Likely Mean Value Theorem of Integrals of Real Mapping Between Fuzzy Bounds

Jong-Won Ryu*, Young-Chel Kwun*, Sung-Mi Kim*

I. Introduction

We study likely mean value theorem with respect to integral of real mapping between fuzzy bounds that is the main purpose of this paper, which investigates ideas in Dubois & Prade ([1, 2, 3])

II. Definitions and Main Results

A fuzzy domain D of the real line R is assumed to be delimited by two fuzzy bounds $\bar{a}$ and $\bar{b}$ in the following sense:

(i) $\bar{a}$ and $\bar{b}$ are fuzzy sets on R, whose membership functions are $\mu_{\bar{a}}$ and $\mu_{\bar{b}}$ from R to $[0, 1]$,

(ii) For all $x \in R$, $\mu_{\bar{a}}(x)$ (resp. $\mu_{\bar{b}}(x)$) evaluates to what extent x can be considered as a greatest lower bound(resp. least upper bound) of D,

(iii) $\bar{a}$ and $\bar{b}$ are normalized, i.e., there exists $a, b \in R$ such that $\mu_{\bar{a}}(x) = 1 = \mu_{\bar{b}}(x)$,

(iv) $\bar{a}$ and $\bar{b}$ are convex fuzzy sets, i.e., $\forall\ \alpha \in (0, 1]$ there $\alpha-$cuts $\bar{a}_\alpha$ and $\bar{b}_\alpha$ are intervals.

D is denoted by $(\bar{a}, \bar{b})$: $\bar{a}$ and $\bar{b}$ are assumed to be ordered in the sense that

$$a_0 = \inf S(\bar{a}) \le \sup S(\bar{b}) = b_0,$$

where $S(\bar{a}) = \{x \mid \mu_{\bar{a}}(x) > 0\}$ is support of $\bar{a}$ (See Dubois & Prade [3]).

Definition 1 ([4]). A fuzzy number $\bar{A}$ is in the real line R is a fuzzy set characterized by a membership function $\mu_{\bar{A}} : R \to [0, 1]$. A fuzzy number $\bar{A}$ is expressed as

$$\bar{A} = \int_{x \in R} \mu_{\bar{A}}(x)/x,$$

with the understanding that $\mu_{\bar{A}}(x) \in [0, 1]$ represents the grade of membership of x in $\bar{A}$ and $\int$ denotes the union of $\mu_A(x)/x$'s.

Definition 2 ([4]). A fuzzy number $\bar{A}$ in R is said to be convex if for any real numbers $x, y, z \in R$ with $x \le y \le z$, $\mu_{\bar{A}}(y) \ge \mu_{\bar{A}}(x) \wedge \mu_{\bar{A}}(z)$ with $\wedge$ standing for min.

A fuzzy number $\bar{A}$ is called normal if the following holds. $\max_x \mu_{\bar{A}}(x) = 1$. A fuzzy number which is normal and convex is refered to as normal convex fuzzy number.

Remark. Let $\bar{a}$ and $\bar{b}$ are normal convex fuzzy number with bounded support. $\bar{a}$ and $\bar{b}$ are assume to be ordered in the sense that

$$a_0 = \inf S(\bar{a}) \le \sup S(\bar{b}) = b_0$$

where $S(\bar{a}) = \{x \mid \mu_{\bar{a}}(x) > 0\}$ is support of $\bar{a}$.

Then $(\bar{a}, \bar{b})$ is satisfy definition of fuzzy domain.

Definition 3 ([6]). Let $\bar{A}$ and $\bar{B}$ be fuzzy sets with membership function $\mu_{\bar{A}}$ and $\mu_{\bar{B}}$, respectively. For every x, $\mu_{\bar{A}}(x) \le \mu_{\bar{B}}(x)$ iff $\bar{A} \subseteq \bar{B}$.

*Dept. of Mathematics, Dong-A Univ.

Definition 4 ([5]) Let $\overline{A}$ and $\overline{B}$ be fuzzy numbers. The membership function of their extended substraction $\overline{A} \ominus \overline{B}$ is defined by

$$\mu_{\overline{A} \ominus \overline{B}}(z) = \sup_{z=x \to y} \min(\mu_{\overline{A}}(x), \mu_{\overline{B}}(y)).$$

Definition 5 ([2]). Let f be a real-valued real mapping, supposedly integrable on the interval $I=[\inf S(\overline{a}), \sup S(\overline{b})]$; then the integral of f over the domain delimited by the fuzzy bounds $\overline{a}$ and $\overline{b}$, denoted by $\int_D f$, is defined according to extension principle by

$$\forall z \in R,\ \mu_{\int_D f}(z) = \sup_{x,\, y \in I} \min(\mu_{\overline{a}}(x), \mu_{\overline{b}}(y))$$

under the constrain $z=\int_x^y f$, where $\int_x^y f$ is short for $\int_x^y f(s)ds$. $\int_D f$ will also be denothed by $\int_{\overline{a}}^{\overline{b}} f$.

