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Asymptotic Gaussian Structures in a Critical Generalized Curie-Weiss Mean Field Model : Large Deviation Approach†

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Abstract

It has been known for mean field models that the limiting distribution reflecting the asymptotic behavior of the system is non-Gaussian at the critical state. Recently, however, Papangelow showed for the critical Curie-Weiss mean field model that there exist Gaussian structures in the asymptotic behavior of the total magnetization. We construct Gaussian structures existing in the internal fluctuation of the system for the critical case of a generalized Curie-Weiss mean field model.

Key Words : Curie-Weiss mean field model; Large deviation rate; Cumulant generating function.

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1. INTRODUCTION

In the area of statistical mechanics, a ferromagnetic crystal is considered as a body consisting of n sites, where n is an extremely large integer. The magnetic spins at these n sites can be modelled by a triangular array of random variables $\{X_i^{(n)} : i = 1, 2, \cdots, n\}(n = 1, 2, \cdots)$ and the total magnetization of the body is given by $S_n = \sum_{i=1}^n X_i^{(n)}$. A standard theory of physics would state that the joint distribution of the spins $(X_1^{(n)}, X_2^{(n)}, \cdots, X_n^{(n)})$ is given by

$$d\mu_n(x_1, x_2, \cdots, x_n) = z_n^{-1} \exp\{-\beta H_n(x_1, x_2, \cdots, x_n)\} \prod_{i=1}^{n} dP(x_i) \ , \qquad (1.1)$$

where z_n is a normalizing constant and $\beta(> 0)$ is a constant which plays the role of inverse temperature. The function H_n is known as the Hamiltonian which represents the energy of the body. When the Hamiltonian takes the particular form $H_n(x_1, \cdots, x_n) = -(\sum x_i)^2/2n$, the model (1.1) is usually called the Curie-Weiss mean field model and a number of probabilistic results have been established for this model. Ellis and Newman(1978b) showed, under appropriate conditions on the probability measure P , that there exist a real number m and a positive integer k so that $(S_n - nm)/n^{1-\frac{1}{2k}}$ converges in distribution to a random variable whose distribution is Gaussian if $k = 1$ and non-Gaussian if $k \geq 2$. It was also shown for the Curie-Weiss mean field model with $k \geq 2$ that the critical value of β, at which a phase transition occurs, equals 1. Chaganty and Sethuraman(1987) extended the result of Ellis and Newman for a more generalized model in which the Hamiltonian takes the following form

$$H_n(x_1, x_2, \cdots, x_n) = -n\psi_n\{(x_1 + x_2 + \cdots + x_n)/n\} \ , \qquad (1.2)$$

where ψ_n is the cumulant generating function of some suitable random variable. Their main tool was the large deviation local limit theorem for arbitrary sequence of random variables of Chaganty and Sethuraman(1985). Choi, Kim and Jeon(1989) also extended the result of Ellis and Newman and obtained same result utilizing the saddlepoint approximation for the probability density function of the sample mean, due to Daniels(1954). It was also revealed for the generalized model (1.2) that the limiting distribution reflecting the asymptotic behavior of S_n is non-Gaussian for the critical case. Recently, however, Papangelow(1989) showed for the critical Curie-Weiss mean field model that there exist Gaussian structures in the asymptotic behavior of S_n.

The purpose of this paper is to construct Gaussian structures in the asymptotic behavior of S_n, utilizing the large deviation local limit theorem of Chaganty and Sethuraman, for the critical case of the generalized Curie-Weiss mean field model (1.2). In section 2, we develop some notations and define the generalized Curie-Weiss mean field model (1.2) precisely. In section 3, the basic limit theorem is established and limit theorems on the asymptotic Gaussian structures are derived. We will prove that if there are $2n$ sites and we let $S_{n,1} = \sum_{i=1}^{n} X_i^{(2n)}$ and $S_{n,2} = \sum_{i=n+1}^{2n} X_i^{(2n)}$, then there exist sequences of real numbers $\{\tau_n\}$ and $\{\sigma_n\}$ such that the difference $(S_{n,1} - n\tau_n)/\sqrt{\sigma_n n} - (S_{n,2} - n\tau_n)/\sqrt{\sigma_n n}$ has an asymptotic normal distribution. To construct Gaussian structures for the generalized model (1.2), we use the results of Changanty and Sethuraman(1985,1987). Thus the proof is a little technical and complicated although many steps in the proof appear to be similar to those of Papangelow. Section 4 gives some examples.

