Journal of the Korean
Statistical Society
Vol. 25, No. 4, 1996

The Ordering of Hitting Times of Multivariate Processes †

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Abstract

In this paper, we introduce a new concept of partial ordering which permits us to compare pairs of the dependence structures of a new hitting times for POD multivariate vector process of interest as to their degree of POD-ness. We show that POD ordering is closed under convolution, limit in distribution, compound distribution, mixture of a certain type and convex combination. Finally, we present several examples of POD ordering processes.

Key Words : Hitting times; POD processes; Associated; POD ordering; Convolution; Limit in distribution; Compound distribution; Mixture of a certain type; Convex.

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†This paper was partially supported by Won Kwang University Grant, 1996.
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1. INTRODUCTION

Lehmann[11] introduced the concepts of positive(negative) dependence together with some other dependent concepts. Since then, a great many papers have been studied on the subject and its extensions and numerous multivariate inequalities have been obtained. In other words, a great many papers have been devoted to various generalizations of Lehmann's concepts to finite-dimensional distributions. For references of available results, see Karlin and Rinott[10], Ebrahimi and Ghosh[7], Shaked[14], Sampson[13], and Baek[2].Whereas a number of dependence notions exist for multivariate processes (see Friday[9]), recently, Ebrahimi[6] introduced some new positively orthant dependence (POD) concepts in terms of the finite-dimensional distributions of the hitting times of the components of a vector process. These concepts not only help us to understand structure of functionals such as hitting times of the given vector process but also have the potentional for new and useful inequalities for stochastic processes. Also, these concepts is a form of qualitative multivariate dependence which has led to many applications in applied probability, reliability, and statistical inference such as analysis of variance, multivariate tests of hypothesis, sequential testing. Like this, since POD processes is a qualitative multivariate form of dependence, it would be seen difficult, or impossible to compare different pairs of stochastic processes as to their "degree of processes". Therefore, the main goal of this paper is to develop a partial ordering which permits us to compare pairs of the dependence structures of a new hitting times for POD multivariate vector process of interest as to their degree of POD-ness. In section 2, we list some of definitions and notations for POD ordering processes. Next in section 3, we prove useful closure properties of the POD ordering. We show that POD ordering is closed under convolution, limit in distribution, compound distribution, mixture of a certain type, transformations of a stochastic processes by increasing functions, and convex combination. Finally in section 4, we present several examples of hitting times possessing various of POD ordering processes.

2. PRELIMINARIES

First, in this section, we present notations and basic facts used in the sequeal. In what follows 'increasing' means non-decreasing and 'decreasing' means non-increasing. Suppose that we are given a n-dimensional($n \geq$

2) stochastic vector processes $\{(X_{11}(t), \cdots, X_{n1}(t))|t \geq 0\}$ and $\{(X_{12}(t), \cdots, X_{n2}(t))|t \geq 0\}$, respectively. The state space of $(X_{11}(t), \cdots, X_{n1}(t))$ and $(X_{12}(t), \cdots, X_{n2}(t))$ will be taken to be a subset, $E = E_1 \times E_2 \times \cdots \times E_n$, of n-dimensional Euclidean space R^n, respectively.

For any states $a_i \in E_i, i = 1, 2, \cdots, n, j = 1, 2$, we define the random times as follows.

$$T_{ij}(a_i) = \inf\{t|X_{ij}(t) \geq a_i, 0 \leq t \leq \infty\}. \tag{2.1}$$

In other words, $T_{ij}(a_i)$ is the hitting times that the ijth component process $X_{ij}(t)$ reaches or goes above a_i (see (6)). If we base the dependence between multivariate processes on the dependence of their hitting times, we then have the following definitions.

Definition 1.(6) The stochastic process $\{(X_{12}(t), \cdots, X_{n2}(t))|t \geq 0\}$ is said to be positively upper orthant dependent $(PUOD)$ if

$$P(\bigcap_{i=1}^{n}(T_{i2}(a_i) > t_i)) \geq \prod_{i=1}^{n} P(T_{i2}(a_i) > t_i) \tag{2.2}$$

for all $t_i \geq 0, a_i \in E_i, i = 1, 2, \cdots, n$.

