Journal of the Korean
Statistical Society
Vol. 25, No. 4, 1996

Estimation of Random Coefficient AR(1) Model for Panel Data [†]

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Abstract

This paper deals with the problem of estimating the autoregressive random coefficient of a first-order random coefficient autoregressive time series model applied to panel data of time series.The autoregressive random coefficients across individual units are assumed to be a random sample from a truncated normal distribution with the space $(-1,\ 1)$ for stationarity. The estimates of random coefficients are obtained by an empirical Bayes procedure using the estimates of model parameters. Also, a Monte Carlo study is conducted to support the estimation procedure proposed in this paper. Finally, we apply our results to the economic panel data in Liu and Tiao(1980).

Key Words : Random coefficient autoregressive model; Panel data; Truncated normal distribution; Maximum likelihood estimate; Modified maximum likelihood estimate; Empirical Bayes estimation.

[†]This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1993.

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1. INTRODUCTION

Data collected on the same individual units in several periods are referred to as pooled cross-sectional and time series data or panel data. Linear regression models with random coefficients for panel data have very extensively been used in the analysis of economic, marketing, management, sociological, geological, environmental, biological, and industrial panel data.

Our interests in this paper are in time series model. The use of time series models with constant coefficients is quite common, but this is restrictive in most real situations. In reality such models are not well fitted to a set of real data. This necessiates the application of coefficients variation in model across the individual units. Thus, it is reasonable to suppose that random coefficients are random drawings from some population distribution.

The steady researches on the random coefficient autoregressive(RCAR) time series model have been shown in Nicholls and Quinn(1980, 1982), and Quinn and Nicholls(1981). This paper deals with RCAR time series model applied to specially panel data. As a research on the RCAR model for panel data Liu and Tiao(1980) discussed the Bayes estimation of random coefficient in the first-order RCAR model, where the autoregressive coefficients across units are regarded as a random sample from a beta distribution rescaled to the space $(-1,1)$ for stationarity. Li and Hui(1983) suggested non-parametric empirical Bayes estimation, similar to that of Martz and Krutchkoff(1969), of random coefficients in the pth-order RCAR model where the prior distribution of random coefficients is generally unknown but its support is restricted to $(-1,1)$ for stationarity. Kim and Basawa(1992) discussed an empirical Bayes estimation of random coefficient for a first-order RCAR model in which random coefficients are assumed to be normally distributed and studied large sample properties of estimates. Here, we note that they did not restrict the random coefficients to the stationary region $(-1,\ 1)$ but permitted to the region $(-\infty,\ \infty)$. All of them assumed in common that random errors in their model are normally distributed.

In this paper, we discuss the estimation of the model parameters and the autoregressive random coefficient of a first-order stationary RCAR model for panel data. We begin the next section by introducing the first-order stationary RCAR model dealed with in this paper in which random coefficients have a truncated normal distribution in $(-1,\ 1)$ for stationarity and random errors have a normal distribution. In Section 3, we discuss the method for estimating the unknown parameters of the model. Also, we obtain an empirical Bayes estimate of random coefficient using the estimates of parameters. We perform a simulation study to investigate the sample behaviors of estimates in Section 4. In Section 5, we applied our estimation method suggested in this paper to panel data of economic time series of Liu and Tiao(1980).

2. THE MODEL

The RCAR model used for panel data of m stationary time series observed from m individual units is as follows:

$$X_{j,t} = \phi_j X_{j,t-1} + \varepsilon_{j,t}, \qquad j = 1, 2, \ldots, m, \quad t = 1, 2, \ldots, n_j, \quad (2.1)$$

where $X_{j,t}$ is the tth observation of the jth series, $\{\varepsilon_{j,t}\}$ is a sequence of i.i.d. N(0, σ^2) random variables, and ϕ_j is the autoregressive random coefficient of the jth series such as $|\phi_j| < 1$ for stationarity. Furthermore, $\{\phi_j\}$ is independent of $\{\varepsilon_{j,t}\}$. Now, we assume that the ϕ_j's are independent drawings from the truncated normal distribution,

$$f(\phi_j; \beta, \delta) = \frac{\exp\{-(\phi_j - \beta)^2/2\delta\}}{\int_{-1}^{1} \exp\{-(t-\beta)^2/2\delta\} dt}, \qquad -1 < \phi_j < 1, \quad (2.2)$$

where $-\infty < \beta < \infty$ and $0 < \delta < \infty$.

3. THE ESTIMATION PROCEDURE

First, we discuss the estimation of parameters, β, δ, and σ^2, of the model (2.1). The likelihood function of $\phi_1, \phi_2, \ldots, \phi_m$, a sample of size m, following the probability density function(p.d.f) (2.2) is

$$L(\beta, \delta | \underline{\phi}) = \frac{\exp\{-\sum_{j=1}^{m} (\phi_j - \beta)^2/2\delta\}}{\left\{\int_{-1}^{1} \exp\{-(t-\beta)^2/2\delta\} dt\right\}^m}, \qquad (3.1)$$

where $-\infty < \beta < \infty$, $0 < \delta < \infty$. In order to obtain the maximum likelihood estimates(MLEs) of β and δ that maximize the likelihood function (3.1) we know that it is convenient to consider a monotone function of log likelihood,

