A Study on Identification of Nonminimum Phase Stable Systems from Partial Impulse Response Sequences

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Abstract

This paper addresses the problem of identifying the class of all stable system transfer functions that interpolate the given partial impulse response sequence. In this context, classical Pade approximations that are also stable, are shown to be a special case of this general formulation. The theory developed in this connection is utilized to obtain a new criterion for determining the model order and system parameters for rational systems, and, further, to generate nonminimum phase optimal stable rational approximations of nonrational systems from its impulse response sequence.

Ⅰ. Introduction

This paper addresses the problem of identifying the class of all stable system transfer functions that interpolate the given partial impulse response sequence. Although classical Pade approximations match the given impulse response sequence to a maximum extent and are optimal in that sense, the systems so obtained need not be stable and hence they may not be attractive from physical considerations. In this context, consider the problem of identifying a linear discrete time-invariant, causal, stable system with an unknown transfer function H (z) from partial information regarding itself. Since the system is causal, it has a one-sided power series expansion given by

$$H(z) = \sum_{k=0}^{\infty} h_k z^k \qquad (1)$$

and stability demands that

$$\sum_{k=0}^{\infty} |h_k| < \infty. \qquad (2)$$

It follows from (1), (2) and uniform convergence that the transfer function H(z) is analytic in $|z| < 1$ and uniformly continuous[1] in $|z| \le 1$ [1, 2]. Clearly, the sequence $\{h_k\}_{k=0}^{\infty}$ represents the impulse response of the system and when the available information is of the form h_k, $k = 0 \rightarrow n$, the system identification problem in the rational case becomes equivalent to a Pade approximation problem. In that case, it is easy to show that rational ARMA(p, q)-type approximations that match the given data are unique provided (Pade approximation) $p + q \le n$[1, 2]. These approximations, however, need not be stable and hence from physical considerations they may not be acceptable. For example, consider the stable (minimum phase) transfer function $H(z) = e^{-3z}$. The ARMA (1, 1) Pade approximation of this function is given by $B(z)/A(z) = (2 - 3z)/(2 + 3z)$, and it represents an unstable system since $A(z)$ has a zero in $|z| < 1$.

In the rational case the identification problem is equivalent to finding the system model order (p, q) and the system parameters. Given the partial impulse response sequence, the system model can be established from the invariance of the rank property associated with certain Hankel matrices generated from this data. Thus, in particular, with h_k, $k \ge 0$, denoting its impulse response sequence as in (1), let

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1 Note that the use of the variable z (rather than z^{-1}) here translates all stability arguments into the compact region $|z| \le 1$. $H(z)$ is said to be minimum phase if it is analytic together with is inverse in $|z| \le 1$. Since stable functions are free of poles in in $|z| \le 1$, in the rational case they are analytic in $|z| \le 1$.

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$$H_k=\begin{bmatrix} h_1 & h_2 & \cdots & h_k \\ h_2 & h_3 & \cdots & h_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ h_k & h_{k+1} & \cdots & h_{2k-1} \end{bmatrix} \quad (3)$$

represent the Hankel matrix of size k×k generated from $h_1, h_2, \cdots h_{2k-1}$. Then, for a rational system with degree p,

$$rank\ H_k = rank\ H_p = p,\ k \geq p \quad (4)$$

and several singular value decomposition technique have been proposed for model order selection based on the above rank condition [3, 4]. Equation (4) shows the linear dependence of $h_{p+1}, h_{p+2}, \cdots$ on their p previous terms, and it represents the finite degree nature of a rational system. Although these techniques have the advantage that they can make use of all available impulse response data, they need not lead to stable systems. Moreover, the above rank condition is not valid in the case of systems that are not rational, since they do not represent finite degree systems. The problem in that case is to obtain equivalent finite degree stable rational approximations that capture all the key features of the original nonrational system in an optimal manner by making use of the given data. Such a rational approximation should interpolate the given information, and preferably be of minimum possible degree.

In this paper, we address this problem and obtain closed form solutions for the class of all stable transfer functions that interpolate the given partial impulse response sequence. Specifically, by making contact with the Schur problem [5] in section II, it is shown in section III that the theory of bounded functions (Schur functions) can be utilized to obtain all stable solutions to this problem. In this context, a new model order selection procedure is proposed here that utilizes the finite degree property of a rational system. Rational and stable approximation of nonrational systems is described in section IV, by making use of ideas developed in section III. Although various authors have addressed related problems in the past utilizing this approach [6]-[13], some interesting new observations will show that rational system identification as well as stable rational approximation of nonrational functions can be realized from the same formulation of the Schur extension problem.

II. The Schur Parametrization

To start with, a function $d(z)$ is said to be bounded (Schur function), if

$i)$ $d(z)$ is analytic in $|z| < 1$

and (5)

$ii)$ $|d(z)| \leq 1$, in $|z| < 1$.

Thus z^k, $1/(2+z)$, $e^{-(1-z)}$ all are bounded functions, the later representing a nonrational one. Because of the analyticity in $|z| < 1$, every bounded function possesses a power series representation of the form

$$d(z) = \sum_{k=0}^{\infty} d_k z^k,\ |z| < 1, \quad (6)$$

that is valid in $|z| \leq 1$. If $d(z)$ is rational, then $|d(z)| \leq 1$ in $|z| < 1$ also implies $d(z)$ is free of poles in $|z| = 1$ and hence $d(z)$ is analytic in $|z| \leq 1$. As a result $d(z)$ represents a stable system.

From Schur's Theorem [5], $d(z)$ given by (6) represents a bounded function iff

$$I - D_k D_k^* \geq 0,\ k = 0 \rightarrow \infty, \quad (7)$$

where

$$D_k = \begin{bmatrix} d_0 & 0 & 0 & \cdots & 0 \\ d_1 & d_0 & 0 & \cdots & 0 \\ d_2 & d_1 & d_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_k & d_{k-1} & d_{k-2} & \cdots & d_0 \end{bmatrix} \quad (8)$$

represents the lower (or upper) triangular Toeplitz matrix generated from d_i, $i = 0 \rightarrow k$. Further, strict inequality is maintained in (7) under the additional constraint

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(1 - |d(e^{j\theta})|^2)\, d\theta > -\infty. \quad (9)$$

Given a partial set of coefficients d_k, $k = 0 \rightarrow n$, that satisfy $I - D_n D_n^* \geq 0$, "the problem of coefficients" is to obtain all bounded functions $d(z)$ such that the power series expansion of $d(z)$ matches the given coefficients, i.e.,

$$d(z) = \sum_{k=0}^{n} d_k z^k + O(z^{n+1}). \quad (10)$$

An algorithm introduced by Schur in this context answers this problem, and as we show below, it forms the basis for our approach to the present parametrization problem. As Schur has first observed, if $d(z)$ represents a bounded function, then, so does the function [5]

$$d_1(z)=\frac{1}{z}\cdot\frac{d(z)-d(0)}{1-d^*(0)d(z)},\ d(0)=d_0. \tag{11}$$

This follows by noting that, since $d(0)<1$ and $|d(z)|\le 1$, in $|z|<1$, the only possible pole of $d_1(z)$ in $|z|<1$ is at $z=0$, and it is cancelled by the zero of

$$\frac{d(z)-d(0)}{1-d^*(0)d(z)} \tag{12}$$

at $z=0$. Thus $d_1(z)$ is analytic in $|z|<1$. To prove its boundedness in $|z|<1$, we can make use of the maximum modulus theorem [14, 15]. At any $z=re^{j\theta}$ in $|z|<1$, let $d(re^{j\theta})=Re^{j\phi(\theta)}$, tne $|R|<1$ and by direct expansion, we get

$$1-\left|\frac{d(re^{j\theta})-d(0)}{1-d^*(0)d(re^{j\theta})}\right|^2=\frac{(1-|R|^2)(1-|d(0)|^2)}{|1-d^*(0)Re^{j\phi}|^2}\ge 0. \tag{13}$$