2. A GENERALIZATION OF THE CURIE-WEISS MEAN FIELD MODEL

Let $\{T_n\}$ be a sequence of nonlattice valued random variables with the corresponding distributions Q_n and moment generating functions $\phi_n(s)$ finite for real s such that $|s| < c \leq \infty$. Assume that $\phi_n(z)$, $n \geq 1$, are analytic and nonvanishing for complex z in $\Omega = \{z : |z| < c_1\}$, where $0 < c_1 \leq c$. Define $I = (-a, a)$ and $\Omega_a = \{z : |z| < a\}$, where $0 < a < c_1$. Let $\gamma_n(t) = \sup_{|s|<c}[\, ts - \psi_n(s)\,]$, for $t \in R$, be the large deviation rate of the distribution Q_n and let $\psi_n(z) = n^{-1} \log \phi_n(z)$, for $z \in \Omega$. Let $A_n = \{\ \psi_n'(s) : s \in I\ \}$. Then, for $t \in A_n$, $\gamma_n(t) = ts_n - \psi_n(s_n)$, where $\psi_n'(s_n) = t$ and $s_n \in I$.

We will now define the generalized Curie-Weiss mean field model.

Definition 1. Let L^* be the class of probability distributions P on $(-c, c)$ such that

$$\int_{-c}^{c} \exp\{\psi_n(x)\}\, dP(x) < \infty\ . \tag{2.1}$$

Let $\{X_i^{(n)} : i = 1, 2, \cdots, n\}(n = 1, 2, \cdots)$ be a triangular array of random variables satisfying $|X_i^{(n)}| < c$ and having the joint distribution of the n-th row $(X_1^{(n)}, X_2^{(n)}, \cdots, X_n^{(n)})$ given by

$$d\mu_n^*(x_1, x_2, \cdots, x_n) = z_n^{-1} \exp\left\{n \cdot \psi_n\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right\} \prod_{i=1}^{n} dP(x_i)\ , \tag{2.2}$$

where $P \in L^*$ and z_n is a normalizing constant.

Remark 1. When T_n is the sum of n *i.i.d.* random variables whose common distribution is standard normal, the model (2.2) becomes the Curie-Weiss mean field model and if common distribution has moment generating function finite for all real values, the model reduces to the generalized model considered by Choi,Kim and Jeon(1989). In this sense, the model (2.2) is a generalized one of the Curie-Weiss mean field model.

For the probability distribution P which belongs to L^*, let $\psi_P(t)$ denote the cumulant generating function of P. Condition (2.1) implies that $\psi_P(t)$ is finite for $t \in B_n = \{ t : \gamma_n(t) < \infty \}$. For the probability distributions Q_n and $P(\in L^*)$, we define

$$G_n(t) = \begin{cases} \gamma_n(t) - \psi_P(t), & t \in B_n, \\ \infty, & t \notin B_n. \end{cases} \tag{2.3}$$

Definition 2. A real number m_n is said to be a global minimum for G_n if $G_n(t) \geq G_n(m_n)$ for all t.

Definition 3. A global minimum m_n for G_n is said to be of type k if

$$G_n(t + m_n) - G_n(m_n) \; = \; \frac{c_{2k,n} t^{2k}}{(2k)!} \; + \; o(|t|^{2k}) \quad \text{as} \quad t \to 0, \tag{2.4}$$

where $c_{2k,n} = G_n^{(2k)}(m_n)$ is strictly positive.