Definition 6 ([3]). A fuzzy point $\overline{c}$ is a convex subset of real line R and its membership function is defined by

$$\forall x,\ \forall y > x,\ \forall z \in [x, y],\ \mu_{\overline{c}}(z) \geq \min(\mu_{\overline{c}}(x), \mu_{\overline{c}}(y)).$$

Theorem 1. Let $\overline{a}$ and $\overline{b}$ are normal convex fuzzy numbers on R with bounded support and f be a real valued maping supposedly integrable on the interval $[\inf S(\overline{a}), \sup S(\overline{b})]$ then there exists a fuzzy point $\overline{c}$ satisfying

$$\int_{\overline{a}}^{\overline{b}} f(s)\, ds \subseteq f(\overline{c})\ (\overline{b} \ominus \overline{a}),$$

where $S(\overline{c}) \subset [\inf S(\overline{a}), \sup S(\overline{b})]$

Proof. By Definition 5,

$$\mu_{\int_{\overline{a}}^{\overline{b}} f}(z) = \sup_{\int_w^u f = z} \min\{\mu_{\overline{a}}(w), \mu_{\overline{b}}(u)\}.$$

Since $\int_w^u f$ is Riemann integral, by ordinary mean value theorem, there exist $t\,(w < t < u)$ satisfy

$$\int_w^u f(s)ds = f(t)\ (u-w).$$

Thus

$$\mu_{\int_{\overline{a}}^{\overline{b}} f}(z) = \sup_{xy=z} \min\left\{ \sup_{\substack{t,\, x=f(t) \\ w<t<u}} \min\{\mu_{\overline{a}}(w), \mu_{\overline{b}}(u)\},\ \sup_{u-w=y} \min\{\mu_{\overline{a}}(w), \mu_{\overline{b}}(u)\} \right\}.$$

We define membership function of $\overline{c}$ such that

$$\mu_{\overline{c}}(w) = \begin{cases} \mu_{\overline{a}}(w), & \text{if } w \in S(\overline{a}) \\ \mu_{\overline{b}}(w), & \text{if } w \in S(\overline{b}). \end{cases}$$

Since $w \in S(\overline{a})$ and $u \in S(\overline{b})$

$$\mu_{\int_{\overline{a}}^{\overline{b}} f}(z) = \sup_{xy=z} \min\left\{ \sup_{\substack{t,\, x=f(t) \\ w<t<u}} \min\{\mu_{\overline{c}}(w), \mu_{\overline{c}}(u)\},\ \sup_{u-w=y} \min\{\mu_{\overline{a}}(w), \mu_{\overline{b}}(u)\} \right\}.$$

By definition of fuzzy point,

$$\min\{\mu_{\overline{c}}(w), \mu_{\overline{c}}(u)\} \leq \mu_{\overline{c}}(t),\ w < t < u,$$

$$\mu_{\int_{\overline{a}}^{\overline{b}} f}(z) \leq$$

$$\sup_{xy=z} \min\left\{ \sup_{\substack{t,\, x=f(t) \\ w<t<u}} \mu_{\overline{c}}(t),\ \sup_{u-w=y} \min\{\mu_{\overline{a}}(w), \mu_{\overline{b}}(u)\} \right\}.$$

By extension principal,

$$\sup_{u-w=y} \min\{\mu_{\overline{a}}(w), \mu_{\overline{b}}(u)\} = \mu_{\overline{b} \ominus \overline{a}}(y)$$

and

$$\sup_{\substack{t,\, x=f(t) \\ w<t<u}} \mu_{\overline{c}}(t) = \mu_{f(\overline{c})}(x).$$

Hence

$$\mu_{\int_{\overline{a}}^{\overline{b}} f}(z) \leq \sup_{xy=z} \min\{\mu_{f(\overline{c})}(x), \mu_{\overline{b} \ominus \overline{a}}(y)\}.$$

Using extension principal,

$$\mu_{\int_{\overline{a}}^{\overline{b}} f} \leq \mu_{f(\overline{c})\,(\overline{b} \ominus \overline{a})}(z).$$

Corollary. Under the assumption of Theorem 1, if we give membership functions of $\bar{a}$ (resp. $\bar{b}$) and there exist y such that $\mu_{\bar{b}}(y)=1$ (resp. $\mu_{\bar{a}}(y)=1$), then we can define membership function of $\bar{b}$. Furthermore, in this case also satisfy Theorem 1.

Proof. It suffices to define membership function of $\bar{b}$. Let $x \in S(\bar{a})$ and $\int_x^y f(s)ds = z$ then there exists $S(\bar{b}) = \left\{ y \mid \int_x^y f(s)ds = z \right\}$. Put $\mu_{\bar{b}}(k)=1$. Define

$$\mu_{\bar{b}}(y) = \begin{cases} \dfrac{y-\inf S(\bar{b})}{k-\inf S(\bar{b})}, & y \langle k \\[2ex] \dfrac{\sup S(\bar{b})-y}{\sup S(\bar{b})-k}, & y \langle k. \end{cases}$$

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류 종 원(Jong-Won Ryu) 정회원
현재 : 동아대학교 자연과학대학 수학과 교수

권 영 철(Young-Chel Kwun) 정회원
현재 : 동아대학교 자연과학대학 수학과 조교수

김 성 미(Sung-Mi Kim) 정회원
1994년 : 동아대학교 수학과(교육학 석사)