$$
\begin{aligned}
L^*(\beta,\delta|\underline{\phi}) &= -2\ln L(\beta,\delta|\underline{\phi}) - m\ln 2\pi \\
&= m\ln\delta + \sum_{j=1}^{m}(\phi_j-\beta)^2/\delta + 2m\ln\{(\Phi(\frac{1-\beta}{\sqrt{\delta}}) - \Phi(\frac{-1-\beta}{\sqrt{\delta}})\},
\end{aligned} \tag{3.2}
$$

where $\Phi(t) = \int_{-\infty}^{t} e^{-y^2/2}/\sqrt{2\pi}dy$. Since we can not know the random coeffcients ϕ_j's, in our practical estimation procedure we substitute $\phi_j, j = 1,2,\ldots,m$, in (3.2) by the least square estimate of ϕ_j,

$$
\hat{\phi}_{j.LSE} = \frac{\sum_{t=2}^{n_j} X_{j,t}X_{j,t-1}}{\sum_{t=2}^{n_j} X_{j,t-1}^2}, \qquad j = 1,2,\ldots,m. \tag{3.3}
$$

Now, the solutions of β and δ which minimize $L^*(\beta,\delta|\hat{\underline{\phi}}_{LSE})$ are the MLEs of β and δ. Exactly speaking, they are the pseudo MLEs of β and δ.

Mittal and Dahiya(1987) showed that for some truncated normal samples the MLEs are nonexistent and become infinite with positive probability. When δ approaches ∞ for fixed β, $Var(\phi_j)$ approaches $0.3333\cdots$. Thus the MLEs don't exist whenever $Var(\phi_j) > 0.3333\cdots$. Mittal and Dahiya(1987) proposed the modified MLEs to rectify the infinity of MLEs. To modify the likelihood function they took the chi-square p.d.f. with the degree of freedom ν as the prior density for $1/\delta$ and the non-informative prior for β. We can also take the gamma p.d.f., the Wald p.d.f. or the Weibull p.d.f. as the prior density for $1/\delta$.

Now, the modified likelihood function which we shall use is

$$
L_M(\beta,\delta|\hat{\underline{\phi}}_{LSE}) = \left\{\frac{\delta^{-\frac{\nu}{2}+1}e^{-\frac{1}{2\delta}}}{\Gamma(\frac{\nu}{2})2^{\nu/2}}\right\}\cdot L(\beta,\delta|\hat{\underline{\phi}}_{LSE}). \tag{3.4}
$$

For appropriate ν, the modified pseudo MLEs of β and δ are the values of β and δ which minimize the monotone log function of (3.4),

$$
\begin{aligned}
L_M^*(\beta,\delta|\hat{\underline{\phi}}_{LSE}) &= -2\ln L_M(\beta,\delta|\hat{\underline{\phi}}_{LSE}) - m\ln 2\pi - 2\ln\{\Gamma(\frac{\nu}{2})2^{\nu/2}\} \\
&= (\nu+m-2)\ln\delta + \{\sum_{j=1}^{m}(\phi_j-\beta)^2+1\}/\delta \\
&\quad + 2m\ln\{(\Phi(\frac{1-\beta}{\sqrt{\delta}}) - \Phi(\frac{-1-\beta}{\sqrt{\delta}})\}.
\end{aligned} \tag{3.5}
$$

It is shown in Mittal and Dahiya(1987) that the value of $\nu = 4$ minimizes the maximum asymptotic bias of the modified MLE of δ for large values of δ.

We define the sample variance of $\hat{\phi}_{j.LSE}$, $j = 1, 2, \ldots, m$, as

$$S^2(\hat{\phi}) = \frac{1}{m-1}\sum_{j=1}^{m}(\hat{\phi}_{j.LSE} - \bar{\hat{\phi}})^2, \tag{3.6}$$

where

$$\bar{\hat{\phi}} = \frac{1}{m}\sum_{j=1}^{m}\hat{\phi}_{j.LSE}. \tag{3.7}$$

In the simulation study of Section 4, we have a strategy of obtaining the MLEs of β and δ which minimize the function $L^*(\beta, \delta|\hat{\underline{\phi}}_{LSE})$ in (3.2) if $S^2(\hat{\phi}) \le (0.95)(0.3333\cdots)$ and the modified pseudo MLEs of β and δ which minimize the function $L^*_M(\beta, \delta|\hat{\underline{\phi}}_{LSE})$ with $\nu = 4$ in (3.5) if $S^2(\hat{\phi}) > (0.95)(0.3333\cdots)$.

Finally, we use as the estimate of σ^2

$$\hat{\sigma}^2 = \frac{\sum_{j=1}^{m}\sum_{t=2}^{n_j}(\hat{\varepsilon}_{j,t} - \bar{\hat{\varepsilon}})^2}{\sum_{j=1}^{m} n_j - m - 1}, \tag{3.8}$$

where $\hat{\varepsilon}_{j,t} = X_{j,t} - \hat{\phi}_{j.LSE}X_{j,t-1}$ and $\bar{\hat{\varepsilon}} = \sum_{j=1}^{m}\sum_{t=2}^{n_j}\hat{\varepsilon}_{j,t}/(\sum_{j=1}^{m} n_j - m)$.