We assume that there exist l, $p_1(> 0)$ such that

$$\int_{B_n} \exp\{- \, l \cdot G_n(t)\} dt \; = \; O(n^{p_1}), \tag{2.5}$$

and the functions G_n have the unique global minimum at some point m_n. Furthermore, assume that there exists $\eta_1 > 0$ such that , for all $0 < \delta < \eta_1$,

$$\inf_{|t|>\delta} [\, G_n(t + m_n) - G_n(m_n) \,] \; = \; \min_{s=-1,1} [\, G_n(m_n + s\delta) - G_n(m_n) \,] \, . \tag{2.6}$$

Remark 2. An easily verifiable sufficient condition for (2.6) is

$$G_n'(t) > 0, \text{ for } t > m_n, \text{ and } G_n'(t) < 0, \text{ for } t < m_n. \tag{2.7}$$

In Example 2, we will verify condition (2.7) instead of condition (2.6).

Remark 3. Suppose that $B_n = (-\infty, \infty)$. If $\gamma_n(t)/|t|$ converges to ∞ as $|t| \to \infty$, then condition (2.1) implies condition (2.6). For the proof, see Remark 3.3 of Chaganty and Sethuraman(1987).

The following theorem of Chaganty and Sethuraman(1985) is a large deviation local limit theorem, which provides an asymptotic expansion for the density function of T_n/n, and applied crucially to obtaining the result of Chaganty and Sethuraman(1987), which is in turn used in Theorem 2. Let $m_n \in A_n$. Then there exists τ_n in I such that $\psi_n'(\tau_n) = m_n$. For $t \in I$, we define $V_n(t) = \psi_n(\tau_n) + itm_n - \psi_n(\tau_n + it)$.

Theorem 1. Assume the following conditions for T_n :
(A1) There exists $\epsilon > 0$ such that $|\psi_n(z)| < \epsilon$ for $z \in \Omega_a$ and $n \geq 1$.
(A2) There exists $\alpha > 0$ such that $\psi_n''(\tau) \geq \alpha$ for $\tau \in I$ and $n \geq 1$.
(A3) There exists $\eta > 0$ such that, for any $0 < \delta < \eta$,

$$\inf_{|t| \geq \delta} \{ \text{ Real}(V_n(t)) \} = \min \{ \text{ Real} (V_n(\delta), V_n(-\delta)) \}, \quad \text{for } n \geq 1.$$

(A4) There exists $p > 0$ such that

$$\sup_{\tau \in I} \int_{-\infty}^{\infty} \left| \frac{\phi_n(\tau + it)}{\phi_n(\tau)} \right|^{\frac{1}{n}} dt = O(n^p).$$

Then the density function k_n of T_n/n at m_n is given by

$$k_n(m_n) = \left[\frac{n}{2\pi \psi_n''(\tau_n)} \right]^{\frac{1}{2}} exp\{-n\gamma_n(m_n)\} \left[1 + O\left(\frac{1}{n}\right) \right]. \tag{2.8}$$

3. ASYMPTOTIC GAUSSIAN STRUCTURES IN THE CRITICAL GENERALIZED CURIE-WEISS MODEL

We construct in this section Gaussian structures of the asymptotic behavior of S_n for the critical case of the generalized model (2.2). We establish basic limit theorem,Theorem 2, and derive from this a few limit theorems which reveal the Gaussian structures of the asymptotic behavior of S_n. The joint distribution μ_n^* can be written as follows :

$$\begin{aligned} d\mu_n^*(x_1, x_2, \cdots, x_n) &= z_n^{-1}\phi_n\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\prod_{i=1}^n dP(x_i) \\ &= z_n^{-1}\left[\int \exp\left\{\left(\sum_{i=1}^n x_i\right)\cdot u\right\} k_n(u)du\right]\prod_{i=1}^n dP(x_i), \end{aligned}$$

where k_n is the density function of T_n/n.

The substitution $u = m_{n,k}(y) = m_n + yn^{-\frac{1}{2k}}$ leads to

$$\begin{aligned} & z_n^{-1}n^{-\frac{1}{2k}}\int\left[\prod_{i=1}^n \exp\{x_i m_{n,k}(y)\}\, dP(x_i)\right] k_n(m_{n,k}(y))dy \\ &= z_n^{-1}n^{-\frac{1}{2k}}\int\left[\prod_{i=1}^n \exp\{x_i m_{n,k}(y) - \psi_P(m_{n,k}(y))\}\, dP(x_i)\right] \\ &\qquad \times k_n(m_{n,k}(y))\cdot\exp\{n\psi_P(m_{n,k}(y))\}\, dy \\ &= \int\left[\prod_{i=1}^n dM_{n,y}(x_i)\right] f_n(y)dy\ , \end{aligned} \tag{3.1}$$

where $dM_{n,y}(x) = \exp\{xm_{n,k}(y) - \psi_P(m_{n,k}(y))\}\, dP(x)$,
and $f_n(y) = z_n^{-1}n^{-\frac{1}{2k}}k_n(m_{n,k}(y))\cdot\exp\{n\psi_P(m_{n,k}(y))\}$.