Using this, (11) gives

$$|d_1(re^{j\theta})|\le\frac{1}{r},\ 0<r<1$$

and as $r\to 1-0$, by maximum modulus theorem, since a function that is analytic in any closed region attains its absolute value only on the boundary not inside that region, we get

$$|d_1(z)|\le 1 \text{ in } |z|<1. \tag{15}$$

i.e., $d_1(z)$ given by (11) represents a bounded function provided $d(z)$ is bounded. The above argument also shows the boundary value $d(e^{j\theta})$ defined by the internal radial limit $\lim_{r\to 1^-} d(re^{j\theta})$ is bounded by unity for almost all θ.

In the rational case, since $z=0$ is not a pole of $d_1(z)$, from (11) we obtain that the degree[2] of the new bounded function $d_1(z)$ never exceeds that of $d(z)$, i.e.,

$$\delta(d_1(z))\le\delta(d(z)), \tag{16}$$

with inequality iff the $1/z$ factor in (11) cancels a pole of (12) [16]. Since this cancellation can occur only at $z=\infty$, from (11)-(12), degree reduction happens iff the denominator term $1-d(0)^*d(z)$ satisfies

$$1-d^*(0)d(z)|_{z=\infty}=0,$$

or

$$\delta(d_1(z))<\delta(d(z))\Longleftrightarrow d(z)d_*(z)|_{z=0}=1, \tag{17}$$

where

$$d_*(z)=d^*(1/z^*) \tag{18}$$

is defined to be the paraconjugate form of $d(z)$. Clearly, the paraconjugate form represents the ordinary complex conjugate operation on the unit circle. In particular, if

$$d(z)=\frac{b_0+b_1z+\cdots+b_pz^p}{a_0+a_1z+\cdots+a_pz^p} \tag{19}$$

represents a degree p rational function, then[3]

$$\delta(d_1(z))=p-1\Longleftrightarrow d(z)d_*(z)|_{z=0}=\frac{b_0b_p^*}{a_0a_p^*}=1. \tag{20}$$

Note that if $\delta(d_1(z))=p$, $d(z)$ does not satisfy (20), then, and rewriting (11), we get

$$d(z)=\frac{d(0)+zd_1(z)}{1+zd^*(0)d_1(z)}, \tag{21}$$

and because of the z-factor that multiplies $d_1(z)$ in (21), it follows that to respect the degree of $d(z)$, the numerator polynomial of $d_1(z)$ must be at most of degree p-1. As a result, its denominator must have degree p and hence, whenever there is no degree reduction we obtain the representation

$$d_1(z)=\frac{f_0+f_1z+\cdots+f_{p-1}z^{p-1}}{g_0+g_1z+g_2z^2+\cdots+g_pz^p}. \tag{22}$$

The bilinear transformation in (11) maps the inside of the unit circle onto itself. Thus, in general, with $d_k(z)$

2 The degree $\delta(H(z))$ of a rational function $H(z)$ equals the totality of its poles (or zeros), with multiplicities counted, including those at infinity.

3 Equation (20) represents the classical Richards' condition [16] for degree reduction.

representing an arbitrary bounded function and with

$$s_k = d_k(0). \quad (23)$$

(11) translates into

$$d_{k+1}(z) = \frac{1}{z} \cdot \frac{d_k(z) - s_k}{1 - s_k^* d_k(z)}, \quad (24)$$

or,

$$d_k(z) = \frac{s_k + z d_{k+1}(z)}{1 + z s_k^* d_{k+1}(z)}, \; k \geq 0, \quad (25)$$

whith the understanding that $d_0(z) \equiv d(z)$. The above Schur algorithm in (25) can be recursively updated and after n such steps, we get

$$d(z) =$$

$$\frac{(b_{n-1}(z) + z s_n \widetilde{a}_{n-1}(z)) + z(z\, \widetilde{a}_{n-1}(z) + s_n^* b_{n-1}(z)) d_{n+1}(z)}{(a_{n-1}(z) + z s_n \widetilde{b}_{n-1}(z)) + z(z\, \widetilde{b}_{n-1}(z) + s_n^* a_{n-1}(z)) d_{n+1}(z)}$$

$$= \frac{b_n(z) + z\, \widetilde{a}_n(z) d_{n+1}(z)}{a_n(z) + z\, \widetilde{b}_n(z) d_{n+1}(z)}, \quad (26)$$

where $a_n(z)$ and $b_n(z)$ are in general two polynomials of degree n, and

$$\widetilde{a}_n(z) = z^n a_{n*}(z) = z^n a_n^*(1/z^*), \quad (27)$$

$$\widetilde{b}_n(z) = z^n b_{n*}(z) = z^n b_n^*(1/z^*) \quad (28)$$

represent polynomials reciprocal to $a_n(z)$ and $b_n(z)$ respectively that satisfy the recursion

$$a_n(z) = a_{n-1}(z) + z s_n \widetilde{b}_{n-1}(z), \; n \geq 1 \quad (29)$$

and

$$b_n(z) = b_{n-1}(z) + z s_n \widetilde{a}_{n-1}(z), \; n \geq 1. \quad (30)$$

$a_n(z)$ and $b_n(z)$ are defined to be the Schur polynomials of the first and second kind respectively. Notice that, if $d(z)$ is rational to start with, application of the above procedure will result in a rational bounded function $d_{n+1}(z)$ for every n, and, further from (16), in that case

$$\delta(d_{n+1}(z)) \leq \delta(d(z)), \; n \geq 0. \quad (31)$$

From (26), the iterations in (29)-(30) start with

$$a_0(z) = 1, \; b_0(z) = s_0 = d_0. \quad (32)$$

Using (29)-(30), it is easy to show that $a_n(z)$, $n = 1 \rightarrow \infty$, represent strict Hurwitz polynomials[4]. To see this, a direct calcuation gives

$$a_n(z) a_{n*}(z) - b_n(z) b_{n*}(z) = (1 - |s_n|^2)(a_{n-1}(z) a_{n-1*}(z) - b_{n-1}(z) b_{n-1*}(z)) > 0, \quad (33)$$

which gives

$$\left| \frac{b_n(e^{j\theta})}{a_n(e^{j\theta})} \right|^2 = 1 - \frac{\prod_{k=0}^{n}(1 - |s_k|^2)}{|a_n(e^{j\theta})|^2} \leq 1, \; n \geq 0, \quad (34)$$

