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On the AR(1) Process with Stochastic Coefficient[1]

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Abstract

This paper is concerned with an estimation problem for the AR(1) process $\{Y_t,\ t=0,\ \pm 1, \cdots\}$ with time varying autoregressive coefficient, where coefficient itself is also stochastic process. Attention is directed to the problem of finding a consistent estimator of ϕ, the mean level of autoregressive coefficient. The asymptotic distribution of the resulting consistent estimator of ϕ is then discussed. We do not assume any time series model for the time varying autoregressive coefficient.

1. Introduction

Consider the following AR(1) process $\{Y_t,\ t=0,\ \pm 1, \cdots\}$ with time varying autoregressive coefficient

$$Y_t = (\phi + \phi_t) Y_{t-1} + \varepsilon_t \ , \qquad (1.1)$$

where $\{\phi_t\}$ represents unobservable random perturbations of the coefficient, $\{\varepsilon_t\}$ and $\{\phi_t\}$ are independent processes with zero mean and finite second moments. When $\{\phi_t\}$ is *iid*, (1.1) reduces to the first order random coefficient autoregressive model (RCA(1)) which has been fully exploited by Nicholls and Quinn (1982). To handle the case where the dependence structure in $\{\phi_t\}$ does exist, several attempts have been made. Among them, see Tjostheim (1986) : MA(1) structure for $\{\phi_t\}$ and Weiss(1985) : AR(1) structure for $\{\phi_t\}$. One may follow state-space-form(SSF) approach when $\{\phi_t\}$ is a Markov process.(See Harvey (1989)).

In this paper, We do not assume any specific models for $\{\phi_t\}$ but impose the following fairly mild restriction on ϕ_t :

$$\{\phi_t\} \text{ is a stationary and ergodic sequence in } L_2 \qquad (1.2)$$

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The stationary condition of the model specified by (1.1) and (1.2) are studied by Pourahmadi (1986). See Pourahmadi (1986) for the condition and several examples therein. However due to complications arising from the interactions between two processes $\{\phi_t\}$ and $\{\varepsilon_t\}$, he did not mention any estimation and inferential problems. To our knowledge, this kinds of problems have not yet been addressed in the literature. The primary goal of this paper is to discuss the estimation problem for this model.

By recursion, Y_t in (1.1) can be written in terms of a casual form

$$Y_t = \sum_{j=0}^{\infty} [\prod_{i=0}^{j-1} (\phi + \phi_{t-i})] \ \varepsilon_{t-j} \ . \tag{1.3}$$

In order for the expression (1.3) to be meaningful not only in L_2-sense but also in almost sure sense, we need the following stationarity condition due to Pourahmadi (1986).

$$\sum_{j=0}^{\infty} E [\prod_{i=0}^{j-1} (\phi + \phi_{t-i})^2] < \infty \ . \tag{1.4}$$

Notice that under the condition (1.4), the $\{Y_t\}$ in (1.3) is strictly stationary and ergodic in L_2.

Remark : For the usual AR(1) process, i.e., $\phi_t = 0$ case, (1.4) is equivalent to $|\phi| < 1$. When $\{\phi_t\}$ is *iid*, the model becomes RCA(1) and (1.4) reduces to

$$\phi^2 + Var(\phi_t) < 1 \tag{1.5}$$

as in the Nicholls and Quinn (1982).

The condition (1.4) will be assumed throughout so that $\{Y_t\}$ is ergodic stationary process in L_2.

2. Estimation Problem for ϕ

Let $\{Y_0, Y_1, \cdots, Y_n\}$ be a given sample from the model defined by (1.1) and (1.2) and the σ-field generated by $\{Y_0, Y_1, \cdots, Y_n\}$ is denoted by $\mathcal{F}_n$.

It must be emphasized that for the special case when $\{\phi_t\}$ is *iid* (RCA(1)), ϕ_t is independent of $Y_{t-1}, Y_{t-2}, \cdots$. However, this is not the case, in general, because Y_{t-1} is a function of $(\phi_{t-1}, \varepsilon_{t-1}), (\phi_{t-2}, \varepsilon_{t-2}), \cdots$. Consequently, $E(\phi_t Y_{t-1}^r)$, $r = 1, 2, 3, \cdots$ can not be factored out of the form $E(\phi_t) \cdot E(Y_{t-1}^r)$. Now, the model in (1.1) can be written as

$$Y_t = \phi Y_{t-1} + \eta_t \quad \text{with} \quad \eta_t = \phi_t Y_{t-1} + \varepsilon_t \tag{2.1}$$

and after simple algebra, it is seen that $E(\phi_t Y_{t-1}) = 0$ and hence $E(\eta_t) = 0$. We are then tempted to estimate ϕ by $\widetilde{\phi_n}$ via least squares method,

$$\widetilde{\phi_n} = \sum Y_t Y_{t-1} / \sum Y_{t-1}^2 , \tag{2.2}$$

where the subscript in $\sum$ runs from t=1 to t=n and this notation will be used throughout.