Since $\int_{R^n} d\mu_n^*(x_1, x_2, \cdots, x_n) = 1$ and $\int dM_{n,y}(x_i) = 1$ for each fixed y, we have $\int f_n(y)dy = 1$. Thus $f_n(y)$ is a density function for each n. The representation (3.1) therefore shows that we can introduce a new random variable W_n with the density function $f_n(y)$ such that, given $W_n = y$, the $X_i^{(n)}$'s are *i.i.d.* random variables with the common distribution $dM_{n,y}(x)$ and hence with the cumulant generating function :

$$\begin{aligned} & \log E_{M_{n,y}}\left[\ \exp\{tX_i^{(n)}\}\ \right] \\ &= \log\int\exp\{tx\}\cdot\exp\{\ x\cdot m_{n,k}(y) - \psi_P(m_{n,k}(y))\ \}\ dP(x) \\ &= \psi_P(t + m_{n,k}(y))\ -\ \psi_P(m_{n,k}(y))\ . \end{aligned} \tag{3.2}$$

For each $r = 1, 2, \cdots$, let $q(r)$ be a positive integer and let $n(r) = r\cdot q(r)$. We assume that the sequence $q(r)$, $r = 1, 2, \cdots$, is nondecreasing, and there is no loss of generality if we further assume that either $q(r)$ is constant or $q(r) \to \infty$ as $r \to \infty$. For convenience we will write n for $n(r)$ and q for $q(r)$ but it must be kept in mind that n and q depend on r. To investigate the joint distribution of

$$X_1^{(n)} + \cdots + X_r^{(n)}, \quad X_{r+1}^{(n)} + \cdots + X_{2r}^{(n)}, \quad \cdots, \quad X_{(q-1)r+1}^{(n)} + \cdots + X_{qr}^{(n)},$$

let $S_n = \sum_{i=1}^n X_i^{(n)}$, $S_{r,i} = S_{r,i}^{(n)} = \sum_{j=1}^r X_{(i-1)r+j}^{(n)}$, $i = 1, 2, \cdots, q$, and $p_j(r) = \sigma_n{}^{-\frac{1}{2}} \; r^{\frac{1}{2}} \; n^{-\frac{j}{2k}} \; \psi_P^{(j+1)}(m_n) \; / \; j\,!$, $r = 1, 2, \cdots$, $j = 1, 2, \cdots, k$. We will write p_j for $p_j(r)$ as with q and n. Further let $\tau_n = \psi_n'^{-1}(m_n) = \psi_P'(m_n)$ and $\sigma_n = \psi_P''(m_n)$.

Theorem 2. Let $(X_1^{(n)}, X_2^{(n)}, \cdots, X_n^{(n)})$ have the joint distribution μ_n^* given by (2.2). Suppose that G_n satisfies conditions (2.5) and (2.6) and has the unique global minimum of type $k(\geq 2)$ at $m_n(\in A_n)$. Let $m_n \to m$ and $c_{2k,n} = G_n^{(2k)}(m_n) \to c_{2k}$ as $n \to \infty$, where m is an interior point of $\cap_{n=1}^{\infty} A_n$. Assume that $\{T_n\}$ satisfies the conditions of Theorem 1. Then, as $r \to \infty$, the random variables

$$\frac{S_n - n\tau_n}{\sigma_n n^{1-\frac{1}{2k}}} \; ; \; \frac{S_{r,1} - r\tau_n}{\sqrt{\sigma_n r}} - \sum_{j=1}^{k} p_j W_n^j \; , \; \cdots, \; \frac{S_{r,q} - r\tau_n}{\sqrt{\sigma_n r}} - \sum_{j=1}^{k} p_j W_n^j \qquad (3.3)$$

are asymptotically independent, the density function of the asymptotic distribution of the first random variable is $\exp\{-c_{2k} y^{2k}/(2k)!\}/\int\{c_{2k} y^{2k}/(2k)!\}dy$, and the asymptotic distribution of $(S_{r,i} - r\tau_n)/\sqrt{\sigma_n r} - \sum_{j=1}^{k} p_j W_n^j$, i fixed, is the standard normal. The statement remains true if we replace the first random variable $(S_n - n\tau_n)/\sigma_n n^{1-\frac{1}{2k}}$ by W_n.