and using this in (29), we obtain

$$0 < \prod_{k=1}^{n}(1 - |s_k|) \leq |a_n(z)| \leq \prod_{k=1}^{n}(1 + |s_k|), \; |z| \leq 1. \quad (35)$$

that shows the strict Hurwitz character of $a_n(z)$. Returning back to (26), $d(z)$ represents a bounded function for every choice of the arbitrary bounded function $d_{n+1}(z)$ there, and in particular also for $d_{n+1}(z) \equiv 0$. Thus $b_n(z)/a_n(z)$ itself is bounded and, moreover, from (34), (26), a direct expansion gives

$$d(z) - \frac{b_n(z)}{a_n(z)} = \frac{z^{n+1} d_{n+1}(z) \prod_{k=0}^{n}(1 - |s_k|^2)}{a_n(z)(a_n(z) + z\, \widetilde{b}_n(z) d_{n+1}(z))} = O(z^{n+1})$$

i.e., the power series expansions of the bounded functions $d(z)$ and $b_n(z)/a_n(z)$ agree upto the first n + 1 terms. However, $d(z)$ in (26) contains an arbitrary bounded function $d_{n+1}(z)$, and hence the above terms must be independent of $d_{n+1}(z)$, and they must depend only on $a_n(z)$ and $b_n(z)$. Thus, for every arbitrary bounded function of $d_{n+1}(z)$, we must have the interpolation property

$$d(z) = \frac{b_n(z) + z\, \widetilde{a}_n(z) d_{n+1}(z)}{a_n(z) + z\, \widetilde{b}_n(z) d_{n+1}(z)} = \sum_{k=0}^{n} d_k z^k + O(z^{n+1}), \quad (36)$$

and the d_k's, $k = 0 \rightarrow n$, can be determined from the Schur polynomials $a_k(z)$, $b_k(z)$ in (26)-(32).

Conversely, (36) is completely specified by the first (n + 1) coefficients $\{d_k\}_{k=0}^{n}$, or from the Schur polynomials $a_n(z)$ and $b_n(z)$. To complete the recurions in (29)-(32), only the coefficients s_k, $k = 0 \rightarrow n$, are required and they can be obtained recursively from the given data $d_k = 0 \rightarrow n$[17]

4 A Hurwitz polynomial is free of zeros in $|z| < 1$, and a strict Hurwitz polynomial is free of zeros in $|z| \leq 1$.

$$s_n = \frac{\sum_{k=0}^{n-1} a_k^{(n-1)} d_{n-k}}{1-\sum_{k=0}^{n-1} b_k^{(n-1)*} d_k} = \frac{\left\{ a_{n-1}(z) \sum_{k=1}^{n} d_k z^k \right\}_n}{1-\left\{ \tilde{b}_{n-1}(z) \sum_{k=1}^{n-1} d_k z^k \right\}_{n-1}},$$

$n \geq 1.$ (37)

where $\{\ \}_n$ represents the coefficient of z^n in $\{\ \}$. Using this, $a_n(z)$ and $b_n(z)$ can be computed recursively, and the class of all bounded functions that interporate the given coefficients d_k, $k=0\rightarrow n$, is given by $d(z)$ in (36)[5].

3. Parametrization of Stable Systems

In general, the given impulse response data h_k, $k=0\rightarrow n$, do not form part of a bounded function, and to make use of the above formulation in section II, it is necessary to 'prepare' this data so that it confirms with a bounded function. To attain this goal, consider the matrix

$$H_n = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_1 & h_{n-1} & \cdots & h_0 \end{bmatrix}, \quad (38)$$

and let $\lambda_1^2(n)$ represent the largest eigenvalue of $H_n H_n^*$. Then, clearly, the sequence

$$d_k = \frac{h_k}{\chi_n},\ \chi_n > \lambda_1(n),\ k=0\rightarrow n, \quad (39)$$

satisfies (7) with inequality, and hence qualifies as the first n + 1 coefficients of a bounded function[6]. Recursive determination of the coefficients s_k, $k=0\rightarrow n$ from (A.7)-(A.8), together with $a_k(z)$, $k=0\rightarrow n$ using (29)-(30), gives

$$H(z) = \chi_n \cdot \frac{b_n(z) + z\,\tilde{a}_n(z) d_{n+1}(z)}{a_n(z) + z\,\tilde{b}_n(z) d_{n+1}(z)} = \sum_{k=0}^{n} h_k z^k + O(z^{n+1}) \quad (40)$$

to be the class of all transfer functions that are analytic in $|z|<1$, free of poles in $|z|\leq 1$ and interpolate the given partial impulse response sequence h_k, $k=0\rightarrow n$. Equation (40) can be given two interesting interpretations: First, if a system transfer function H(z) is rational to start with, then its representation as in (40) after n steps of the Schur algorithm will imply that $d_{n+1}(z)$ must be a rational function. Similiarly if $H(z)$ is nonrational to start with, then $d_{n+1}(z)$ must be nonrational in (40).

The alternate interpretation shows that given h_0, $h_1,\cdots h_n$, equation (40) represents all stable system transfer functions both rational and nonrational that interpolate the given data, and they can be obtained by varying $d_{n+1}(z)$ over all bounded functions. Thus even if the given data corresponds to a nonrational system, the freedom present in the choice of $d_{n+1}(z)$ in (40) can be utilized for rational approximation of H(z) by appropriate choice of rational bounded functions $d_{n+1}(z)$.

The above discussion shows that $d_{n+1}(z)$ can be utilized for rational system identification as well as rational approximation of nonrational systems. In particular, if $d_{n+1}(z)$ is chosen to be a rational bounded function, then since $d_{n+1}(z)$ and $H(z)$ are free of poles in $|z|\leq 1$, $H(z)$ in (40) represents a stable regular rational transfer function (analytic in $|z|\leq 1$) that matches the given coefficients[7]. As a result, the class of all stable rational functions that interpolate the given impulse response sequence is obtained from (40) by varying $d_{n+1}(z)$ over all rational bounded functions.

Ⅳ. The Rational Case

If $d(z)$ in (26) is rational to start with, as remarked earlier repeated application of the Schur procedure will result in rational bounded functions $d_{n+1}(z)$ that satisfy the degree constraint in (31). As a

5 If we let $d_{n+1}(z) = A(z)$, a regular (analytic in $|z|\leq 1$) all-pass function in (37), then $d(z)$ generates every all-pass function that satisfies the above interpolation property. In general, if $A(z)$ is a regular rational all-pass function, the denominator in (37) is only Hurwitz, and, hence, it can possess zeros on the unit circle. However, the reciprocal nature of the pole/zero pairs in an allpass function makes $d(z)$ in its irreducible form a regular rational all-pass function. Notice that the minimum degree of such a regular rational all-pass function is n + 1 and it corresponds to $d_{n+1}(z) = \pm 1$. Since only (n + 1) coefficients are matched by any such all-pass function, it follows that no stable all-pass approximations are possible in the z-domain in the Pade sense.