It can also be shown that

$$\tilde{\phi}_n - \phi = \sum(\phi_t Y_{t-1}^2 + \varepsilon_t Y_{t-1}) / \sum Y_{t-1}^2$$

Via the ergodic theorem, it then follows

$$\tilde{\phi}_n \xrightarrow{p} \phi + E(\phi_t Y_{t-1}^2) / \gamma(0) , \tag{2.3}$$

where $\gamma(0)$ stands for the autocorrelation function of lag 0, i.e.,

$$\gamma(0) = E(Y_t^2) .$$

Thus, from (2.3), we have the following.

Lemma 2.1 : For the process modelled by (1.1) and (1.2) with (1.4), the asymptotic bias of the least squares estimator $\tilde{\phi}_n$ in (2.2) is $E(\phi_t Y_{t-1}^2) / \gamma(0)$.

Remark : If ϕ_t is uncorrelated with Y_{t-1}^2 which is true for RCA(1), $\tilde{\phi}_n$ is a consistent estimator of ϕ. However, the bias never vanishes, in general, under our framework and it seems hard to find a closed form of the bias due to the lack of information about the dependence structure of $\{\phi_t\}$.

$\tilde{\phi}_n$ can be improved further by considering

$$\tilde{\phi}_n - \sum \phi_t Y_{t-1}^2 / \sum Y_{t-1}^2 . \tag{2.4}$$

From (2.3) and the ergodic theorem the term in (2.4) is seen to be consistent for ϕ.

However, due to the fact that $\{\phi_t\}$ is unobservable, ϕ_t in (2.4) must be replaced by sample information.

It is then natural to consider the best predictor (in the MSE sense) of ϕ_t based on $Y_0, Y_1, \cdots , Y_n$, which leads to

$$\widehat{\phi_n^*} = \tilde{\phi}_n - \sum\{E(\phi_t|\mathcal{F}_n) \cdot Y_{t-1}^2\} / \sum Y_{t-1}^2 . \tag{2.5}$$

The consistency of $\widehat{\phi_n^*}$ is presented in the following theorem.

Theorem 2.1 : Under the same conditions as in lemma 2.1, we have

$$\widehat{\phi_n^*} \xrightarrow{p} \phi \,, \quad \text{as} \quad n \to \infty \,. \tag{2.6}$$

proof : It suffices to show that

$$\begin{aligned} & n^{-1}\sum E(\phi_t|\mathscr{F}_n) \cdot Y_{t-1}^2 \\ & = n^{-1}\sum E(\phi_t Y_{t-1}^2|\mathscr{F}_n) \xrightarrow{p} E(\phi_t Y_{t-1}^2) \,. \end{aligned} \tag{2.7}$$

Using the conditional Jensen's inequality, we have with probability one,

$$\begin{aligned} & |n^{-1}\sum E(\phi_t Y_{t-1}^2|\mathscr{F}_n) - E(\phi_t Y_{t-1}^2)| \\ & \leq E(|n^{-1}\sum \phi_t Y_{t-1}^2 - E(\phi_t Y_{t-1}^2)|\,|\mathscr{F}_n) \,. \end{aligned} \tag{2.8}$$

By taking the expectation on both sides of (2.8), it follows that

$$\begin{aligned} & E|n^{-1}\sum E(\phi_t Y_{t-1}^2|\mathscr{F}_n) - E(\phi_t Y_{t-1}^2)| \\ & \leq E|n^{-1}\sum \phi_t Y_{t-1}^2 - E(\phi_t Y_{t-1}^2)| \,. \end{aligned} \tag{2.9}$$

Furthermore, the ergodic theorem gives

$$n^{-1}\sum \phi_t Y_{t-1}^2 \xrightarrow{L_1} E(\phi_t Y_{t-1}^2)$$

which leads to via (2.9)

$$n^{-1}\sum E(\phi_t Y_{t-1}^2|\mathscr{F}_n) \xrightarrow{L_1} E(\phi_t Y_{t-1}^2)$$

and hence (2.7) holds, which completes the proof ▨

From the practical point of view, $\widehat{\phi_n^*}$ in (2.5) still suffers two drawbacks. First, the direct calculation of $E(\phi_t|\mathscr{F}_n)$ seems complicated since the joint distribution of ϕ_t and $Y_0, Y_1, \cdots, Y_n$ is not known. Second, even when a simple form of $E(\phi_t|\mathscr{F}_n)$ is available, it may contain some parameters to be estimated.

In the next section, by assuming linear structure on the conditional expectation of ϕ_t given the sample, these drawbacks will be circumvented at the cost of losing the generality.

3. Limiting distribution of a consistent estimator of ϕ

Recalling the representation in (2.1) and motivated by the format of ARCH modelling in Engles (1982), we postulate that $E(\phi_t|\mathscr{F}_{t-1})$ is described by the following linear predictor type condition.