Proof. We first prove the second case with W_n in place of the first random variable $(S_n - n\tau_n)/\sigma_n n^{1-\frac{1}{2k}}$ in (3.3). The asymptotic distribution of W_n is established in Theorem 3.7 of Chaganty and Sethuraman(1987). Let $\{y_r\}$ be a sequence of real numbers such that $y_r \to y$ as $r \to \infty$. Then, by the representation (3.1) of the joint distribution μ_n^* , given $W_n = y_r$, the random variables

$$\frac{S_{r,1} - r\tau_n}{\sqrt{\sigma_n r}} - \sum_{j=1}^{k} p_j W_n^j, \; \cdots, \; \frac{S_{r,q} - r\tau_n}{\sqrt{\sigma_n r}} - \sum_{j=1}^{k} p_j W_n^j$$

are conditionally *i.i.d.*, each with cumulant generating function :

$$\log E\left[\exp\left\{ t\left(\frac{S_{r,1} - r\tau_n}{\sqrt{\sigma_n r}} - \sum_{j=1}^{k} p_j y_r^j \right) \right\} \right]$$

$$= r\left\{ \psi_P\left(\frac{t}{\sqrt{\sigma_n r}} + m_n + y_r n^{-\frac{1}{2k}}\right) - \psi_P\left(m_n + y_r n^{-\frac{1}{2k}}\right) \right\}$$

$$-\frac{r\tau_n t}{\sqrt{\sigma_n r}} - t\sum_{j=1}^{k} p_j y_r^j \qquad \text{by (3.2)}$$

$$= \frac{t}{\sqrt{\sigma_n}} r^{\frac{1}{2}} \left\{ \psi'_P(m_n + y_r n^{-\frac{1}{2k}}) - \psi'_P(m_n) \right\} + \frac{1}{2}\psi''_P(m_n + y_r n^{-\frac{1}{2k}})\,\frac{t^2}{\sigma_n}$$

$$-t\sum_{j=1}^{k} p_j y_r^j + o(1)$$

$$= \frac{t}{\sqrt{\sigma_n}} r^{\frac{1}{2}} \left\{ \sum_{j=1}^{k} \frac{\psi_P^{(j+1)}(m_n)}{j!} n^{-\frac{j}{2k}} y_r^j + o(r^{-\frac{1}{2}}) \right\} + \frac{1}{2}\psi''_P(m_n + y_r n^{-\frac{1}{2k}})\frac{t^2}{\sigma_n}$$

$$-t\sum_{j=1}^{k} p_j y_r^j + o(1)$$

$$= \frac{1}{2}\psi''_P(m_n + y_r n^{-\frac{1}{2k}})\frac{t^2}{\sigma_n} + o(1) \quad \text{as } r \to \infty.$$

This converges to $\frac{1}{2}t^2$ as $r \to \infty$ and hence it follows that the conditional distribution of $(S_{r,i} - r\tau_n)/\sqrt{\sigma_n r}$, given $W_n = y_r$, is asymptotically standard normal, *independently of* y. Thus the second half of the theorem now follows from Proposition of Papangelow(1989). In the special case where $q(r) = 1$, for all r, what we have just proved implies that

$$\frac{S_n - n\tau_n}{\sqrt{\sigma_n n}} - \sum_{j=1}^{k} p_j W_n^j \xrightarrow{d} N(0,1)\ , \tag{3.4}$$

and it can be easily shown by (3.4) that $\{S_n - n\tau_n\}/\sigma_n n^{1-\frac{1}{2k}} - W_n \xrightarrow{p} 0$. Since this implies the first part of the theorem, the proof is completed.

