Fuzzy Weakly Implicative Ideals of Bck-Algebras

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ABSTRACT

In this paper, we investigated the relation between the ideals of BCK-algebras and fuzzy ideals. We defined the weakly implicative ideals of BCK-algebras and obtained some properties. We proved some results for the fuzzy weakly implicative ideals of bounded commutative BCK-algebras. We also investigated that the weakly implicative ideals are similar to the fuzzy positive implicative ideals.

I. INTRODUCTION

In [11] Ougen extended the concept of fuzzy sets to BCK-algebras in general. In [3] Hoo has seen some general properties for fuzzy ideals of BCI and MV-algebras (or bounded commutative BCK-algebras). We shall adopt the definition and teminology of [2] and [3].

We review some fuzzy logic concepts. We shall write $a \wedge b$ for $min\{a, b\}$ and $a \vee b$ for $max\{a, b\}$ for any two real numbers a, b and denote the closed unit interval by [0, 1]. A fuzzy subset of a BCI-algebra X is a funtion $\mu : X \rightarrow [0, 1]$.

Definition 1.1. Let $(X, *, 0)$ be a BCK-algebra. A fuzzy set μ in X is called a ***fuzzy ideal*** of X if it satisfies the following conditions:

(1) $\mu(0) \geq \mu(x)$ for all $x \in X$,
(2) $\mu(x) \geq \mu(x * y) \wedge \mu(y)$ for all $x \in X$.

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Proposition 1.2. [3] *Let X be a BCK-algebra and μ a fuzzy ideal in X. Then*

(1) $x \leq y$ *implies* $\mu(x) \geq \mu(y)$,
(2) $\mu(x * y) \geq \mu(x * z) \wedge \mu(z * y)$,
(3) $\mu(x * y) = \mu(0)$ *implies* $\mu(x) \geq \mu(y)$,
(4) $\mu(x * y) \wedge \mu(y) = \mu(x) \wedge \mu(y)$,
(5) *if X is bounded, then* $\mu(x) \wedge \mu(1 * x) = \mu(1)$ *for all* $x \in X$,
(6) *if* $x \geq y$, *then* $\mu(x) = \mu(x * y) \wedge \mu(y)$.

For a given fuzzy set μ and $t \in [0, 1]$, let $\mu_t = \{x \in X | \mu(x) \geq t\}$. This could be an empty set. In fact, μ is a fuzzy ideal if and only if for each $t \in [0, 1]$, μ_t is either empty or an ideal of X.

For a given fuzzy ideal of a BCK-algebra X, $X_\mu = \{x \in X | \mu(x) = \mu(0)\}$ is an ideal of X. And if I is an ideal of a BCK-algebra X, then the characteristic function of I, $\chi_I \rightarrow [0, 1]$, is a fuzzy ideal of X, and $I = X_{\chi_I}$. We can check these facts easily.

Definition 1.3. Let X be a commutative BCK-algebra and μ a fuzzy set in X. μ is a *fuzzy prime ideal* of X if it is non-constant and $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ for all $x, y \in X$.

Proposition 1.4. [3] *If X is commutative then a non-constant fuzzy subset α of X is a fuzzy prime ideal of X if and only if for each $t \in [0, 1]$, α_t is either empty or a prime ideal of X if it is proper.*

Proposition 1.5. *Let X be a BCK-algebra and μ be a fuzzy ideal of X.*

Then we always have

$$\mu((x*z)*z) \geq \mu((x*y)*z) \wedge \mu(y*z) \text{ for all } x, y, z \in X.$$

Hence every ideal of X is weakly implicative.

Proof: By (2) Proposition 1.2, we have $\mu((x*z)*z) \geq \mu((x*z)*y) \wedge \mu(y*z) = \mu((x*y)*z) \wedge \mu(y*z)$. Suppose that I is an ideal of X and $(x*y)*z \in I$ and $y*z \in I$. Consider the fuzzy ideal χ_I of X. Then, $I = X_{\chi_I}$. Thus $\chi_I((x*y)*z) = \chi_I(0) = \chi_I(y*z)$. This means that $\chi_I((x*y)*z) = 1 = \chi_I(y*z)$. Hence $\chi_I((x*z)*z) \geq \chi_I((x*y)*z) \wedge \chi_I(y*z) = 1$. Therefore $(x*z)*z \in X_{\chi_I} = I$, proving that I is weakly implicative.