Next, we turn to the problem of obtaining the estimates of random coefficient. The joint p.d.f. of $\underline{X}_j = (X_{j,1}, X_{j,2}, \ldots, X_{j,n_j})$ for each j is

$$f(\underline{X}_j|\phi_j) = \prod_{t=1}^{n_j} f(\varepsilon_{j,t}) = (2\pi\sigma^2)^{-\frac{n_j}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{t=1}^{n_j}(X_{j,t} - \phi_j X_{j,t-1})^2\right\}. \tag{3.9}$$

From (2.2) and (3.9), the posterior p.d.f. of ϕ_j given $\underline{X}_j = (X_{j,1}, X_{j,2}, \cdots, X_{j,n_j})$ is

$$f(\phi_j|\underline{X}_j) = \frac{f(\phi_j)f(\underline{X}_j|\phi_j)}{\int_{-1}^{1} f(\phi_j)f(\underline{X}_j|\phi_j)d\phi_j} = \frac{\exp\{-\frac{1}{2\psi}(\phi_j - \xi)^2\}}{\int_{-1}^{1}\exp\{-\frac{1}{2\psi}(t - \xi)^2\}dt}, \tag{3.10}$$

where

$$\xi = \frac{\beta\sigma^2 + \delta\sum_{t=2}^{n_j} X_{j,t}X_{j,t-1}}{\sigma^2 + \delta\sum_{t=2}^{n_j} X_{j,t-1}^2} \quad \text{and} \quad \psi = \frac{\sigma^2\delta}{\sigma^2 + \delta\sum_{t=2}^{n_j} X_{j,t-1}^2}. \tag{3.11}$$

That is, the posterior p.d.f. of ϕ_j has a truncated normal distribution $N(\xi, \psi)$, where $-1 < \phi_j < 1$. Hence, from (3.10) and Johnson and Kotz(1970, p80-83) the Bayes estimate of ϕ_j with respect to the quadratic loss is given by

$$E(\phi_j|\underline{X}_j) = \xi + \sqrt{\psi}\left\{\frac{Z(\frac{-1-\xi}{\sqrt{\psi}}) - Z(\frac{1-\xi}{\sqrt{\psi}})}{\Phi(\frac{1-\xi}{\sqrt{\psi}}) - \Phi(\frac{-1-\xi}{\sqrt{\psi}})}\right\}, \tag{3.12}$$

where $Z(t) = \exp(-t^2/2)/\sqrt{2\pi}$. Substituting β, δ, and σ^2 of (3.12) by the estimates $\hat{\beta}, \hat{\delta}$, and $\hat{\sigma}^2$ obtained previously, we finally obtain an empirical Bayes estimate of ϕ_j,

$$\hat{\phi}_{j.EB} = E(\phi_j|\underline{X}_j)|_{\beta=\hat{\beta},\delta=\hat{\delta},\sigma^2=\hat{\sigma}^2}, \qquad j = 1, 2, \cdots, m. \tag{3.13}$$

Now, let's introduce other previous estimate of ϕ_j for the first-order stationary RCAR model which can be compared with $\hat{\phi}_{j.EB}$ in the next section. It is shown in Li and Hui(1983) that for sufficiently large n_j a nonparametric empirical Bayes estimate of ϕ_j is given by

$$\hat{\phi}_{j.EBLH} = \hat{\phi}_{j.LSE} + \frac{(1-\hat{\phi}^2_{j.LSE})(\bar{\hat{\phi}} - \hat{\phi}_{j.LSE})}{n_j S^2(\hat{\phi})}, \qquad j = 1, 2, \cdots, m, \tag{3.14}$$

where $S^2(\hat{\phi})$ and $\bar{\hat{\phi}}$ are defined in (3.6) and (3.7), respectively. The performance of three estimates, $\hat{\phi}_{j.EB}$, $\hat{\phi}_{j.EBLH}$, and $\hat{\phi}_{j.LSE}$ is investigated by a Monte Carlo study in the next section.

4. SIMULATION STUDY

In order to examine the sample behaviors of estimates, panel data of time series $\{X_{j,t}\}$, $t = 1, 2, \cdots, n_j$, for each $j = 1, 2, \cdots, m$, were simulated according to the model (2.1) with each values of $(\beta, \delta, \sigma^2) = (-0.8,\ 0.15,\ 1.0)$, $(0.5,\ 0.25,\ 0.5)$, and $(0.0,\ 0.35,\ 1.0)$. All experiments were accomplished using 1000 repititions on samples with different lengths, i.e., $(m, n) = (5, 6)$, $(10, 11)$, $(20, 21)$, $(30, 31)$, $(50, 51)$, $(100, 101)$, and $(300, 301)$, letting $n_j = n$ for simplicity. All except the last observation from individual units on each sample are used for estimation and the last observation is remained to be compared with the one-step ahead forecast obtained using the estimated coefficients. The exact procedure by which data were generated is presented in the Appendix.

Now, let's discuss the estimation of model parameters, β, δ, and σ^2. First, when $|\hat{\phi}_{j.LSE}| \geq 1.0$, $\hat{\phi}_{j.LSE}$ is corrected by its boundary value. Second, the equation (3.2) or (3.5) with $\nu = 4$ is minimized about β and δ according as $S^2(\hat{\phi}) \leq (0.95)(0.3333\cdots)$ or $S^2(\hat{\phi}) > (0.95)(0.3333\cdots)$ by the BCONG subroutine of IMSL running on UNIX workstation, which minimizes a function of two parameters subject to bounds on the parameters using a quasi-Newton method and the first-derivatives of the function. Since this subroutine requires starting values, a number of experiments were carried on some discrete grids of $(\beta,\ \delta)$. As a result, we can know that different starting values have little or no effect on minimizing the monotone log likelihood function.