6 If $x_n = \lambda_1(n)$ in (49), then the singularity of J_n in (42) forces a unique all pass solution with degree equal to the rank of J_n.

7 $d_{n+1}(z)$ nonrational implies $H(z)$ is also nonrational. In that case, although is analytic $H(z)$ in $|z|<1$, as an example due to Fejer shows, a priori, $H(z)$ need not represent a stable system [14, 19].

result, from (36) and (40), it follows that every stable rational function $H(z)$ can be represented as in (40) where $d_{n+1}(z)$ is a unique rational bounded function that satisfies

$$\delta(d_{n+1}(z)) \leq \delta(H(z)). \tag{41}$$

and degree reduction in (41) happens according to (19) (20). Thus if $H(z)$ represents a stable ARMA(p, q) system with $p > q$ in (40), then $\delta(H(z)) = p$ and since $(b_0 b_p^*) = 0$, it follows from (20), (31) that

$$\delta(d_{n+1}(z)) = p \tag{42}$$

and further using the degree arguments as in (21)-(22), we obtain[8]

$$d_{n+1}(z) = \frac{f(z)}{g(z)} = \frac{f_0 + f_1 z + \cdots + f_{p-1} z^{p-1}}{1 + g_1 z + g_2 z + \cdots + g_p z^p},\ n \geq 1. \tag{43}$$

(Since $g(z)$ is strict Hurwitz, $g_0 \neq 0$ and it is normalized here to unity.) Substituting (43) into (40) we get

$$H(z) = \chi_n \cdot \frac{b_n(z) g(z) + z f(z) \widetilde{a}_n(z)}{a_n(z) g(z) + z f(z) \widetilde{b}_n(z)} = \sum_{k=0}^{n} h_k z^k + O(z^{n+1}). \tag{44}$$

Since every rational system after repeated application of the Schur procedure has the above representation for any n, where f(z)/g(z) is a unique rational bounded function as in (43), we can make use of the degree constraint of H(z) in (44) to obtain this unknown bounded function. Towards this, notice that the formal degree of both the numerator and denominator of (44) is n + p, and to respect the ARMA(p, q) nature of H(z), we first equate the denominator coefficients of $z^{p+1}, z^{p+2}, \cdots, z^{p+n}$ to zero. However, as shown in [17], equating the coefficients of $z^{p+1}, z^{p+2}, \cdots, z^{p+n}$ in the denominator to zero implies that the respective coefficients in the numerator are also zeros. As a result, we obtain n equations from the denominator coefficients of $z^{p+1}, z^{p+2}, \cdots, z^{p+n}$ and p-q equations from the remaining numerator coefficients of $z^{q+1}, \cdots, z^p$. Thus we have n + p-q equations and 2p unknowns $g_k = 1 \rightarrow p$ and $f_k = 0 \rightarrow p-1$. Clearly the minimum value of n is given by n = p + q and in that case the resulting 2p equations in 2p unknowns can be represented in matrix form as

$$Ax = b \tag{45}$$

where
A=

$$\begin{bmatrix} a_{p+q} & 0 & \cdots & 0 & b_0^* & 0 & \cdots & 0 \\ a_{p+q-1} & a_{p+q} & \cdots & 0 & b_1^* & b_0^* & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{q+1} & a_{q+2} & \cdots & a_{p+q} & b_{p-1}^* & b_{p-2}^* & \cdots & b_0^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & \cdots & a_p & b_{p+q-1}^* & b_{p+q-2}^* & \cdots & b_q^* \\ b_0 & b_1 & \cdots & b_{p-1} & a_{p+q}^* & a_{p+q-1}^* & \cdots & a_{q+1}^* \\ 0 & b_0 & \cdots & b_{p-2} & 0 & a_{p+q}^* & \cdots & a_{q+2}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{q+1} & 0 & 0 & \cdots & a_{p-1}^* \\ 0 & 0 & \cdots & b_q & 0 & 0 & \cdots & a_p^* \end{bmatrix}, \tag{46}$$

$$x = [g_p\ g_{p-1} \cdots g_2\ g_1\ f_{p-1}\ f_{p-2} \cdots f_1\ f_0] \tag{47}$$

and

$$b = [0\ 0 \cdots 0\ a_{p+q} \cdots a_{p+1}\ b_p\ b_{p-1} \cdots b_{q+1}] \tag{48}$$

Here a_k, b_k, $k = 0 \rightarrow p+q$ represent the coefficients of the p + q degree Schur polynomials $a_{p+q}(z)$ and $b_{p+q}(z)$ respectively, i.e.,

$$a_{p+q}(z) = a_0 + a_1 z + a_2 z + \cdots + a_{p+q} z^{p+q}, \tag{49}$$

and

$$b_{p+q}(z) = b_0 + b_1 z + b_2 z + \cdots + b_{p+q} z^{p+q}. \tag{50}$$

Note that p + q represents the minimum value for the available data points n, and if a larger number of such data is available, then these equations can be modified to accomodate that situation leading to an overdetermined system of equations in (45). At the correct stage, (45) is guaranteed to have a unique solution that results in a bounded function for $f(z)/g(z)$, and the unknown system parameters of $H(z)$ can be expressed in terms of the g_k's and f_k's so obtained. In fact, from (44), with

$$H(z) = \frac{Q(z)}{P(z)} = \frac{Q_0 + Q_1 z + \cdots + Q_q z^q}{P_0 + P_1 z + \cdots + P_p z^p} = \sum_{k=0}^{p+q} h_k z^k + O(z^{p+q+1}) \tag{51}$$

8 The coefficients of $f(z)$ and $g(z)$ in (53) are generic, and they differ from that in (22).

we get

$$P_k = \sum_{i=0}^{k} a_i g_{k-i} + \sum_{i=0}^{k-1} b_{n-i}^{*} f_{k-1-i},\ k=0 \rightarrow p \quad (52)$$

and

$$Q_k = \chi_{p+q} \left(\sum_{i=0}^{k} b_i g_{k-i} + \sum_{i=0}^{k-1} a_{n-i}^{*} f_{k-1-i} \right),\ k=0 \rightarrow q. \quad (53)$$

Clearly, stability of $H(z)$ and the interpolation property follows from the bounded character of $d_{n+1}(z)$, and since $P(z)$ are computed without involving any spectral factorization, the nonminimum phase characteristics of $H(z)$ if any is also preserved here. Finally, to show the uniqueness of the ARMA(p, q) from in (41), it can be shown that no further degree reduction is possible since common factor cancellation in its numerator and denominator does not occur [17].

Equations (43)-(53) can be implemented provided p and q are known. Usually the model order (p, q) is unknown, and that will have to be evaluated from the given data. As we show below, the invariant characteristics of the rational bounded function $d_{n+1}(z)$ in (42), together with the Schur update rule in (25) can be used to determine the model order.