A number of consequences can be derived from Theorem 2. To state them we now introduce a sequence of independent random variables. Let $Y, Y_1, Y_2, \cdots$ be a sequence of independent random variables such that the density of Y is $const \cdot \exp\{-c_{2k}y^{2k}/(2k)!\}dy$, where $const$ is a normalizing constant, and each of $Y_1, Y_2, \cdots$ has the standard normal distribution.

Theorem 3. The random vector

$$\left(\frac{S_n - n\tau_n}{\sigma_n n^{1-\frac{1}{2k}}}\ ;\ \frac{S_{r,2} - r\tau_n}{\sqrt{\sigma_n r}} - \frac{S_{r,1} - r\tau_n}{\sqrt{\sigma_n r}},\ \cdots,\ \frac{S_{r,q} - r\tau_n}{\sqrt{\sigma_n r}} - \frac{S_{r,1} - r\tau_n}{\sqrt{\sigma_n r}}\right)$$

converges in distribution to $(Y; Y_2 - Y_1, Y_3 - Y_1, \cdots)$, where the latter contains finitely many or infinitely many components according as $q(r)$ is fixed or tends to ∞ as $r \to \infty$. If $q(r)$ is fixed, then the random vector

$$\left(\frac{S_n - n\tau_n}{\sigma_n n^{1-\frac{1}{2k}}}\ ;\ \frac{S_{r,1} - n\tau_n}{\sqrt{\sigma_n r}} - \frac{1}{\sqrt{q}}\frac{S_n - n\tau_n}{\sqrt{\sigma_n n}},\ \cdots,\ \frac{S_{r,q} - r\tau_n}{\sqrt{\sigma_n r}} - \frac{1}{\sqrt{q}}\frac{S_n - n\tau_n}{\sqrt{\sigma_n n}}\right)$$

converges in distribution to $\left(Y\ ;\ Y_1 - \frac{1}{q}\sum_{i=1}^q Y_i,\ \cdots,\ Y_q - \frac{1}{q}\sum_{i=1}^q Y_i\right)$.

Proof. The first part of the theorem follows from Theorem 2. For the last part, let $V_i^{(r)} = \{S_{r,i} - r\tau_n\} / \sqrt{\sigma_n r} - \sum_{j=1}^k p_j W_n^j$, $i = 1, 2, \cdots, q$. Then we have, for each i, $V_i^{(r)} - q^{-1}\sum_{i=1}^q V_i^{(r)} \xrightarrow{d} Y_i - q^{-1}\sum_{i=1}^q Y_i$ and

$$\begin{aligned} V_i^{(r)} - \frac{1}{q}\sum_{i=1}^q V_i^{(r)} &= \frac{S_{r,i} - r\tau_n}{\sqrt{\sigma_n r}} - \sum_{j=1}^k p_j W_n^j - \frac{1}{q}\sum_{i=1}^q\left(\frac{S_{r,i} - r\tau_n}{\sqrt{\sigma_n r}} - \sum_{j=1}^k p_j W_n^j\right) \\ &= \frac{S_{r,i} - r\tau_n}{\sqrt{\sigma_n r}} - \frac{1}{\sqrt{q}}\frac{S_n - n\tau_n}{\sqrt{\sigma_n n}}. \end{aligned}$$

Thus the last part is again an obvious consequence of Theorem 3.

The rate of the growth of $q(r)$ is crucial for the asymptotic behavior of partial sums. If $r^k/n \to 0$ as $r \to \infty$, then $p_j \to 0$ as $r \to \infty$, for each j, and if $n = o(r^k)$ as $r \to \infty$, then $q^{\frac{1}{2k}}\{S_{r,i} - r\tau_n\} / \sigma_n r^{1-\frac{1}{2k}} - W_n \xrightarrow{p} 0$. If $r^k/n \to c$ as $r \to \infty$, where $0 < c < \infty$, then $\sum_{j=1}^k p_j W_n^j \xrightarrow{d} \sqrt{\sigma} c^{\frac{1}{2k}} Y$, where $\sigma = \psi_P''(m)$. Thus we have the following corollaries as the consequences of Theorem 2.