In order to compare the estimates of parameters for various samples, (a) the true parameter values in the model, (b) the sample means of the estimates for the 1000 repititions, and (c) the sample standard deviation(s.d.)s of the corresponding sample means are calculated and summarized in Table 4.1. Also, the number within $(\cdot)$ is the number of repititions that $S^2(\hat{\phi}) > (0.95)(0.3333\cdots)$.

As for the models considered in this paper, we can see as follows: first, most of true parameters fail to be included within two sample s.d.'s of the corresponding sample means of estimates. But, the estimates of parameters generally converge to their true parameter values and their biases for δ and σ^2 consistently decrease with the increase of both m and n . Second, the sample s.d.'s of the corresponding sample means consistently decrease as m and n become larger at the same time. Third, in a number of experiments for other models in addition to the models in this paper we can know that for fixed β and σ^2 the proportion of out of range $S^2(\hat{\phi})$ increases as δ is larger or the sample size is smaller .

Next, let's discuss the estimation of random coefficient ϕ_j. We obtained three estimates: $\hat{\phi}_{j.EB}$ in (3.13), $\hat{\phi}_{j.EBLH}$ in (3.14), and $\hat{\phi}_{j.LSE}$ in (3.3). Now, for the purpose of evaluating the performance of estimates, we computed two criterion statistics used in Liu and Tiao(1980). One of two criterions is the average squared deviation(ASD) between the estimate $\hat{\phi}$ and the real value ϕ,

$$ASD(\hat{\phi}) = \frac{1}{1000m} \sum_{i=1}^{1000} \sum_{j=1}^{m} (\hat{\phi}_{i,j} - \phi_{i,j})^2, \tag{4.1}$$

where $\phi_{i,j}$ is a random coefficient for the jth individual unit on the ith repetition and $\hat{\phi}_{i,j}$ is an estimate of $\phi_{i,j}$. Another is the average squared prediction error(ASPE) between actual and one-step ahead prediction values under the estimate $\hat{\phi}$ considered,

$$ASPE(\hat{\phi}) = \frac{1}{1000m}\sum_{i=1}^{1000}\sum_{j=1}^{m}\{X_{i,j,t+1} - X_{i,j,t}^{(1)}\}^2, \quad (4.2)$$

where $X_{i,j,t+1}$ is an observation at time $t+1$ from the jth individual unit on the ith repitition and $X_{i,j,t}^{(1)} = \hat{\phi}_{i,j}X_{i,j,t}$ is an one-step ahead forcast at time t for the jth individual unit on the ith repitition.

The results of (a) the ASDs and (b) the ASPEs recorded in Table 4.2 indicate that $\hat{\phi}_{j.EB}$ is better or not worse in terms of the ASD and the ASPE than other estimates. Specially, for small samples of size $(m,n) = (5,5)$, $(10,10)$, $(20,20)$ $\hat{\phi}_{j.EB}$ performs better than other estimates do.

Table 4.1 Results of parameters estimaion on simulated samples.

m	n		β	δ	σ^2		β	δ	σ^2	
		(a)	-0.8000	0.1500	1.0000					
5	5	(b)	-0.9018	0.4559	0.8260	(c)	0.0171	0.0119	0.0091	(127)
10	10		-0.8980	0.3163	0.9015		0.0130	0.0081	0.0043	(2)
20	20		-0.8329	0.2136	0.9515		0.0094	0.0045	0.0022	(0)
30	30		-0.8118	0.1838	0.9674		0.0077	0.0034	0.0015	(0)
40	40		-0.8084	0.1740	0.9749		0.0064	0.0029	0.0011	(0)
50	50		-0.7955	0.1670	0.9799		0.0059	0.0026	0.0009	(0)
100	100		-0.7948	0.1564	0.9898		0.0040	0.0017	0.0004	(0)
300	300		-0.7973	0.1503	0.9968		0.0020	0.0009	0.0001	(0)
		(a)	0.5000	0.2500	0.5000					
5	5	(b)	0.5868	0.5351	0.4125	(c)	0.0183	0.0129	0.0046	(212)
10	10		0.6005	0.4314	0.4525		0.0137	0.0118	0.0022	(38)
20	20		0.5556	0.3410	0.4742		0.0096	0.0089	0.0011	(5)
30	30		0.5239	0.2950	0.4829		0.0073	0.0062	0.0007	(1)
40	40		0.5110	0.2782	0.4868		0.0059	0.0050	0.0005	(0)
50	50		0.5057	0.2686	0.4895		0.0051	0.0042	0.0004	(0)
100	100		0.4984	0.2561	0.4948		0.0032	0.0024	0.0002	(0)
300	300		0.4995	0.2504	0.4983		0.0017	0.0012	0.0001	(0)
		(a)	0.0000	0.3500	1.0000					
5	5	(b)	-0.0117	0.5742	0.8110	(c)	0.0201	0.0134	0.0089	(310)
10	10		-0.0135	0.5197	0.9089		0.0133	0.0131	0.0044	(125)
20	20		-0.0110	0.4462	0.9542		0.0076	0.0107	0.0022	(36)
30	30		-0.0009	0.4048	0.9668		0.0053	0.0088	0.0014	(16)
40	40		-0.0036	0.3928	0.9757		0.0043	0.0072	0.0011	(5)
50	50		-0.0047	0.3838	0.9810		0.0036	0.0066	0.0009	(2)
100	100		0.0022	0.3687	0.9906		0.0026	0.0042	0.0004	(0)
300	300		0.0005	0.3509	0.9966		0.0014	0.0018	0.0001	(0)