Definition 2.(6) The stochastic process $\{(X_{12}(t), \cdots, X_{n2}(t))|t \geq 0\}$ is said to be positively lower orthant dependent $(PLOD)$ if

$$P(\bigcap_{i=1}^{n}(T_{i2}(a_i) \leq t_i)) \geq \prod_{i=1}^{n} P(T_{i2}(a_i) \leq t_i) \tag{2.3}$$

for all $t_i \geq 0, a_i \in E_i, i = 1, 2, \cdots, n$.

We say that the stochastic process $\{(X_{12}(t), \cdots, X_{n2}(t))|t \geq 0\}$ is said to be positively orthant dependent (POD) if they satisfy both (2.2) and (2.3).

Definition 3. The stochastic process $\{(X_{12}(t), \cdots, X_{n2}(t))|t \geq 0\}$ is said to be associated if $Cov(f(T_{12}(a_1), \cdots, T_{n2}(a_n)), g(T_{12}(a_1), \cdots, T_{n2}(a_n))) \geq 0$ for all increasing functions f and g for which the covariance exists and $a_i \in E_i, i = 1, 2, \cdots, n$.

Before we state more definitions, we let $\beta = \beta(F_1, \cdots, F_n)$ denote the class of multivariate distribution function H having specified marginal distribution functions $F_1, F_2, \cdots$, and F_n, where $F_1, F_2, \cdots$, and F_n are nondegenerate, and we then consider β^+, a subclass of β, defined by

$$\beta^+ = \{H(t_1,\cdots,t_n) \mid H \text{ is } POD \quad , \quad H(t_1,\infty,\cdots,\infty) = F_1(t_1), \ \cdots, \\ H(\infty,\cdots,\infty,t_n) = F_n(t_n)\}\,.$$

Let H_1, H_2, H_3 and H_4 belong to β^+ and use the notation $\bar{H}_1(t_1,\cdots,t_n) = P(T_{11}(a_1) > t_1,\cdots,T_{n1}(a_n) > t_n), \bar{H}_2(t_1,\cdots,t_n) = P(T_{12}(a_1) > t_1,\cdots,T_{n2}(a_n) > t_n), \bar{H}_3(t_1,\cdots,t_n) = P(T_{11}(a_1) \le t_1,\cdots,T_{n1}(a_n) \le t_n), \bar{H}_4(t_1,\cdots,t_n) = P(T_{12}(a_1) \le t_1,\cdots,T_{n2}(a_n) \le t_n)$.

Definition 4. The multivariate distribution H_2 is said to be more positively upper orthant dependent than H_1 if

$$\bar{H}_2(t_1,t_2,\cdots,t_n) \ge \bar{H}_1(t_1,t_2,\cdots,t_n) \tag{2.4}$$

for all $t_i \ge 0, i = 1,2,\cdots,n$. We write $H_2 > (PUOD)H_1$.

Definition 5. The multivariate distribution H_4 is said to be more positively lower orthant dependent than H_3 if

$$\bar{H}_4(t_1,t_2,\cdots,t_n) \ge \bar{H}_3(t_1,t_2,\cdots,t_n) \tag{2.5}$$

for all $t_i \ge 0$, $i = 1,2,\cdots,n$. We write $H_4 > (PLOD)H_3$.

Moreover, we say that the stochastic processes $\{(X_{12}(t),\cdots,X_{n2}(t))|t \ge 0\}$ is said to be more positively orthant dependent than $\{(X_{11}(t),\cdots,X_{n1}(t))|t \ge 0\}$ if they satisfy both (2.4) and (2.5). We write $(X_{12}(t),\cdots,X_{n2}(t)) > (POD)(X_{11}(t),\cdots,X_{n1}(t))$.

3. CLOSURE PROPERTIES OF $(\bar{\beta}, > (POD))$

In this section, we establish preservation of the POD ordering under convolution, limit in distribution, compound distribution, mixture of a certain type, transformations of stochastic processes by increasing functions and convex combination. First note that $(X_{12}(t),\ \cdots,\ X_{n2}(t)) > (POD)\ (X_{11}(t),\ \cdots,\ X_{n1}(t))$ if and only if $E(f(T_{12}(a_1),\cdots,T_{n2}(a_n))g(T_{12}(a_1),\cdots,T_{n2}(a_n))) \ge E(f(T_{11}(a_1),\cdots,T_{n1}(a_n))g(T_{11}(a_1),\cdots,T_{n1}(a_n)))$ for all increasing functions f and g.

