (a_1', \cdots, a_l')N$, where a_i' denotes the images of a_i in I/mI for $i = 1, \cdots, l$.

1. Introduction

Let (R, m) be a Cohen-Macaulay local ring with infinite residue field and let I be a proper ideal of R. We define

$$gr_I(R) := R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

and call it the associated graded ring of I. $gr_I(R)/mgr_I(R)(= R/m \oplus I/mI \oplus I^2/mI^2 \oplus \cdots$ and call it the fiber cone of I.) is a basic geometric object but it has scarcely been studied algebraically for its own sake. (cf. [2], [5])

Let $J = (a_1, \cdots, a_l)$ be a minimal reduction of an equimultiple ideal I of R. M. Kim ([2], Theorem 3.1) proved that if $I^2 = JI$, then $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal

$$N = I/mI \oplus I^2/mI^2 \oplus \cdots .$$

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The present paper has its origin in the effort of showing the converse of Kim's result. To wit, we shall prove that the following conditions are equivalent:

(1) $I^2 = JI$.

(2) $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal N.

(3) $N^2 = (a'_1, \cdots, a'_l)N$, where a'_i denotes the images of a_i in I/mI.

This paper has divided into three sections. In section two, we obtain a relationship between a minimal reduction of I and a minimal reduction of N. Finally in section three we prove our main theorem.

2. Preliminaries

In this paper all rings are assumed to be commutative with identity. By a local ring (R, m), we mean a Noetherian ring R which has a unique maximal ideal m. If I is an ideal of a local ring (R, m), the analytic spread of I, denoted $l(I)$, is defined to be $dim(R[It]/mR[It])$. In [3], it is shown that $ht(I) \leq l(I) \leq dim(R)$. An ideal I is called equimultiple if $ht(I) = l(I)$. In particular, all m-primary ideals are equimultiple. If R/m is infinite and $l = l(I)$, then there exist elements $a_1, \cdots, a_l$ in I such that

$$(a_1, \cdots, a_l)I^n = I^{n+1}$$

for some $n > 0$. The ideal $J = (a_1, \cdots, a_l)$ is called a minimal reduction of I.

For an m-primary ideal I in a local ring (R, m), the integer $e(I)$ will denote the multiplicity of I. That is, if $d = dim(R)$, then

$$e(I) = \lim_{n \to \infty} \frac{d!}{n^d} \lambda_R(R/I^n)$$

where $\lambda_R(-)$ denotes the length of R-module. $e(R)$ means the multiplicity of the maximal ideal m of R. We shall use μ for the number of elements in a minimal basis of an ideal.

If (R, m) is a Cohen-Macaulay local ring, then it is true that

$$\dim(R) \leq \mu(m) \leq e(R) + \dim(R) - 1.$$

The first inequality is the Krull principal ideal theorem and the second inequality is a result of Abhyankar [1].

DEFINITION 2.1. A Cohen-Macaulay local ring (R, m) is said to have minimal multiplicity if $\mu(m) = e(R) + \dim(R) - 1$.

LEMMA 2.2.([4], THEOREM 1). *Let (R, m) be a Cohen-Macaulay local ring with an infinite field R/m. Suppose that J is a minimal reduction of m. Then $m^2 = Jm$ if and only if R has minimal multiplicity.*

Let (R, m) be a d-dimensional local ring with an infinite field R/m. Let $J = (a_1, \cdots, a_l)R$ be a minimal reduction of an equimultiple ideal I of R. Then $a_i \notin mI$ for all $i = 1, \cdots, l$, ([3], Lemma 3). Let a_i' denote the image of a_i in I/mI in

$$T = R/m \oplus I/mI \oplus I^2/mI^2 \oplus \cdots (= gr_I(R)/mgr_I(R)).$$

It then follows easily that $(a_1', \cdots, a_l')T$ is a minimal reduction of the maximal homogeneous ideal

$$N = I/mI \oplus I^2/mI^2 \oplus \cdots .$$

PROPOSITION 2.3. *Let I be an equimultiple ideal of a local ring (R, m) with an infinite field R/m. Let N be the maximal homogeneous ideal of $T = gr_I(R)/mgr_I(R)$ and let $n > 0$ be integer. Suppose that $J = (a_1, \cdots, a_l)$ is a minimal reduction of I. Then $I^{n+1} = (a_1, \cdots, a_l)I^n$ holds if and only if $N^{n+1} = (a_1', \cdots, a_l')N^n$ holds where a_i' denotes the image of a_i in I/mI for all $i = 1, \cdots, l$.*