But, for the samples of larger size there are not significant differences among three estimates from the ASD and the ASPE viewpoint. In comparing the three estimates of random coefficient in the light of the simulation experiment results $\hat{\phi}_{EB}$ is better than other estimates. In conclusion, we can recommend $\hat{\phi}_{EB}$ for a small sample.

Table 4.2 Results of empirical Bayes estimation on simulated samples.

m	n		$\hat{\phi}_{EB}$	$\hat{\phi}_{EBLH}$	$\hat{\phi}_{LSE}$		$\hat{\phi}_{EB}$	$\hat{\phi}_{EBLH}$	$\hat{\phi}_{LSE}$
			$(\beta,\ \delta,\ \sigma^2) = (-0.800,\ 0.150,\ 1.000)$						
5	5	(a)	0.110	0.261	0.215	(b)	1.112	1.305	1.245
10	10		0.053	0.068	0.092		1.090	1.122	1.145
20	20		0.028	0.031	0.039		1.021	1.027	1.036
30	30		0.019	0.020	0.024		1.029	1.031	1.036
40	40		0.014	0.015	0.017		1.022	1.024	1.027
50	50		0.011	0.012	0.013		1.023	1.024	1.026
100	100		0.006	0.006	0.006		1.009	1.010	1.010
300	300		0.002	0.002	0.002		1.002	1.002	1.002
			$(\beta,\ \delta,\ \sigma^2) = (0.500,\ 0.250,\ 0.500)$						
5	5	(a)	0.134	0.254	0.220	(b)	0.571	0.647	0.643
10	10		0.070	0.081	0.091		0.554	0.565	0.572
20	20		0.037	0.039	0.043		0.524	0.525	0.528
30	30		0.025	0.025	0.027		0.521	0.522	0.523
40	40		0.019	0.019	0.020		0.512	0.513	0.513
50	50		0.015	0.015	0.016		0.509	0.510	0.510
100	100		0.007	0.007	0.008		0.505	0.505	0.505
300	300		0.002	0.002	0.002		0.503	0.503	0.503
			$(\beta,\ \delta,\ \sigma^2) = (0.000,\ 0.350,\ 1.000)$						
5	5	(a)	0.152	0.295	0.214	(b)	1.145	1.344	1.274
10	10		0.078	0.088	0.090		1.117	1.141	1.153
20	20		0.041	0.042	0.043		1.047	1.051	1.054
30	30		0.027	0.028	0.028		1.039	1.040	1.041
40	40		0.020	0.020	0.021		1.019	1.019	1.020
50	50		0.016	0.016	0.016		1.017	1.018	1.018
100	100		0.008	0.008	0.008		1.015	1.015	1.015
300	300		0.003	0.003	0.003		1.001	1.001	1.001

It will be interesting to observe the sample behaviors of the estimates according to several sizes, i.e. $m = 5, 15, 30, 50, 100, 300$, of individual units with fixed small sample size $n = 8$, 16, or 24, or vice versa, since the length of panel data often is short in size n. Because of the space limited we present the results of such simulations as previously conducted only for the model with $(\beta, \delta, \sigma^2) = (0.5, 0.25, 0.5)$ in Table 4.3-4.4.

Now we summarize the results as follows: first, the biases of the estimates for β and δ increase till the size $m = 15$ for $n = 16$ or $n = 24$ and $m = 50$ for $n = 8$. But they decrease with the increase of m since the size of m. As the reason for these results, we can guess that $S^2(\hat{\phi})$ also increases with the increase of relatively small size m, since the estimates $\hat{\phi}_j$'s are very rough in

Table 4.3 Results of parameters estimaion on samples simulated for the model with parameter $(\beta, \delta, \sigma^2) = (0.500,\ 0.250,\ 0.500)$.

		sample means			sample s.d.'s of means			
m	n	β	δ	σ^2	β	δ	σ^2	
5	8	0.548	0.400	0.438	0.016	0.014	0.004	(110)
15	8	0.653	0.524	0.443	0.014	0.013	0.002	(50)
30	8	0.654	0.543	0.442	0.012	0.012	0.001	(17)
50	8	0.660	0.556	0.442	0.011	0.011	0.001	(1)
100	8	0.627	0.517	0.440	0.008	0.008	0.001	(0)
300	8	0.576	0.469	0.441	0.004	0.004	0.000	(0)
5	16	0.529	0.293	0.467	0.013	0.012	0.002	(46)
15	16	0.571	0.368	0.470	0.011	0.011	0.001	(13)
30	16	0.538	0.348	0.470	0.009	0.008	0.001	(2)
50	16	0.515	0.326	0.469	0.007	0.006	0.001	(0)
100	16	0.491	0.305	0.469	0.004	0.003	0.001	(0)
300	16	0.496	0.293	0.469	0.002	0.002	0.000	(0)
5	24	0.516	0.274	0.479	0.012	0.010	0.002	(30)
15	24	0.568	0.338	0.479	0.011	0.009	0.001	(4)
30	24	0.524	0.303	0.480	0.008	0.007	0.001	(0)
50	24	0.507	0.291	0.479	0.006	0.005	0.001	(0)
100	24	0.496	0.277	0.479	0.004	0.003	0.000	(0)
300	24	0.498	0.269	0.480	0.002	0.001	0.000	(0)