In below, we show that the ordering is preserved under convolution. We need the following Lemma 1 which is of independent interest.

Lemma 1. Let (a) $\{(X_{11}(t),\cdots,X_{n1}(t))\ \mid t\geq 0\}$ and $\{(X_{12}(t),\cdots,X_{n2}(t))\ \mid t\geq 0\}$ have distributions $H_1(H_3)$ and $H_2(H_4)$, where $H_1(H_3)$, $H_2(H_4)$ belong to β^+, respectively, (b) $\{(X_{12}(t),\cdots,X_{n2}(t))\mid t\geq 0\} > (POD)\{(X_{11}(t),\cdots,X_{n1}(t))|t\geq 0\}$, and (c) $(Z_1(t),\cdots,Z_n(t))$ with an arbitrary POD distribution function H independent of both of $\{(X_{11}(t),\cdots,X_{n1}(t)))\mid\ t\geq 0\}$ and $\{(X_{12}(t),\cdots,X_{n2}(t))|t\geq 0\}$. Then $(X_{12}(t)+Z_1(t),\cdots,X_{n2}(t)+Z_n(t)) > (POD)(X_{11}(t)+Z_1(t),\cdots,X_{n1}(t)+Z_n(t))$.

Proof. The proof will be given for the case $n=2$. For the general n, the proof is similar. First we will show that $(X_{12}(t)+Z_1(t),X_{22}(t)+Z_2(t))$ is PQD.

$$\begin{aligned}
&Cov[f(X_{12}(t)+Z_1(t)),g(X_{22}(t)+Z_2(t))]\\
&\quad= Cov[\ E\{f(X_{12}(t)+Z_1(t))|(Z_1(t),Z_2(t))\},\\
&\qquad\qquad E\{g(X_{22}(t)+Z_2(t))|(Z_1(t),Z_2(t))\}\ ]\\
&\qquad +E[Cov\{f(X_{12}(t)+Z_1(t)),g(X_{22}(t)+Z_2(t))\}|(Z_1(t),Z_2(t))].
\end{aligned}$$

Note that the first and second terms are greater than or equal to zero for any increasing functions f and g. Thus $(X_{12}(t)+Z_1(t),X_{22}(t)+Z_2(t))$ is PQD. Similarly we can show that $(X_{11}(t)+Z_1(t),X_{21}(t)+Z_2(t))$ is also PQD.

Next, we will show that $(X_{12}(t)+Z_1(t),X_{22}(t)+Z_2(t)) > (PQD)(X_{11}(t)+Z_1(t),X_21(t)+Z_2(t))$, i.e. $E(f(X_{12}(t)+Z_1(t))g(X_{22}(t)+Z_2(t)))\geq E(f(X_{11}(t)+Z_1(t))g(X_{21}(t)+Z_2(t)))$ for any increasing functions f and g. Now,

$$\begin{aligned}
&E(f(X_{12}(t)+Z_1(t))g(X_{22}(t)+Z_2(t)))\\
&\quad= E(E(f(X_{12}(t)+Z_1(t))g(X_{22}(t)+Z_2(t))|(Z_1(t),Z_2(t)))\\
&\quad= E(E(f(X_{12}(t)+Z_1(t))g(X_{22}(t)+Z_2(t)))))\\
&\quad\geq E(E(f(X_{11}(t)+Z_1(t))g(X_{21}(t)+Z_2(t))))\\
&\quad= E(f(X_{11}(t)+Z_1(t))g(X_{21}(t)+Z_2(t))).
\end{aligned}$$