PROOF. For every integer $i \geq 1$,

$$N^i = I^i/mI^i \oplus I^{i+1}/mI^{i+1} \oplus \cdots$$

where $N = I/mI \oplus I^2/mI^2 \oplus \cdots$. Then, for every integer $i \geq 1$,

$$(a_1', \cdots, a_l')N^i = \frac{JI^i + mI^{i+1}}{mI^{i+1}} \oplus \frac{JI^{i+1} + mI^{i+2}}{mI^{i+2}} \oplus \cdots .$$

$(\Rightarrow)$ Suppose that $I^{n+1} = (a_1, \cdots, a_l)I^n$. Then, since $I^{n+k} = JI^{n+(k-1)}$

for all $k \geq 1$, we see that

$$\begin{aligned}(a'_1, \cdots, a'_l)N^n &= \frac{JI^n + mI^{n+1}}{mI^{n+1}} \oplus \frac{JI^{n+1} + mI^{n+2}}{mI^{n+2}} \oplus \cdots \\ &= \frac{I^{n+1} + mI^{n+1}}{mI^{n+1}} \oplus \frac{I^{n+2} + mI^{n+2}}{mI^{n+2}} \oplus \cdots \\ &= \frac{I^{n+1}}{mI^{n+1}} \oplus \frac{I^{n+2}}{mI^{n+2}} \oplus \cdots \\ &= N^{n+1}.\end{aligned}$$

($\Leftarrow$) Suppose that $N^{n+1} = (a'_1, \cdots, a'_l)N^n$. That is,

$$\frac{I^{n+1}}{mI^{n+1}} \oplus \frac{I^{n+2}}{mI^{n+2}} \oplus \cdots = \frac{JI^n + mI^{n+1}}{mI^{n+1}} \oplus \frac{JI^{n+1} + mI^{n+2}}{mI^{n+2}} \oplus \cdots$$

Then, since $I^{n+k} = JI^{n+(k-1)} + mI^{n+k}$ for all $k \geq 1$, we see that $I^{n+k} = JI^{n+(k-1)}$ for all $k \geq 1$, by Nakayama Lemma. Hence $I^{n+1} = JI^n$. □

COROLLARY 2.4. *Let (R, m) be a d-dimensional local ring with an infinite field R/m and let Q be an m-primary ideal of R. Let M be the maximal homogeneous ideal of $gr_Q(R)/mgr_Q(R)$ and let $n > 0$ be integer. Suppose that $(x_1, \cdots, x_d)$ is a minimal reduction of Q. Then $Q^{n+1} = (x_1, \cdots, x_d)Q^n$ holds if and only if $M^{n+1} = (x'_1, \cdots, x'_d)M^n$ holds where x'_i denotes the image of x_i in Q/mQ for all $i = 1, \cdots, d$.*

With the same argument of the proof in Proposition 2.3, we get following remark.

REMARK 2.5. Let $(R, m), I$, and J be as in Proposition 2.3. Let a^*_i denote the image of a_i in I/I^2 of

$$gr_I(R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

and let $n > 0$ be integer. Then $I^{n+1} = (a_1, \cdots, a_l)I^n$ if and only if $(G^+)^{n+1} = (a^*_1, \cdots, a^*_l)(G^+)^n$ where $G^+ = I/I^2 \oplus I^2/I^3 \oplus \cdots$.