Table 4.4 Results of empirical Bayes estimation on samples simulated for the model with parameter $(\beta, \delta, \sigma^2) = (0.500,\ 0.250,\ 0.500)$.

		ASD			ASPE		
m	n	ϕ_{EB}	ϕ_{EBLH}	ϕ_{LSE}	ϕ_{EB}	ϕ_{EBLH}	ϕ_{LSE}
5	8	0.091	0.151	0.121	0.566	0.608	0.593
15	8	0.083	0.097	0.121	0.541	0.558	0.572
30	8	0.081	0.091	0.122	0.551	0.567	0.584
50	8	0.079	0.089	0.121	0.542	0.559	0.577
100	8	0.079	0.088	0.121	0.548	0.564	0.584
300	8	0.079	0.088	0.121	0.548	0.564	0.584
5	16	0.049	0.068	0.054	0.529	0.544	0.536
15	16	0.045	0.048	0.054	0.534	0.536	0.542
30	16	0.045	0.047	0.055	0.529	0.532	0.538
50	16	0.045	0.047	0.055	0.529	0.533	0.538
100	16	0.045	0.046	0.055	0.525	0.528	0.534
300	16	0.044	0.046	0.054	0.528	0.531	0.536
5	24	0.032	0.039	0.034	0.532	0.538	0.534
15	24	0.031	0.032	0.034	0.520	0.522	0.524
30	24	0.031	0.032	0.035	0.515	0.517	0.519
50	24	0.031	0.032	0.034	0.516	0.518	0.519
100	24	0.030	0.031	0.034	0.518	0.520	0.521
300	24	0.030	0.031	0.034	0.518	0.519	0.521

the samples of small size n. Second, it is shown that there are little differences among the estimates of σ^2 in spite of the increase of m with fixed n. Third, in Table 4.4 $\hat{\phi}_{EB}$ is better or not worse in terms of the ASD and the ASPE than others. Specially, $\hat{\phi}_{EB}$ is superior to others for smaller sizes of m and n. Fourth, the sample s.d.'s of the corresponding sample means and the proportion of out of range $S^2(\hat{\phi})$ generally decrease with the increase of m for fixed n or vice versa.

5. APPLICATION

For a practical illustration, we used the annual average hourly earnings in non-durable goods manufacturing of $m = 14$ metropolitan areas in California from 1945 to 1977 studied in Liu and Tiao(1980).

Following after their job, we begin with the natural logarithm, $\{W_{j,t}\} = \{\ln X_{j,t}\}$, of the original series $\{X_{j,t}\}$ by the means of remedy for increasing variance with time. Also, in order to obtain a stationary panel of time series we find the difference series $\{Y_{j,t}\}$ of $W_{j,t}$'s, where $Y_{j,t} = W_{j,t} - W_{j,t-1}$. The difficulties in numerical optimization often occur when the data values are very small. Thus, the series with which we finally deal is the standardized series $\{Z_{j,t}\}$ with $Z_{j,t} = (Y_{j,t} - \bar{Y}_j)/S_j$, where $\bar{Y}_j = \sum_{t=2}^{n_j} Y_{j,t}/(n_j - 1)$ and $S_j = \sqrt{\sum_{t=2}^{n_j}(Y_{j,t} - \bar{Y}_j)^2/(n_j - 1)}$. Now, we construct our model as follows

$$Z_{j,t} = \phi_j Z_{j,t-1} + \varepsilon_{j,t}, \qquad j = 1, 2, \cdots, 14, \qquad t = 1, 2, \cdots, n_j. \tag{5.1}$$

Applying the results in Section 3 to model (5.1), we compute such ASPEs between actual and one-step ahead prediction values under three estimates, $\hat{\phi}_{j.EB}$, $\hat{\phi}_{j.EBLH}$, and $\hat{\phi}_{j.LSE}$, as

$$ASPE(\hat{\phi}) = \frac{1}{14}\sum_{j=1}^{14}\{W_{j,t+1} - W_{j,t}^{(1)}\}^2, \tag{5.2}$$

where $W_{j,t}^{(1)} = W_{j,t} + \bar{Y}_j + \hat{\phi}_j Z_{j,t} S_j$ is an one-step ahead forcast of $W_{j,t+1}$ at time t.