Corollary 1. If $r^k/n \to 0$ as $r \to \infty$, then the random vector

$$\left(\frac{S_n - n\tau_n}{\sigma_n n^{1-\frac{1}{2k}}}\ ;\ \frac{S_{r,1} - r\tau_n}{\sqrt{\sigma_n r}}, \frac{S_{r,2} - r\tau_n}{\sqrt{\sigma_n r}},\ \cdots,\ \frac{S_{r,q} - r\tau_n}{\sqrt{\sigma_n r}}\right)$$

converges in distribution to the random sequence $(Y\ ;\ Y_1, Y_2, \cdots)$.

Corollary 2. If $r^k/n \to c$ as $r \to \infty$, where $0 < c < \infty$, then the random vector

$$\left(\frac{S_n - n\tau_n}{\sigma_n n^{1-\frac{1}{2k}}} \; ; \; \frac{S_{r,1} - r\tau_n}{\sqrt{\sigma_n r}}, \frac{S_{r,2} - r\tau_n}{\sqrt{\sigma_n r}}, \; \cdots, \; \frac{S_{r,q} - r\tau_n}{\sqrt{\sigma_n r}} \right)$$

converges in distribution to the random sequence

$$\left(Y \; ; \; \sqrt{\sigma} c^{\frac{1}{2k}} Y + Y_1, \sqrt{\sigma} c^{\frac{1}{2k}} Y + Y_2, \; \cdots \right).$$

Corollary 3. If $n = o(r^k)$ as $r \to \infty$, then

$$\left(\frac{S_n - n\tau_n}{\sigma_n n^{1-\frac{1}{2k}}} \; ; \; q^{\frac{1}{2k}} \frac{S_{r,1} - r\tau_n}{\sigma_n r^{1-\frac{1}{2k}}}, \cdots, \; q^{\frac{1}{2k}} \frac{S_{r,q} - r\tau_n}{\sigma_n r^{1-\frac{1}{2k}}} \right)$$

converges in distribution to $(Y \; ; \; Y, \; Y, \; \cdots)$, where either finitely many or infinitely many components are contained as in Theorem 3. In particular, if q is fixed, then the random vector

$$\left(\frac{S_n - n\tau_n}{\sigma_n n^{1-\frac{1}{2k}}} \; ; \; \frac{S_{r,1} - r\tau_n}{\sigma_n n^{1-\frac{1}{2k}}}, \cdots, \; \frac{S_{r,q} - r\tau_n}{\sigma_n n^{1-\frac{1}{2k}}} \right)$$

converges in distribution to $(Y \; ; \; q^{-1}Y, \; q^{-1}Y, \; \cdots, q^{-1}Y)$.

4. EXAMPLES

In all the examples, to simplify matters, let T_n be the sum of n *i.i.d.* random variables with common distribution function F. Then the functions $\psi_n \equiv \psi$, $\gamma_n \equiv \gamma$, $V_n \equiv V$, and $G_n \equiv G$ are independent of n and therefore it is straightforward to verify the conditions of Theorem 1.

Example 1. Let F be the standard normal and P be symmetric Bernoulli. Then the model (2.2) becomes Curie-Weiss mean field model as already mentioned. In this case, $\psi_n(t) \equiv \psi(t) = t^2/2$, $\gamma_n(t) \equiv \gamma(t) = \sup_s[ts - t^2/2] = t^2/2$, and $G_n(t) \equiv G(t) = t^2/2 - \log(\cosh(t)), t \in R$. Thus conditions (2.1) and (2.5) holds clearly. Since $B_n = (-\infty, \infty)$, condition (2.6) holds by Remark 3. Also, it can be shown by simple calculation that $G(t)$ has the unique global minimum of type 2 at the origin. Since $\tau_n \equiv \tau = \psi_P'(0) = 0$, $\sigma_n \equiv \sigma = \psi_P''(0) = 1$ and $c_4 = G^{(4)}(0) = 2$, we have, by Theorem 3, $S_n \,/\, n^{3/4} \xrightarrow{d} Y$, where the density of Y is $const \cdot \exp\{-y^4/12\}dy$ and

$$\frac{S_{r,i}}{\sqrt{r}} - \frac{S_{r,1}}{\sqrt{r}} \xrightarrow{d} N(0,2) \,, \quad \text{for each } i = 2, 3, \cdots, q.$$