Theorem 1. Suppose that the stochastic process (a) $\{(X_{12}(t),\cdots,X_{n2}(t))\mid t\geq 0\} > (POD)\{(X_{11}(t),\cdots,X_{n1}(t))|\ t\geq 0\}$, (b) $\{(Y_{12}(t),\cdots,Y_{n2}(t))\mid t\geq 0\} > (POD)\{(Y_{11}(t),\cdots,Y_{n1}(t))\mid t\geq 0\}$, and (c) $\{(X_{12}(t),\cdots,X_{n2}(t))\mid t\geq 0\}$ and $\{(Y_{12}(t),\cdots,Y_{n2}(t))|t\geq 0\}$ are independent and have increasing sample paths, $\{(X_{11}(t),\cdots,X_{n1}(t))|t\geq 0\}$ and $\{(Y_{11}(t),\cdots,Y_{n1}(t))|t\geq 0\}$ are independent and have increasing sample paths. Then $\{(X_{12}(t)+Y_{12}(t),\cdots,X_{n2}(t)+Y_{n2}(t))|t\geq 0\} > (POD)\{(X_{11}(t)+Y_{11}(t),\cdots,X_{n1}(t)+Y_{n1}(t))|t\geq 0\}$.

Proof. By assumption, $(X_{12}(t),\cdots,X_{n2}(t)) > (POD)(X_{11}(t),\cdots,X_{n1}(t))$. Specifying $(Z_1(t),\cdots,Z_n(t))$ to be $(Y_{12}(t),\cdots,Y_{n2}(t))$, we apply Lemma 1 to obtain

$$\begin{aligned}(X_{12}(t)+Y_{12}(t),\cdots,X_{n2}(t)+Y_{n2}(t))\\ >(POD)(X_{11}(t)+Y_{12}(t),\cdots,X_{n1}(t)+Y_{n2}(t)).\end{aligned}\tag{3.1}$$

Next, we use the assumption $(Y_{12}(t),\cdots,Y_{n2}(t))>(POD)(Y_{11}(t),\cdots,Y_{n1}(t))$, specifying $(Z_1(t),\cdots,Z_n(t))$ to be $(X_{11}(t),\cdots, X_{n1}(t))$, and again use Lemma 1 yielding

$$\begin{aligned}(X_{11}(t)+Y_{12}(t),\cdots,X_{n1}(t)+Y_{n2}(t))\\ >(POD)(X_{11}(t)+Y_{11}(t),\cdots.,X_{n1}(t)+Y_{n1}(t))\end{aligned}\tag{3.2}$$

By combining (3.1) and (3.2),

$$\begin{aligned}(X_{12}(t)+Y_{12}(t),\cdots,X_{n2}(t)+Y_{n2}(t))\\ >(POD)(X_{11}(t)+Y_{12}(t),\cdots,X_{n1}(t)+Y_{n2}(t))\\ >(POD)(X_{11}(t)+Y_{11}(t),\cdots,X_{n1}(t)+Y_{n1}(t)).\end{aligned}$$

Thus,

$$\begin{aligned}(X_{12}(t)+Y_{12}(t),\cdots,X_{n2}(t)+Y_{n2}(t))\\ >(POD)(X_{11}(t)+Y_{11}(t),\cdots,X_{n1}(t)+Y_{n1}(t)).\end{aligned}$$

Thus we complete the proof.

The next theorem demonstrates that, under suitable conditions, limits of more POD processes inherit the more POD structure.

Theorem 2. Let (a) $\{(X_{n1}(t),\cdots,X_{nk}(t))|t\geq 0\}$ and $\{(Y_{n1}(t),\cdots,Y_{nk}(t))\mid t\geq 0\}$, be a sequence of k-dimensional with distribution H_k and H'_k, respectively for every n. (b) $(X_{n1}(t),\cdots,X_{nk}(t))>(POD)(Y_{n1}(t),\cdots,Y_{nk}(t))$ for every n, (c) $H_k\xrightarrow{w}H$ where H is the distribution function of a processes $(X_1(t),\cdots,X_k(t))$ and $H'_k\xrightarrow{w}H'$ where H' is the distribution function of a processes $(Y_1(t),\cdots,Y_k(t))$, (d) $\{(X_{n1}(t),\cdots,X_nk(t)|t\geq 0\}$, $\{(Y_{n1}(t),\cdots,Y_{nk}(t)|t\geq 0\}$, $\{(X_1(t),\cdots,X_k(t)|t\geq 0\}$ and $\{(Y_1(t),\cdots,Y_k(t)|t\geq 0\}$ have all sample paths and they are right continuous on $[0,\infty)$ with finite left limits at all t. Then $(X_1(t),\cdots,X_k(t))>(POD)(Y_1(t),\cdots,Y_k(t))$.