(C1) $$E(\phi_t|\mathscr{F}_{t-1}) = \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \cdots + \beta_p Y_{t-p} \,,$$

Denoting

$$\beta = (\beta_1, \beta_2, \cdots, \beta_p)' : \text{p}\times 1 \text{ vector of unknown constants} \tag{3.1}$$

and

$$Y(t-1) = (Y_{t-1}, Y_{t-2}, \cdots, Y_{t-p})' : \text{p}\times 1 \text{ vector} \tag{3.2}$$

(C1) can be written as

$$E(\phi_t | \mathcal{F}_{t-1}) = \beta' Y(t-1) .$$

The conditional least squares (CLS) estimators $\hat{\phi}_n$ and $\hat{\beta}_n$ of ϕ and β respectively are then obtained by minimizing

$$Q = \Sigma [Y_t - E(Y_t | \mathcal{F}_{t-1})]^2$$

$$= \Sigma [Y_t - \phi Y_{t-1} - \beta' Y(t-1) Y_{t-1}]^2$$

with the understanding that $Y_{-p+1} = \cdots Y_{-1} = 0$.

It can then be shown that

$$\begin{pmatrix} \hat{\phi}_n \\ \hat{\beta}_n \end{pmatrix} = V_n^{-1} X_n , \tag{3.3}$$

where

$$V_n = \Sigma Y_{t-1} \begin{pmatrix} 1 & Y(t-1)' \\ Y(t-1) & Y(t-1)Y(t-1)' \end{pmatrix} : \text{(p+1)}\times\text{(p+1) matrix} \tag{3.4}$$

$$X_n = \Sigma Y_t Y_{t-1} \begin{pmatrix} 1 \\ Y(t-1) \end{pmatrix} : \text{(p+1)}\times 1 \text{ vector} \tag{3.5}$$

In order to obtain the asymptotic distribution of $(\hat{\phi}_n, \hat{\beta}_n')'$, we need the following condition.

(C2) $\qquad E Y_t^8 < \infty$

We now turn our attention to the estimation bias

$$\begin{pmatrix} \hat{\phi}_n \\ \hat{\beta}_n \end{pmatrix} - \begin{pmatrix} \phi \\ \beta \end{pmatrix} = V_n^{-1} \left[X_n - V_n \begin{pmatrix} \phi \\ \beta \end{pmatrix} \right]$$

After some algebra, it can be shown that

$$\begin{pmatrix} \hat{\phi}_n \\ \hat{\beta}_n \end{pmatrix} - \begin{pmatrix} \phi \\ \beta \end{pmatrix} = (n^{-1} V_n)^{-1} \frac{1}{\sqrt{n}} \left[\Sigma (Y_t - E(Y_t | \mathcal{F}_{t-1}) Y_{t-1} \begin{pmatrix} 1 \\ Y(t-1) \end{pmatrix} \right] . \tag{3.6}$$

By applying martingale CLT for stationary martingale differences (see Hall and Heyde (1980)), the second term on the right of (3.6) converges in distribution to (p+1) variate normal distribution with mean zero and variance-covariance matrix Γ with

$$\Gamma = Var\{(Y_t - E(Y_t|\mathcal{F}_{t-1})\,Y_{t-1}\begin{pmatrix}1\\ Y(t-1)\end{pmatrix})\}$$

$$= E[\ Y_{t-1}^2\, Var(Y_t|\mathcal{F}_{t-1})\begin{pmatrix}1 & Y(t-1)' \\ Y(t-1) & Y(t-1)Y(t-1)'\end{pmatrix}]\ . \qquad (3.7)$$

It may be noted that the existence of Γ can be guarantted by (C2). We are now in a position to present the following theorem.

Theorem 3.1 : Under conditions (C1) and (C2), we have

$$\sqrt{n}\left[\begin{pmatrix}\hat{\phi}_n\\ \hat{\beta}_n\end{pmatrix} - \begin{pmatrix}\phi\\ \beta\end{pmatrix}\right] \xrightarrow{d} N(0,\ V^{-1}\Gamma V^{-1})\ , \qquad (3.8)$$

where

$$V = plim\{n^{-1} V_n\}$$

and Γ is defined as in (3.7).

proof : First note that the consistency of $\hat{\phi}_n$ is a consequence of the theorem. Combining (3.6) and (3.7), and using the ergodic theorem the proof is immediate. ▧

4. Concluding Remarks

In this paper, we have discussed the estimation problem for the AR(1) process with time varying autoregressive coefficient ϕ_t, where any time series model has not been assumed for ϕ_t. The usual estimator for ϕ, the mean level of ϕ_t, turns out to be inconsistent and a consistent estimator of ϕ is suggested and relevant limiting distributions can be obtained. The generalization of this study is twofold: First, one may extend the results to p-order process. Second, it could be possible to derive a consistent estimator of ϕ without imposing linear predictor type condition(C1). These extentions will be left open problems for the future study.

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