4.1 *Model Order Selection*

Having determined $d_{p+q+1}(z) = f(z)/g(z)$ as in (45)-(48), the bounded function $d_{p+q+2}(z)$ at the next stage $(n=p+q+1)$ can be evaluated in a similar manner from the Schur polynomials $a_{p+q+1}(z)$ and. $b_{p+q+1}(z)$. In fact, letting

$$d_{p+q+2}(z) = \frac{c(z)}{e(z)} = \frac{\sum_{k=0}^{p-1} c_k z^k}{\sum_{k=0}^{p-1} e_k z^k} \quad (54)$$

from (44), we also have

$$H(z) = \chi_{p+q+1} \cdot \frac{b_{p+q+1}(z)\, e(z) + z\, \tilde{a}_{p+q+1}(z)\, c(z)}{a_{p+q+1}(z)\, e(z) + z\, \tilde{b}_{p+q+1}(z)\, c(z)} \quad (55)$$

and as before $e(z)$ and $c(z)$ can be evaluated by equating the coefficients of $z^{p+1}, z^{p+2}, \cdots, z^{2p+q+1}$ in the denominator and $z^{q+1}, z^{q+2}, \cdots, z^{p}$ in the numerator to zero[9]. Once again, these equations possess a unique solution at the correct stage for the unknowns e_k, $k=1 \rightarrow p$, c_k, $k=0 \rightarrow p-1$, (with $e_0=1$), and they result in a bounded function in (44). Notice that both these bounded functions $d_{p+q+1}(z)$ and $d_{p+q+2}(z)$ are of degree p, have the same form as in (43), and are related through the Schur rule as in (25). $d_{p+q+2}(z)$ makes use of additional information h_{p+q+1} about the system, through the new Schur polynomials. Substituting these two bounded functions into (25) and simplifying, we obtain

$$\frac{f(z)}{g(z)} = \frac{f_0 e(z) + z c(z)}{e(z) + z f_0^{*} c(z)} \quad (56)$$

Equation (46) relates the coefficients of the bounded functions at two consecutive stages, and equating the ratios of like powers on both sides of (46) and rearranging, we obtain the conditions

$$\varepsilon_k(p, q) = 0,\ k=0 \rightarrow p-1,$$

where

$$\varepsilon_0(p, q) = f_0 e_p + c_{p-1} \quad (57)$$

and

$$\varepsilon_k(p, q) = \frac{f_0 e_k + c_{k-1}}{e_k + f_0^{*} c_{k-1}} - \frac{f_k}{g_k},\ k=1 \rightarrow p-1. \quad (58)$$

These conditions are a direct consequence of (42), and reflect the ARMA(p, q) nature of the problem. Since the first stage where (47) and (48) are satisfied occurs at the correct stage, by updating p and q sequentially starting with $p \geq 1$ and $q \leq p$, the true model order can be found as the smallest integers p and q that satisfy $\varepsilon_0(p, q) = 0$, or, more genreally

$$\varepsilon_0(p, q) = \sqrt{\sum_{k=0}^{m} |\varepsilon_k(p, q)|^2} = 0. \quad (59)$$

The key feature of a rational system -- its degree -- is exploited here in determining the true model order and system parameters.

4.2 Numerical Results

The nonminimum phase stable rational transfer function examples in Figs. 1-3 highlight all important aspects of the algorithm described earlier. In the first step, if $H(e^{j\theta})$ is known, h_k, $k=0 \rightarrow r$, can be

9 Although this results in (2p + 1) equations in 2p unknowns, since the coefficient of z^{2p+q+1} is the same as (77), the remaining 2p equations are implemented in our computations.

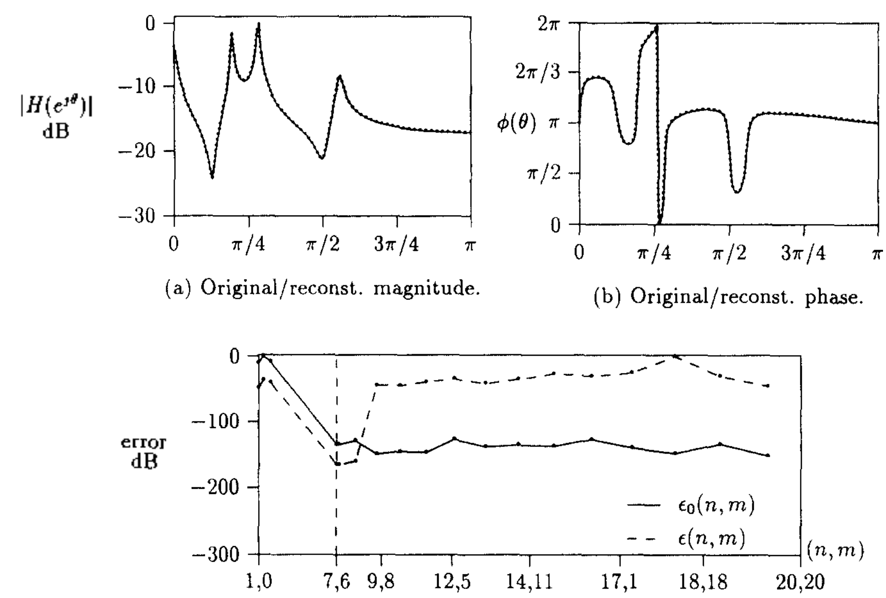
computed using the formula (Figs. 1 and 3)

$$h_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\theta}) e^{-jk\theta} d\theta, \tag{60}$$

and, if it is unknown, they can be estimated from the input-output data samples as (Fig. 2)

$$\hat{h}_k = \frac{1}{N} \sum_{n=1}^{N-k} x(n+k) w^*(n), \tag{61}$$

wher $x(n)$ represents the output of the unknown system $H(z)$ that has been excited by a stationary zero-mean white noise process $w(n)$. The unknown system is assumed to be ARMA(n, m) with $n \ge m$, and initialization begins with $n=1$, $m=0$. Preparation of the given impulse response sequence h_k, $k=0 \to n+m+1$, is first carried out to generate $d_k = h_k / x_{n+m+1}$, $k=0 \to n+m+1$ as in (38)-(39). Computation of the Schur polynomials $a_{n+m+1}(z)$, $b_{n+m+1}(z)$ using (29)-(30) and (37), followed by those of the bounded functions $d_{n+m+1}(z) = f(z)/g(z)$ and $d_{n+m+2}(z) = c(z)/e(z)$ then allow $\varepsilon_0(n, m)$ and $\varepsilon(n, m)$ to be evaluated using (57), (59), *provided both* $d_{n+m+1}(z)$ and $d_{n+m+2}(z)$ exist as bounded functions. The heavy dots on all curves in Figs. 1(c)-3(c) indicate the presence of such a stage (n, m), and if such is not the case that particular stage is skipped and the indices n and m are updated. Notice that $\varepsilon_0(n, m)$ and $\varepsilon(n, m)$ are guaranteed to exist at the correct stage $n=p$ and $m=q$, and since the first place where $\varepsilon_0(n, m)$ and $\varepsilon(n, m)$ equal zero also occurs at the correct stage, sequential updating of n and m continues until substantial relative minima in the values of $\varepsilon_0(n, m)$ and $\varepsilon(n, m)$ are observed to occur for the first time. The corresponding pair (n, m) is then identified as the model order (p, q) and the system parameters are computed from (51)-(53). Finally, to facilitate comparison, the exact magnitude $|H(e^{j\theta})|$ and its reconstructed counterpart (dotted) are plotted in Figs. 1(a)-3(a). Similarly, the exact phase $\phi(\theta)$ and the reconstructed phase (dotted) are plotted in Figs. 1(b)-3(b). Fig. 2 shows the reconstruction of an ARMA(5, 4) system from its input-output data



(a) Original/reconst. magnitude.