3. Main Result

THEOREM 3.1. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring with an infinite field R/m and let I be an equimultiple ideal of R. Let N be the maximal homogeneous ideal of $gr_I(R)/mgr_I(R)$. Suppose that $J = (a_1, \cdots, a_l)$ is a minimal reduction of I. Let $a'_1, \cdots, a'_l$ denote the images of $a_1, \cdots, a_l$ in I/mI. Then the following conditions are*

(1) $I^2 = (a_1, \cdots, a_l)I$.

(2) $gr_I(R)/mgr_I(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal N.

(3) $N^2 = (a'_1, \cdots, a'_l)N$.

PROOF. (1) $\Rightarrow$ (2): See the proof of Theorem 3.1 in [2] or the proof of Theorem 1 in [5].

(2) $\Rightarrow$ (3): Since $J = (a_1, \cdots, a_l)$ is a minimal reduction of I, there is a positive integer n with $I^{n+1} = JI^n$. Hence we get that $N^{n+1} = (a'_1, \cdots, a'_l)N^n$ by Proposition 2.3. We put $T = gr_I(R)/mgr_I(R)$. Localizing at N, we have that

$$(NT_N)^{n+1} = (a'_1, \cdots, a'_l)T_N(NT_N)^n.$$

That is, $(a'_1, \cdots, a'_l)T_N$ is a minimal reduction of NT_N. Thus we get that

$$N^2T_N = ((a'_1, \cdots, a'_l)N)T_N \qquad (*)$$

by Lemma 2.2.

Suppose that $f \in N^2$. We will show that $f \in (a'_1, \cdots, a'_l)N$. Of course we may assume that f is homogeneous. We denote by T_n the n th graded component of T, i.e., $T_n = I^n/mI^n$ for $n \geq 0$ and $T_n = (0)$ for $n < 0$. Let us express $f = c$, where $c \in T_n (n \geq 2)$. Then we find that $g \in T \backslash N$ such that $gc \in (a'_1, \cdots, a'_l)N$, which follows from the equation $(*)$. This allows us to express

$$gc = \sum_{i=1}^{l} a'_i h_i \qquad (**)$$

with $h_i \in N$. We denote by $h_i^{(n)}$ the homogeneous term of h_i of degree n. Then, comparing the elements of degree n in both sides of the equation

(**), we obtain

$$g^{(0)}c = \sum_{i=1}^{l} a'_i h_i^{(n-1)}$$

with $h_i^{(n-1)} \in I^{n-1}/mI^{n-1}$. Since $g^{(0)}$ is a unit of R/m, we have the required fact that $c \in (a'_1, \cdots, a'_l)N$. Thus $N^2 = (a'_1, \cdots, a'_l)N$.

(3) ⇔ (1): See Proposition 2.3. □

COROLLARY 3.2. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring with an infinite field R/m. Let $(x_1, \cdots, x_d)$ be a minimal reduction of an m-primary ideal Q of R. Let M be the maximal homogeneous ideal of $gr_Q(R)/mgr_Q(R)$. Let $x'_1, \cdots, x'_d$ denote the images of $x_1, \cdots, x_d$ in Q/mQ. Then the following conditions are equivalent.*

(1) $Q^2 = (x_1, \cdots, x_d)Q$.

(2) $gr_Q(R)/mgr_Q(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal M.

(3) $M^2 = (x'_1, \cdots, x'_d)M$.

PROOF. Recall that any m-primary ideal is equimultiple. □

COROLLARY 3.3. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring with an infinite field R/m. Then the following conditions are equivalent.*

(1) R has minimal multiplicity.

(2) $m^2 = (x_1, \cdots, x_d)m$ for some minimal reduction $x_1, \cdots, x_d$ of m.

(3) $gr_m(R)$ is Cohen-Macaulay with minimal multiplicity at its maximal homogeneous ideal $K = m/m^2 \oplus m^2/m^3 \oplus \cdots$.

(4) $K^2 = (x'_1, \cdots, x'_d)K$, where x'_i denotes the image of x_i in m/m^2 for all $i = 1, \cdots, d$.

PROOF. (1) ⇔ (2): See Lemma 2.2.

(2) ⇔ (3) ⇔ (4): This follows immediately from Theorem 3.1. □

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