Table 5.1 Parameter estimates and average squared prediction errors(ASPE) in the analyses of natural logarithms of average hourly earnings data.

data	$\hat{\beta}$	$\hat{\delta}$	$\hat{\sigma}^2$	$\hat{\phi}_{EB}$	$\hat{\phi}_{EBLH}$	$\hat{\phi}_{LSE}$	$\hat{\phi}_{B}$
overall series(m=14)							
1945-68	0.597	0.080	0.121	0.000266	0.000788	0.000337	0.000349
1945-69	0.536	0.040	0.116	0.000404	0.000503	0.000421	0.000425
1945-70	0.514	0.036	0.112	0.000400	0.000481	0.000417	0.000691
1945-71	0.500	0.028	0.114	0.000721	0.001055	0.000740	0.001016
1945-72	0.478	0.027	0.116	0.000405	0.000668	0.000408	0.000428
1945-73	0.438	0.035	0.112	0.000417	0.000661	0.000364	0.000550
1945-74	0.409	0.041	0.115	0.001479	0.001428	0.001508	0.001804
1945-75	0.486	0.045	0.141	0.001423	0.001515	0.001374	0.001493
1945-76	0.530	0.076	0.144	0.000766	0.001255	0.001171	0.000894
total ASPE				0.006281	0.008353	0.006741	0.007650
long series(m=4)							
1945-68	0.529	0.012	0.177	0.000134	0.000255	0.000105	0.000128
1945-69	0.496	0.018	0.171	0.000067	0.000065	0.000070	0.000235
1945-70	0.498	0.018	0.167	0.000193	0.000168	0.000211	0.000396
1945-71	0.515	0.014	0.172	0.000201	0.000214	0.000204	0.000287
1945-72	0.521	0.014	0.170	0.000086	0.000100	0.000080	0.000071
1945-73	0.505	0.016	0.164	0.000158	0.000154	0.000159	0.000196
1945-74	0.500	0.015	0.171	0.002177	0.002202	0.002161	0.002349
1945-75	0.523	0.018	0.212	0.000077	0.000082	0.000126	0.000097
1945-76	0.523	0.012	0.208	0.000127	0.000134	0.000138	0.000118
total ASPE				0.003220	0.003373	0.003256	0.003877
short series(m=10)							
1945-68	0.694	0.137	0.053	0.000342	0.000691	0.000429	0.000438
1945-69	0.557	0.050	0.054	0.000543	0.000643	0.000561	0.000501
1945-70	0.523	0.044	0.058	0.000488	0.000554	0.000499	0.000809
1945-71	0.495	0.034	0.059	0.000928	0.001201	0.000955	0.001308
1945-72	0.461	0.031	0.068	0.000525	0.000752	0.000540	0.000571
1945-73	0.411	0.040	0.068	0.000499	0.000732	0.000446	0.000691
1945-74	0.372	0.046	0.070	0.001236	0.001179	0.001247	0.001586
1945-75	0.473	0.056	0.086	0.001943	0.002039	0.001873	0.002051
1945-76	0.560	0.057	0.096	0.001129	0.001658	0.001585	0.001204
total ASPE				0.007632	0.009450	0.008135	0.009159

When the prior distribution of ϕ_j is supposed to be a rescaled beta distribution, Liu and Tiao(1980) computed the ASPEs using the Bayes estimate $\hat{\phi}_{j.B}$ for three groups of data, i.e., the overall group(m=14) including all series, the long-series group(m=4) including series 1, 2, 5, 9, and the short-series group(m=10) including series 3, 4, 6, 7, 8, 10, 11, 12, 13, and 14. Table 5.1 shows not only the estimates $\hat{\beta}$, $\hat{\delta}$, $\hat{\sigma}^2$ obtained by the estimation method in Section 3 but also the ASPEs for one-step ahead forecast obtained using the estimates $\hat{\phi}_{j.EB}$, $\hat{\phi}_{j.EBLH}$, $\hat{\phi}_{j.LSE}$, and the Bayes estimate $\hat{\phi}_B$ of Liu and Tiao, respectively, for various time periods and three series groups.

We can observe that the $\hat{\phi}_{EB}$ generally gives smaller ASPEs than other estimates for an overall group or a short-series group of which series are relatively short in length. There is only a little difference among the ASPEs of three $\hat{\phi}$'s for a long-series group of which series are long in length as seen in the previous simulation results.

6. CONCLUDING REMARKS

We have discussed the estimation of model parameters and random coefficient in the first-order RCAR model where the prior distribution of random coefficients is a truncated normal distribution in $(-1,\ 1)$ for stationarity.

In this paper, the parameters are indirectly estimated using the estimate $\hat{\phi}_{LSE}$. To directly obtain the MLEs of β, δ, and σ^2 we find the conditional likelihood function $L(\beta, \delta, \sigma^2|\underline{X})$ given $X_{1,j}, j = 1, 2, \ldots, m$, which is more convenient than the likelihood function in the random coefficient model, where

$$\begin{aligned} L(\beta, \delta, \sigma^2|\underline{X}) &= \prod_{j=1}^{m} f(x_{j,2}, x_{j,3}, \cdots, x_{j,n_j}|x_{j,1}) \\ &= \prod_{j=1}^{m}\prod_{t=2}^{n_j} f(x_{j,t}|x_{j,t-1}) \\ &= \prod_{j=1}^{m}\prod_{t=2}^{n_j} \Pr\ (\phi_j x_{j,t-1} + \varepsilon_{j,t} = x_{j,t}), \qquad (6.1) \end{aligned}$$