(b) Original/reconst. phase.

(c) Test criteria for model order selection

Fig 1. Reconstruction of a nonminimum phase ARMA(7, 6) model from its partial impulse response sequence. The original model corresponds to

$$H(z) = \frac{0.602 - 2.206z + 4.160z^2 - 4.952z^3 + 4.366z^4 - 2.721z^5 + 0.888z^6}{1.0 - 3.511z + 6.438z^2 - 8.052z^3 + 7.875z^4 - 6.018z^5 + 3.186z^6 - 0.888z^7}.$$

The reconstructed model is given by

$$H_r(z) = \frac{0.602 - 2.206z + 4.160z^2 - 4.952z^3 + 4.366z^4 - 2.721z^5 + 0.888z^6}{1.0 - 3.511z + 6.438z^2 - 8.052z^3 + 7.875z^4 - 6.018z^5 + 3.186z^6 - 0.888z^7}.$$

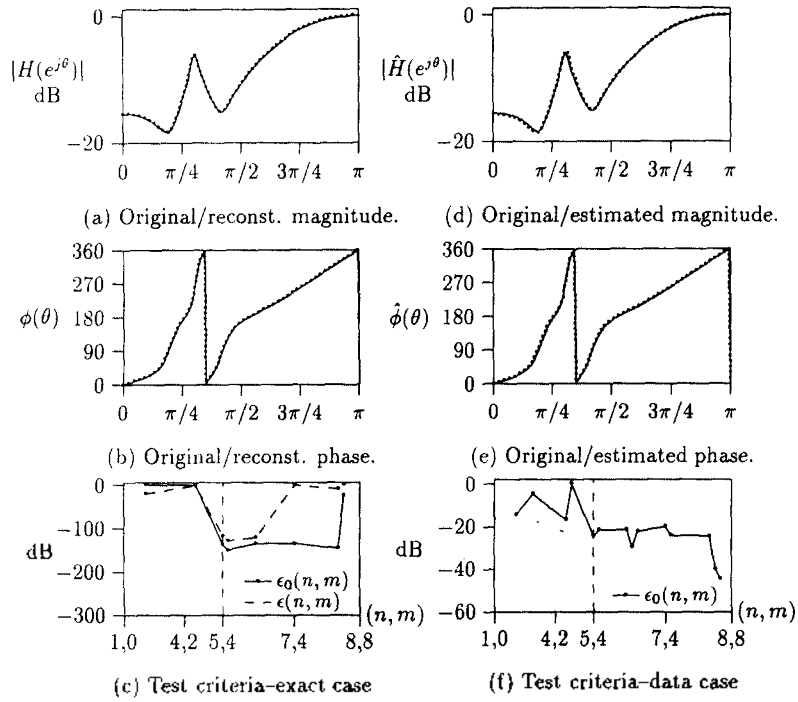


Fig 2. Reconstruction/estimation of a nonminimum phase ARMA(5, 4) system. The original model corresponds to

$$H(z)=\frac{0.0569-0.1365z+0.2038z^2-0.1724z^3+0.0907z^4}{1+0.0458z+0.2303z^2+0.5220z^3+0.3741z^4+0.0907z^5}\,.$$

The reconstructed model in (a)-(c) is given by

$$H_r(z)=\frac{0.0569-0.1365z+0.2038z^2-0.1724z^3+0.0907z^4}{1+0.0458z+0.2303z^2+0.5220z^3+0.3741z^4+0.0907z^5}\,.$$

The estimated model in (d)-(f) is given by

$$\hat{H}(z)=\frac{0.0572-0.1372z+0.2033z^2-0.1721z^3+0.0898z^4}{1+0.0208z+0.2354z^2+0.5284z^3+0.3724z^4+0.0779z^5}\,.$$

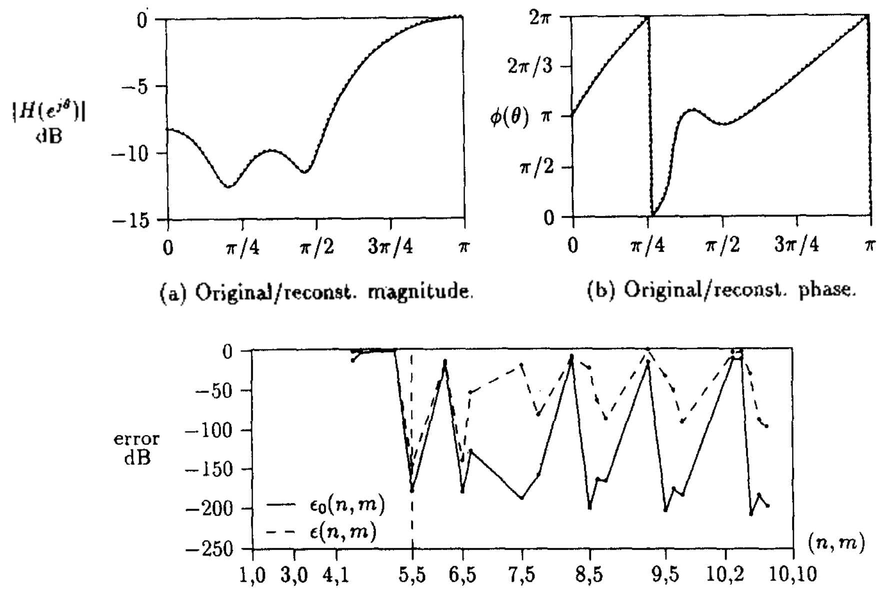
The coefficients of $\hat{H}(z)$ are estimated from 12 realizations each consisting of 600 data samples.

samples with $\hat{h}_k$ estimated using (61). Although the theoretical development in section III assumes $p \geq q$, as the MA(5) example in Fig. 3 shows, every case where $q < p$ can be detected as an ARMA(q, q) system. This means of course that some of the reconstructed coefficients in the denominator are filled in automatically as zeros, to raise the denominator degree to q.

V. Stable Rational Approximation of Nonrational Systems

A nonrational system has a transfer function that, unlike the rational systems, cannot be expressed as the ratio of two polynomials of finite degree. If such a system is stable, then it admits a power series expansion in $|z| < 1$, and the problem is to represent this by a rational system in some optimal manner. As remarked in the introduction, although Pade approximations can achieve this goal, such approximations need not guarantee stability. Moverover, the Hankel matrices generated from the impulse response data has no particular rank invariant structure in this case. In this context, once again we can make use of (40) to obtain all stable rational solutions to this problem.

To start with notice that the Schur extraction principle in (24)-(25) is perfectly general and hence if $H(z)$ is nonrational, then the representation in (40) is still valid, where $d_{n+1}(z)$ in that case represents



(a) Original/reconst. magnitude. (b) Original/reconst. phase.