Proof. Denote by $C(H)$ and $C(H')$ the sets of continuity points of H and H', respectively. Let $D=C(H)\cap C(H')$. It follows from our assumption that $H(t_1,t_2,\cdots,t_k)\geq H'(t_1,t_2,\cdots,t_k)$ for all $(t_1,t_2,\cdots,t_k)\in D$.

Since D is a dense set in R^k, $H > (POD)H'$ i.e. $(X_1(t), \cdots, X_k(t)) > (POD)(Y_1(t), \cdots, Y_k(t))$.

The following theorem is another application of Theorem 2 which is very important in recognizing more POD in compound distributions which arise naturally in stochastic processes.

Theorem 3. Let $Z_{i2}(t) = \sum_{j=1}^{N(t)} Y_{ij}, i = 1, 2, \cdots, n$ and $Z_{i1}(t) = \sum_{j=1}^{N(t)} X_{ij}, i = 1, 2, \cdots, n$, (a) $(Y_{11}, \cdots, Y_{n1}), (Y_{12}, \cdots, Y_{n2}), \cdots$ are independent random processes, (b) $(X_{11}, \cdots, X_{n1}), (X_{12}, \cdots, X_{n2}), \cdots$ are independent random processes, (c) $(Y_{1i}, \cdots, Y_{ni}) > (POD)(X_{1i}, \cdots, X_{ni}), i = 1, 2, \cdots$ and (d) $N(t)$ be a Poisson process which is independent of $(Y_{1i}, \cdots, Y_{ni})$ and $(X_{1i}, \cdots, X_{ni})$, $i = 1, 2, \cdots$. Then

$$\begin{aligned}(Z_{12}(t) &= \sum_{j=1}^{N(t)} Y_{1j}, \cdots, Z_{n2}(t) = \sum_{j=1}^{N(t)} Y_{nj}) \\ &> (POD)(Z_{11}(t) = \sum_{j=1}^{N(t)} X_{1j}, \cdots, Z_{n1}(t) = \sum_{j=1}^{N(t)} X_{nj}).\end{aligned}$$

Proof. We will show that the more $PLOD$ case is proved. Let $T_{ij}(a_i)$ be the hitting times of $Z_{ij}(t), i = 1, 2, \cdots, n, j = 1, 2$. Then

$$\begin{aligned}&P(T_{12}(a_1) \le t_1, \cdots, T_{n2}(a_n) \le t_n) \\ &= P(\sum_{j=1}^{N(s)} Y_{1j} \ge a_1, t_1 \le s < \infty, \cdots, \sum_{j=1}^{N(s)} Y_{nj} \ge a_n, t_n \le s < \infty) \\ &= P(\sum_{j=1}^{N(t_1)} Y_{1j} \ge a_1, \cdots, \sum_{j=1}^{N(t_n)} Y_{nj} \ge a_n) \\ &= \sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} P(N(t_1) = l_1, \cdots, N(t_n) = l_n) P(\sum_{j=1}^{l_1} Y_{1j} \ge a_1, \cdots, \sum_{j=1}^{l_n} Y_{nj} \ge a_n) \\ &\ge \sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} P(N(t_1) = l_1, \cdots, N(t_n) = l_n) P(\sum_{j=1}^{l_1} X_{1j} \ge a_1, \cdots, \sum_{j=1}^{l_n} X_{nj} \ge a_n) \\ &= P(\sum_{j=1}^{N(t_1)} X_{1j} \ge a_1, \cdots, \sum_{j=1}^{N(t_n)} X_{nj} \ge a_n) \\ &= P(T_{11}(a_1) \le t_1, \cdots, T_{n1}(a_n) \le t_n).\end{aligned}$$

Similarly, the more $PUOD$ case is proved.

Our next result deals with the preservation of the POD ordering under mixture. In order to motivate our definition of a subclass of $beta^+$ in which the POD ordering is preserved under mixture we need a definition and a similar result of Ebrahimi and Ghosh[7].