Since $k = 2$, from Corollaries 1, 2 and 3, we have the following results:

$$\text{if } r^2/n \to 0 \text{ as r} \to \infty, \text{ then } \frac{S_{r,i}}{\sqrt{r}} \xrightarrow{d} N(0,1) \text{ for } i = 1,2,\cdots,q, \tag{4.1}$$

$$\text{if } r^2/n \to c \text{ as r} \to \infty, \text{ then } \frac{S_{r,i}}{\sqrt{r}} \xrightarrow{d} Y + N(0,1) \text{ for } i = 1,2,\cdots,q, \tag{4.2}$$

$$\text{and if } n = o(r^2) \text{ as } r \to \infty, \text{ then } q^{1/4} \cdot \frac{S_{r,i}}{r^{3/4}} \xrightarrow{d} Y \text{ for } i = 1,2,\cdots,q. \tag{4.3}$$

Example 2. Let F be the triangular distribution on the interval $(-2b, 2b)$ with $b = \sqrt{3/2}$ and P be the standard normal distribution. Then the joint distribution (2.2) is given by

$$d\mu_n^*(x_1, x_2, \cdots, x_n) = z_n^{-1} \left[\frac{n \cdot \sinh(b \sum x_i/n)}{b \sum x_i} \right]^{2n} \prod_{i=1}^{n} dP(x_i) .$$

In this case condition (2.1) is satisfied since we have, with $A > 1$,

$$\int_{-\infty}^{\infty} \exp\{\psi_n(x)\}\, dP(x)$$

$$\leq 2 \left\{ \int_0^A \left[\frac{\sinh(bx)}{bx} \right]^2 dP(x) + \int_A^{\infty} \sinh^2(bx) dP(x) \right\} < \infty .$$

To prove that $G(t)$ has the unique global minimum at the origin, note that $G(t) = \gamma(t) - t^2/2 = ts_o(t) - \psi(s_o(t)) - t^2/2$, where $s_o(t)$ is the unique solution of $\psi'(s) = t$, and $G'(t) = \gamma'(t) - t = s_o(t) - t$. Since $G(t)$ is an even function and $G(0) = 0$, it is sufficient to show that $G(t) > 0$ for $t > 0$. Since $G'(t) = s_o(t) - t > 0$ if and only if $t > \psi(t)$ and since $\psi(t) = 2\{b \cdot coth(bt) - 1/t\}$, we have only to prove that $(x^2+3)\sinh(x) - 3x \cdot \cosh(x) > 0$, for $x > 0$. Now we have, for $x > 0$,

$$\begin{aligned}(x^2+3)\sinh(x) - 3x \cdot \cosh(x) &= (x^2+3)\sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!} - 3x \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \\ &= 3 \sum_{j=1}^{\infty} \frac{(2j)^2}{3(2j+1)!} x^{2j+1} > 0 .\end{aligned}$$

Furthermore it can be easily shown by simple calculation that $G'(0) = G''(0) = G'''(0) = 0$ and $G^{(4)}(0) = \frac{3}{5} > 0$. Thus $G(t)$ has the unique global minimum of type 2 at the origin. Since $G(t)$ is even and have the unique global minimum at zero and since $G'(t) > 0$ for $t > 0$, condition (2.6) holds by (2.7). Since P is standard normal, $\tau_n \equiv \tau = \psi'_P(0) = 0$ and $\sigma_n \equiv \sigma = \psi''_P(0) = 1$. Thus, from Theorem 3, we get $S_n \,/\, n^{3/4} \xrightarrow{d} Y$, where the density function of Y is $const \cdot \exp\{\, -y^4/40 \,\}\, dy$ and

$$\frac{S_{r,i}}{\sqrt{r}} - \frac{S_{r,1}}{\sqrt{r}} \xrightarrow{d} N(0,2) \quad \text{for each } i = 1, 2, \cdots, q.$$

From Corollaries 1, 2 and 3, we have the same results as those in (4.1), (4.2) and (4.3).

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