(c) Test criteria for model order selection

Fig 3. Reconstruction of an MA(5) model from its partial impulse response sequence. The original model corresponds to

$$H(z)=0.8704-2.4714z+3.5782z^2-3.6560z^3+2.0357z^4-1.0z^5.$$

The reconstructed model is given by

$$H_r(z)=\frac{0.8704-2.4714z+3.5782z^2-3.6560z^3+2.0357z^4-1.0z^5}{1+2.1\times10^{-11}z-1.5\times10^{-14}z^2+4.1\times10^{-15}z^3+2.6\times10^{-14}z^4+7.2\times10^{-14}z^5}.$$

a nonrational bounded function. Interestingly, as remarked there, if $d_{n+1}(z)$ is replaced by any rational bounded function, we obtain a stable rationa transfer function $H_r(z)$ that interpolates the given impulse response sequence $\{h_k\}_{k=0}^{n}$. This key observation can be used to determine interpolating rational systems with minimum degree.

Since Pade approximations preserve the optimal degree character, if such approximations are also stable, then they must follow from (40) for a specific rational bounded function $d_{n+1}(z)$. To determine such bounded functions, let $d_{n+1}(z)=f(z)/g(z)$ represent a degree m bounded function that when substituted into (40) generates an ARMA(p, q) Pade-approximation $H_r(z)$. Thus

$$H_r(z)=\frac{Q(z)}{P(z)}=x_n\cdot\frac{b_n(z)\,g(z)+z\,\tilde{a}_n(z)\,f(z)}{a_n(z)\,g(z)+z\,\tilde{b}_n(z)\,f(z)}$$
$$=\sum_{k=0}^{n}h_kz^k+O(z^{n+1}). \quad (62)$$

For (62) to represent the Pade approximation, we must have $p+q\le n$, and once again to respect the ARMA(p, q) nature of $H_r(z)$, the polynomial $f(z)$ must be of degree $m-1$ and hence $g(z)$ must be of degree m. Thus the formal degrees of $P(z)$ and $Q(z)$ in (62) are $n+m$, provided $\delta(a_n(z))=n$, and hence the coefficients of $z^{p+1},\cdots,z^{n+m}$ in the denominator, and the coefficients of $z^{q+1},\cdots,z^{n+m}$ in the numerator must be zeros. As shown in Appendix-B, the coefficients of $z^{m+1},\cdots,z^{n+m}$ in the numerator and denominator generate the same equations and hence this gives $n+(m-p)+(m-q)=2m-(p+q)$ equations in $2m$ unknowns. Since $n\ge p+q$, there are at least $2m$ equations and they can be used to solve for the unknowns. From the above degree argument $m-p\ge0$ and $m-q\ge0$, or $m\ge\max(p,q)$, and hence for a given p, q(with $p\ge q$), the least complex bounded function $d_{n+1}(z)$ is also of degree p. In that case, the desired bounded function $d_{n+1}(z)$ has exactly the same form as in (43), and the system of equations so obtained has the functional representation in (45)-(48). However, unlike the rational case, the system of equations so obtained need not yield a solution for $g(z)$ and $f(z)$, and even if a solution exists there, $g(z)$ so obtained need not be strict Hurwitz, and further $f(z)/g(z)$ need not turn out to be a bounded function. However, for some p, q if $f(z)/g(z)$ turns out to be a bounded function, then $H_r(z)$ in (62) represents a stable ARMA(p,q) Pade approximation to the given nonrational function. Thus every stable Pade approximation to the given data has the representation

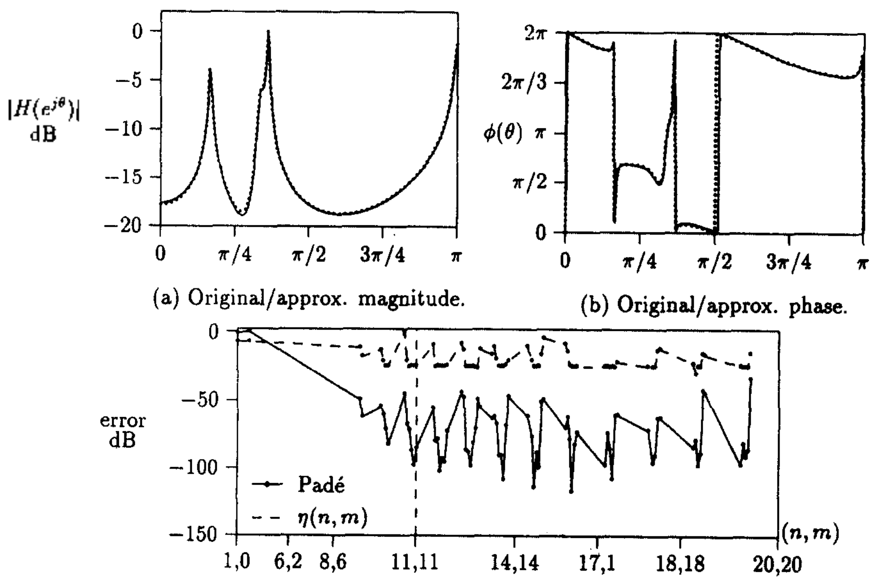
$$H_r(z) = \frac{Q(z)}{P(z)} = x_{p+q} \frac{b_{p+q}(z)\, g(z) + z\, \tilde{a}_{p+q}(z)\, f(z)}{a_{p+q}(z)\, g(z) + z\, \tilde{b}_{p+q}(z)\, f(z)} = \sum_{k=0}^{n} h_k z^k + O(z^{n+1}). \qquad (63)$$

where $n \geq p+q$, and $f(z)/g(z)$ represents a bounded function given by (45)-(48). We summarize the above observations as follows:

The necessary and sufficient condition for the existence of a stable ARMA(p, q) Pade approximation to the impulse response sequence $\{h_k\}_{p+q}^{k=0}$ is that the system of linear equations in (45)-(48) generated from the associated Schur polynomial coefficients yield a bounded solution of degree q for $f(z)/g(z)$ in (43). In that case, (51)-(53) and (63) represent the d esired stable transfer function.

Interestingly, the above remarks raise the following question: Given an $H(z)$ that represents a stable (nonrational) system transfer function, does there always axist a stable ARMA(p, q) Pade approximation for some p and q? Clearly, if such a solution exists, then that must follow from (63) with $n \geq p+q$ for a rational bounded function $f(z)/g(z)$ of degree p, that is obtained by solving the system of equations in (45)-(53). Note that due to the presence of the z factor in (63), $\delta(f(z)) \leq p-1$, $\delta(g(z)) = p$ are necessary conditions, provided $\delta(a_n(z)) = n$. Thus

$$f(z) = f_0 + f_1 z + \cdots + f_{p-1} z^{p-1}$$



(a) Original/approx. magnitude.

(b) Original/approx. phase.