Definition 6. A stochastic process $\{X_{22}(t)|t \geq 0\}$ is stochastically increasing (SI) in $\{X_{12}(t)|t \geq 0\}$ if $E(f(T_{22}(a_2))|T_{12}(a_1) = t_1)$ is increasing in t_1 for all $a_i \in E_i, i = 1, 2$, and real valued function f.

Proposition 1. Let $\{(X_{12}(t), \cdots, X_{n2}(t))|t \geq 0\}$ given λ, be conditionally POD processes, and $\{X_{i2}(t)|t \geq 0\}$ be stochastically increasing (SI) in λ for $i = 1, 2, \cdots, n$. Then $\{(X_{12}(t), \cdots, X_{n2}(t))|t \geq 0\}$ are POD processes.

We may now define the class β^+_λ by

$$\begin{aligned}\beta^+_\lambda = \{H_\lambda | H(t_1, \infty, \cdots, \infty|\lambda) = F_1(t_1|\lambda), \cdots, H(\infty, \cdots, \infty, t_n|\lambda) \\ = F_n(t_n|\lambda), H_\lambda \mid \lambda \text{ is } POD, \text{ and } F_1, \cdots, F_n \text{ are } SI \text{ in } \lambda \}.\end{aligned}$$

Now consider $(\beta^+_\lambda, > (POD))$. The following theorem shows that if two elements of β^+_λ are ordered according to $> (POD)$, then after mixing λ, the resulting element in β^+ preserves the same order.

Proposition 2. Let $(X_{12}(t), \cdots, X_{n2}(t))|\lambda$ and $(X_{11}(t), \cdots, X_{n1}(t))|\lambda$ belong to β^+_λ, and $(X_{12}(t), \cdots, X_{n2}(t))|\lambda > (POD)(X_{11}(t), \cdots, X_{n1}(t))|\lambda$ for all λ. Then, unconditionally, $(X_{12}(t), \cdots, X_{n2}(t)), (X_{11}(t), \cdots, X_{n1}(t))$ belong to β^+ and $(X_{12}(t), \cdots, X_{n2}(t)) > (POD)(X_{11}(t), \cdots, X_{n1}(t))$.

Proof. From the Proposition 1, $(X_{12}(t), \cdots, X_{n2}(t))$ and $(X_{11}(t), \cdots, X_{n1}(t))$ are POD. Now,

$$\begin{aligned}&E(f(T_{12}(a_1), \cdots, T_{n2}(a_n))g(T_{12}(a_1), \cdots, T_{n2}(a_n))) \\ &\quad = E_\lambda(E(f(T_{12}(a_1), \cdots, T_{n2}(a_n))g(T_{12}(a_1), \cdots, T_{n2}(a_n)))|\lambda)) \\ &\quad \geq E_\lambda(E(f(T_{11}(a_1), \cdots, T_{n1}(a_n))g(T_{11}(a_1), \cdots, T_{n1}(a_n)))|\lambda)) \\ &\quad = E(f(T_{11}(a_1), \cdots, T_{n1}(a_n))g(T_{11}(a_1), \cdots, T_{n1}(a_n))).\end{aligned}$$

The inequality comes from the assumption that

$$(X_{12}(t), \cdots, X_{n2}(t))|\lambda > (POD)(X_{11}(t), \cdots, X_{n1}(t)) \mid \lambda$$

for all λ.

Theorem 4. Let (a) $\{(X_{i1}(t), \cdots, X_{ik}(t))|t \geq 0\}$ and $\{(Y_{i1}(t), \cdots, Y_{ik}(t))|t \geq 0\}$ be a sequence of k -variate processes with random increasing sample paths, respectively $i = 1, 2, \cdots, n$, (b) $(X_{i1}(t), \cdots, X_{ik}(t)) > (POD)(Y_{i1}(t), \cdots, Y_{ik}(t))$ for each $i = 1, 2, \cdots, n$, (c) $g_j : R^n \to R, j = 1, 2, \cdots, k$ are increasing functions in each of their arguments when all other arguments are fixed. Then the processes

$$Y_j(t) = g_j(X_{1j}(t), \cdots, X_{nj}(t)) > (POD)Z_j(t) = g_j(y_{1j}(t), \cdots, Y_{nj}(t))$$

for $j = 1, 2, \cdots, k$.