since the process of (2.1) is the first-order Markovian process. The monotone log function of the likelihood (6.1) is given by

$$\begin{aligned} L^*(\beta, \delta, \sigma^2|\underline{X}) &= -2\ln L(\beta, \delta, \sigma^2|\underline{X}) - (\sum_{j=1}^{m} n_j - m)\ln 2\pi \\ &= \sum_{j=1}^{m}\sum_{t=2}^{n_j} \ln(\sigma^2 + \delta x_{j,t-1}^2) + \sum_{j=1}^{m}\sum_{t=2}^{n_j} \frac{(x_{j,t} - \beta x_{j,t-1})^2}{\sigma^2 + \delta x_{j,t-1}^2} \\ &\quad -2\sum_{j=1}^{m}\sum_{t=2}^{n_j} \ln\{\Phi(c_1) - \Phi(c_2)\} \\ &\quad +2(\sum_{j=1}^{m} n_j - m)\ln\{\Phi(\frac{1-\beta}{\sqrt{\delta}}) - \Phi(\frac{-1-\beta}{\sqrt{\delta}})\}, \qquad (6.2) \end{aligned}$$

where

$$c_1 = \frac{\delta x_{j,t-1}^2 x_{j,t} + \beta\sigma^2 x_{j,t-1} + |x_{j,t-1}|(\sigma^2 + \delta x_{j,t-1}^2)}{|x_{j,t-1}|\sqrt{\delta\sigma^2(\sigma^2 + \delta x_{j,t-1}^2)}},$$

$$c_2 = \frac{\delta x_{j,t-1}^2 x_{j,t} + \beta\sigma^2 x_{j,t-1} - |x_{j,t-1}|(\sigma^2 + \delta x_{j,t-1}^2)}{|x_{j,t-1}|\sqrt{\delta\sigma^2(\sigma^2 + \delta x_{j,t-1}^2)}}.$$

The MLE of σ^2 obtained by minimizing the function (6.2) is relatively exact, but the MLEs of β and δ show the results of blowing up. To rectify the problem of blowing up the chi-square density is incorporated as the prior density for $1/\delta$ with the likelihood function (6.1), that is,

$$L_M(\beta, \delta, \sigma^2|\underline{X}) = \left\{ \frac{\delta^{-\frac{\nu}{2}+1} e^{-\frac{1}{2\delta}}}{\Gamma(\frac{\nu}{2}) 2^{\nu/2}} \right\}^m \cdot L(\beta, \delta, \sigma^2|\underline{X}). \quad (6.3)$$

The monotone log function of modified likelihood (6.3) is then as follows

$$\begin{aligned} L_M^*(\beta, \delta, \sigma^2|\underline{X}) &= -2\ln L_M(\beta, \delta, \sigma^2|\underline{X}) - (\sum_{j=1}^m n_j - m)\ln 2\pi \\ &\quad -2m \ln\{\Gamma(\frac{\nu}{2})2^{\frac{\nu}{2}}\} \\ &= m(\nu - 2)\ln\delta + m/\delta + L^*(\beta, \delta, \sigma^2|\underline{X}). \end{aligned} \quad (6.4)$$

When $1/\delta$ is distributed with $\chi^2(\nu)$, the degree of freedom ν and the mode M_o of $\chi^2(\nu)$ is in the relation of $M_o = \nu - 2$. In our tentative simulation we can observe that the function (6.4) with $\nu = (1/\delta) + 2$ considerably exactly produce the MLEs of β, δ, and σ^2. The problem of obtaining the exact MLEs from (6.4) is finally to estimate relatively exactly initial value of δ. This problem remains to be solved.

A number of Monte Carlo experiments were carried to illustrate the procedures developed in our study. On conclusion, the results of the empirical Bayes estimate $\hat{\phi}_{EB}$ are specially satisfactory on small samples. It is very worth since the panel data of time series are generally short in length. Further research into the exact estimation of model parameters should be preceded to obtain an even better empirical Bayes estimate of random coefficient in the truncated normal RCAR model.

APPENDIX

Data used in the simulation study are generated using the following procedures. Note that all subroutines used here are those of IMSL running on UNIX workstation. For each j:

1. The standard normal distributed random errors, $\varepsilon_{j,t}$'s, were generated by the RNNOA subroutine.

2. To generate ϕ_j's, independent of $\varepsilon_{j,t}$'s, from a truncated normal distribution $N(\beta, \delta)$ in $(-1,\ 1)$ we obtain the following algorithm using the rejection method(see, Ross(1990)):

   STEP 1: Generate independent pseudo random numbers, U_1 and U_2 which are uniformly distributed in the interval $(0, 1)$ using the RNUN subroutine.

   STEP 2: Set $Y = 2U_1 - 1$.

   STEP 3: If $U_2 \leq \exp\left\{-\frac{(Y-\beta)^2}{2\delta}\right\}$, set $\phi_j = Y$, otherwise return to Step 1.

3. Let $X_{j,1} = \varepsilon_{j,1}$ in order to obtain an initial value $X_{j,1}$ of $\{X_{j,t}\}$.

4. Finally, $X_{j,2}, X_{j,3}, \cdots, X_{j,n_j}$ are recursively generated as soon as ϕ_j and $\varepsilon_{j,t}$, $t = 1, 2, \cdots, n_j$, are determined.

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