(c) Test criteria for model order selection

Fig 4. Rational approximation of a nonrational system. The original nonrational transfer function is given by

$$H(z) = H_1(z)e^{-s} + H_2(z)e^{-(s-1)}$$

where

$$H_1(z) = \frac{1.1+z}{1.02-1.75z+z^2}, \quad H_2(z) = \frac{1.04-1.31z+z^2}{1.09-0.91z+1.06z^2+1.03z^3-0.86z^4+z^5}.$$

The stable Pad\'e approximation ARMA(11,11) model is given by $H_r(z) = B_m(z)/A_n(z)$, where

$$A_n(z) = 1-2.111z+2.351z^2-0.265z^3-1.869z^4+2.459z^5-1.078z^6$$
$$= +0.119z^7+0.228z^8+0.0557z^9+0.0059z^{10}+0.00026z^{11}.$$

$$B_m(z) = 3.571-9.609z+15.720z^2-12.974z^3+7.440z^4-2.243z^5+1.227z^6$$
$$= -0.555z^7+0.126z^8-0.0188z^9+0.0016z^{10}-0.00006z^{11}.$$

and

$$g(z) = g_0 + g_1 z + \cdots + g_p z^p$$

subject to the constraints $g_p \neq 0$ and $n \geq p+q$.

In fact, given h_k, $k=0 \rightarrow n$, there might exist no stable Pade approximation, and, moreover, the lowest nontrivial stable rational approximation that matches the given data might be of ARMA(n, n) form which corresponds to $d_{n+1} \equiv 0$ in (40). However, so long as $f(z)/g(z)$ is chosen to be a rational bounded function in (62), it represents the class of all stable rational transfer functions that interpolate $h_0, h_1, \cdots h_n$. Thus even if stable Pade approximations and absent in a particular situation, by relaxing the Pade con straint, other stable rational approximations can be obtained.

Figs. 4-5 as well as our extensive computations involving nonrational systems containing logarithmic and essential singularities seem to indicate that stable Pade approximations always exist. In the simulations presented here, the original nonrational function $H(z)$ is used to compute h_k, $k=0 \rightarrow p+q$, and it is first 'prepared' to generate $d_k = 0 \rightarrow p+q$, as in (38)-(39) and thereby the Schur polynomials $a_{p+q}(z)$ and $b_{p+q}(z)$ are generated recursively. From the above theorem, since a stable ARMA(p, q) Pad\'e approximation to this data must follow from (63) for a bounded solution of $f(z)/g(z)$ given by (45)-(48), those equations are verified for such a solution, and the indices are updated in a sequential manner. The heavy dots in Figs 4(c)-5(c) indicate the presence of such a stage, where (45)-(48) yield a bounded solution for $d_{p+q+1}(z) = f(z)/g(z)$ that results in a stable Pade approximation.

Further, if the two consecutive rational functions $d_{p+q+1}(z)$ and $d_{p+q+2}(z)$ turn out to be bounded and are related as in the Schur algorithm, then $\varepsilon_0(p, q)$, $\varepsilon_k(p, q)$ will be zero indicating the "near rational" nature of the data. When $H(z)$ is known in advance, the percent spectral error

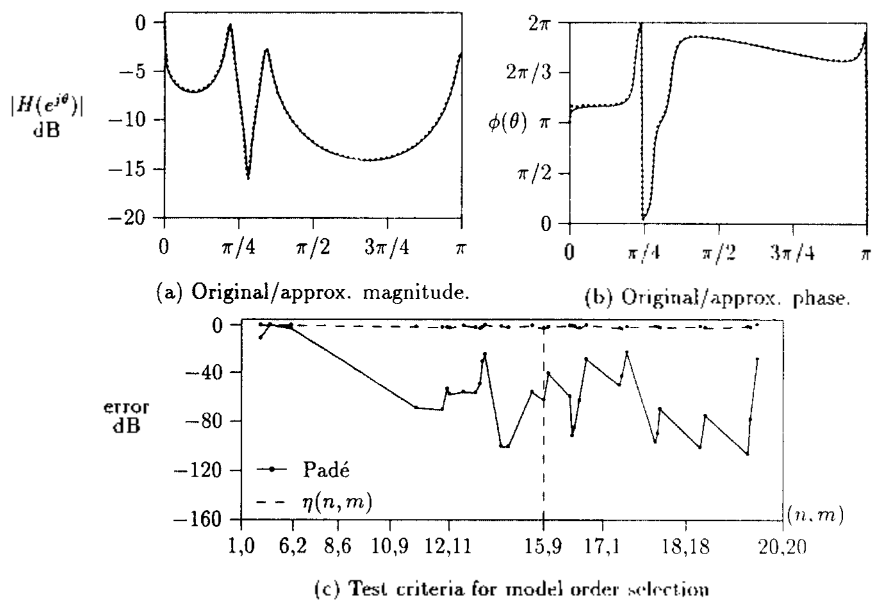


Fig 5. Rational approximation of a nonrational system with a logarithmic singularity at $z=1$. The original nonrational transfer function is given by

$$H(z) = \frac{0.797 - 1.036z + 0.829z^2}{1 - 1.470z + 0.946z^2 + 0.841z^3 - 1.277z^4 + 0.829z^5} \ln(1-z)$$

The stable Pade ARMA(15, 9) model is given by $H_r(z) = B_m(z)/A_n(z)$, where

$$A_n(z) = 1 - 4.979z + 10.732z^2 - 11.928z^3 + 4.428z^4 + 6.348z^5 - 11.090z^6 + 7.886z^7 - 2.597z^8 - 0.022z^9 + 0.293z^{10} - 0.075z^{11} + 0.0038z^{12} + 0.0003z^{13} + 0.000003z^{14} - 7.0 \times 10^{-7} z^{15},$$

$$B_m(z) = -0.0093 - 0.759z + 3.372z^2 - 6.456z^3 + 6.828z^4 - 3.958z^5 + 0.898z^6 + 0.211z^7 - 0.147z^8 + 0.020z^9.$$

$$\eta(p, q) = \sup_{\theta} \frac{\left| |H(e^{j\theta})|^2 - |H_r(e^{j\theta})|^2 \right|}{|H(e^{j\theta})|^2} \quad (64)$$

also may be used for model order selection. Notice that Fig. 5 represents a transcendental nonminimum phase system (zero at the origin) with a logarithmic singularity at $z = 1$. Nevertheless, as seen from Figs. 5(a)-(c), the ARMA(15, 9) Pade approximation is stable and preserves the nonminimum phase character of the original system. The abundance of stable Pade approximations are evident in Figs 4(c) 5(c). Nevertheless a rigorous proof is still lacking in the general case regarding the bounded character of $f(z)/g(z)$ for some p, q, and the issue remains unresolved.

V. Conclusions

This paper investigates the problem of obtaining all stable rational solutions that interpolate the given partial impulse response sequence by making use of the well known theory of bounded (Schur) functions. In this context, a new test criterion is developed to determine the model order of rational systems, and thereby determine their system parameters from the given impulse response sequence. The theory developed is further utilized to obtain the necessary and sufficient conditions for stable Pade approximations of nonrational systems. A practical algorithm is developed that translates the stability condition into the bounded character of a rational function generated from a set of linear equations obtained from the Schur polynomial coefficients associated with the given impulse response sequence. Interestingly, since the present technique does not make use of any factorization procedure, the nonminimum phase characteristics of the original system are preserved here.

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