We now turn our attention to a simple but important property of the class β^+.

Result 1. The class

$$\beta^+ = \{H \mid H(t_1, \cdots, t_n) \text{ is } POD,\ H(t_1, \infty, \cdots, \infty) = F_1(t_1), \\ \cdots,\ H(\infty, \cdots, \infty, t_n) = F_n(t_n)\}$$

is convex.

Proof. Let $H_1, H_2 \in \beta^+$ and for $\alpha \in (0,1), H = \alpha H_1 + (1-\alpha)H_2$. Then we will show that H is convex combination of H_1 and H_2. Since each of the H_1 and $H_2 \in \beta^+$,

$$\begin{aligned} &P_H(T_{12}(a_1) > t_1, \cdots, T_{n2}(a_n) > t_n) \\ &\quad = \alpha P_{H_1}(T_{12}(a_1) > t_1, \cdots, T_{n2}(a_n) > t_n) \\ &\qquad\qquad +(1-\alpha)P_{H_2}(T_{12}(a_1) > t_1, \cdots, T_{n2}(a_n) > t_n) \\ &\quad \geq \alpha P_H(T_{12}(a_1) > t_1) \cdots P_H(T_{n2}(a_n) > t_n) \\ &\qquad\qquad +(1-\alpha)P_H(T_{12}(a_1) > t_1) \cdots P_H(T_{n2}(a_n) > t_n) \\ &\quad = P_H(T_{12}(a_1) > t_1) \cdots P_H(T_{n2}(a_n) > t_n). \end{aligned} \tag{3.3}$$

Hence H is $PUOD$. The proof of $PLOD$ is similar to the proof of $PUOD$. Moreover,

$$\begin{aligned} H(t_1, \infty, \cdots, \infty) &= \alpha F_1(t_1) + (1-\alpha)F_1(t_1) = F_1(t_1), \\ H(\infty, t_2, \cdots, \infty) &= \alpha F_2(t_2) + (1-\alpha)F_2(t_2) = F_2(t_2) \\ &\ \vdots \\ H(\infty, \cdots, \infty, t_n) &= \alpha F_n(t_n) + (1-\alpha)F_n(t_n) = F_n(t_n). \end{aligned} \tag{3.4}$$

It follows from (3.3), (3.4) that $H \in \beta^+$. Thus β^+ is convex.

4. EXAMPLES

Example 1. Consider a multivariate processes $\{(X_{n1}, Y_{n1}, Z_{n1})|n \geq 1\}$, $\{(X_{n2}, Y_{n2}, Z_{n2})|n \geq 1\}$ such that (X_{11}, Y_{11}, Z_{11}),$(X_{21}, Y_{21}, Z_{21}), \cdots$ are independent and (X_{12}, Y_{12}, Z_{12}), $(X_{22}, Y_{22}, Z_{22}), \cdots$ are independent processes. Then it is easy to check that $\{(X_{n2}, Y_{n2}, Z_{n2})|n \geq 1\} > (POD)\{(X_{n1}, Y_{n1}, Z_{n1}) |n \geq 1\}$ whenever $(X_{i2}, Y_{i2}, Z_{i2}) > (POD)(X_{i1}, Y_{i1}, Z_{i1})$, for each $i = 1, 2, \cdots$.

Example 2. Consider a system with components which is subjected to shocks. Let $N(t)$ be the number of shocks received by time t and $\{(X_k, Y_k, S_k) \mid k = 1, 2, \cdots\}$ and $\{(X'_k, Y'_k, S'_k) \mid k = 1, 2, \cdots\}$ are sequence of damages to components $1, 2, \cdots, 5$ and 6 by shock k, respectively. Define the compound Poisson processes by $Z_{11}(t) = \sum_{k=1}^{N(t)} X'_k$, $Z_{12}(t) = \sum_{k=1}^{N(t)} X_k$, $Z_{21}(t) = \sum_{k=1}^{N(t)} Y'_k$, $Z_{22}(t) = \sum_{k=1}^{N(t)} Y_k$, $Z_{31}(t) = \sum_{k=1}^{N(t)} S'_k$, $Z_{32}(t) = \sum_{k=1}^{N(t)} S_k$ are the total damages to components $1, 2, \cdots, 5$ and 6 by time t, respectively. Then we obtain using Theorem 3 that $(Z_{12}(t), Z_{22}(t), Z_{32}(t)) > (POD)(Z_{11}(t), Z_{21}(t), Z_{31}(t))$ for every $t \geq 0$, whenever $(X_i, Y_i, S_i) > (POD)(X'_i, Y'_i, S'_i)$, for each $i = 1, 2, 3, \cdots$.

Example 3. Consider a parallel system with $2n$ components. Assume that the ith component fails if the total damages to the component exceeds a threshold $a_i, i = 1, 2, \cdots, n$. Let $X_{ij}(t)$ be the total damages to the ijth component at time t, $i = 1, 2, \cdots, n$, $j = 1, 2$ and $(X_{12}(t), \cdots, X_{n2}(t)) > (POD)(X_{11}(t), \cdots, X_{n1}(t))$. Then, the life time of the system is given by the random variable $T = \max_{1 \leq i \leq n} T_{ij}(a_i)$, where $T_{ij}(a_i)$ is the hitting time defined in (2.1), $i = 1, 2, \cdots, n, j = 1, 2$. Hence, we get the useful bound

$$P(\max_{1 \leq i \leq n} T_{i2}(a_i) < t) \geq P(T_{11}(a_1) < t, \cdots, T_{n1}(a_n) < t),$$

for all $t \geq 0$. Similar bounds can be obtained for series system.

Example 4. Consider the following stress-strength model for $2n$ systems. Let the $Z_{ij}(t), i = 1, 2, \cdots, n$, $j = 1, 2$, be the strength of ijth systems at time t, respectively. We will assume that the $2n$ systems receive shocks from a common source. Using a cumulative damage shock model(see Barlow and Proschan(1975)), we now let $N(t)$ be the number of shocks occuring by time t and U_k be i.i.d. positive random variables denoting the damage to either system due to the kth shock, $k = 1, 2, \cdots$. Hence, the stress experienced by either system at time t is given by the processes $X_{ij}(t) = \sum_{k=1}^{N(t)} U_k$, $i = 1, 2, \cdots, n$, $j = 1, 2$. Using the Example 2, we can obtain that $(X_{12}(t), \cdots, X_{n2}(t)) > (POD)(X_{11}(t), \cdots, X_{n1}(t))$. Assume that $(Z_{12}(t), \cdots,$

$Z_{n2}(t)) > (POD)\ (Z_{11}(t), \cdots, Z_{n1}(t))$ such that $(Z_{11}(t), \cdots, Z_{n1}(t))$, and $(Z_{12}(t), \cdots, Z_{n2}(t))$ are independent processes with decreasing sample paths and that $(Z_{11}(t), \cdots, Z_{n1}(t))$, $(Z_{12}(t), \cdots, Z_{n2}(t))$ and $X_{ij}(t), i = 1, 2, \cdots, n, j = 1, 2,$ are independent processes. Then we obtain using Theorem 1 that $(X_{12}(t) - Z_{12}(t), \cdots, X_{n2}(t) - Z_{n2}(t)) > (POD)(X_{11}(t) - Z_{11}(t), \cdots, X_{n1}(t) - Z_{n1}(t))$. Consequently, the life times of the systems, namely, $T_{i2}(0) = \inf\{t \mid X_{i2}(t) - Z_{i2}(t) \geq 0\}$ are more POD random variables than $T_{i1}(0) = \inf\{t | X_{i1}(t) - Z_{i1}(t) \geq 0\}$, $i = 1, 2, \cdots, n$, respectively. Useful bounds on the joint survival of $2n$ dependent systems are therefore given by $(T_{i2}(0) > t_i, i = 1, 2, \cdots, n) > (POD)(T_{i1}(0) > t_i, i = 1, 2, \cdots, n)$.

ACKNOWLEGEMENTS

We would like to thank to the editor and the referees for components and suggestions in improving this paper.

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