From above Proposition, for the weakly implicativity of ideals of BCK-algebra, we can define followings.

Definition 1.6. Given a BCK-algebra $(X, *, 0)$, a nonempty subset I of X is said to be a *weakly implicative ideal* of X if it satisfies the followings:

(1) $0 \in I$,

(2) $(x*y)*z \in I$ and $y*z \in I$ imply $(x*z)*z \in I$, for all $x, y, z \in X$.

Definition 1.7. Lex X be a BCK-algebra. A fuzzy set μ in X is called a *fuzzy weakly implicative ideal* of X if it satisfies the followings:

(1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(2) $\mu((x*z)*z) \geq \mu((x*y)*z) \wedge \mu(y*z)$ for all $x, y, z \in X$.

Definition 1.8. Let X be a BCK-algebra. A fuzzy set μ in X is called a *fuzzy positive implicative ideal* of X if it satisfies the following conditions:

(1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(2) $\mu(x*z) \geq \mu((x*y)*z) \wedge \mu(y*z)$ for all $x, y, z \in X$.

Proposition 1.9. *Let X be a BCK-algebra. Then*

(1) *I is a positive implicative ideal of X if and only if χ_I is a fuzzy positive implicative ideal of X,*

(2) *if X is commutative, then I is a prime ideal of X if and only if χ_I is a fuzzy prime ideal of X.*

Proof: (1) Suppose that I is apositive implicative ideal of X. Then $0 \in I$. So $\chi_I(0) = 1$. We claim that $\chi_I(x*z) \geq \chi_I(x*y)*z) \wedge \chi_I(y*z)$. If $(x*y)*z$, $y*z \in I$, then $x*z \in I$. Then $\chi_I(x*z) = \chi_I((x*y)*z) = \chi_I(y*z) = 1$. Thus $\chi_I(x*z) \geq \chi_I((x*y)*z) \wedge \chi_I(y*z)$. If $((x*y)*z) \notin I$ or $(y*z) \notin I$, $\chi_I((x*y)*z) = 0$ or $\chi_I(y*z) = 0$. Then $\chi_I((x*y)*z) \wedge \chi_I(y*z) = 0$. Thus we always have, for $x*z$, $\chi_I(x*z) \geq \chi_I((x*y)*z) \wedge \chi_I(y*z)$. Hence χ_I is a positive implicative ideal of X. Conversely, suppose the χ_I is a positive implicative ideal of X. Since $\chi_I(0) \geq \chi_I(x)$, for all $x \in X$, $\chi_I(0) = 1$. So $0 \in I$. If $(x*y)*z$, $y*z \in I$, then $\chi_I((x*y)*z) = \chi_I(y*z) = 1$. Then $\chi_I(x*z) = 1$. So $x*z \in I$. Therefore I is a positive implicative ideal of X.

(2) Suppose that I is a prime ideal of X. We already know that χ_I is a fuzzy ideal of X. Since I is proper, χ_I is non-constant. Let $x, y \in X$. If $x \in I$ or $y \in I$ then $x \wedge y \in I$ and hence $\chi_I(x \wedge y) = 1 = \chi_I(x) \vee \chi_I(y)$. If $x \notin I$ and $y \notin I$ then $x \wedge y \notin I$ and hence $\chi_I(x \wedge y) = 0 = \chi_I(x) \vee \chi_I(y)$. Hence χ_I is a prime ideal of X. Conversely, let χ_I be a fuzzy prime ideal of X. Since $I = X_{\chi_I}$, I is a prime ideal of X.

Theorem 1.10. *A nonempty subset I of a BCK-algebra X is a weakly implicative ideal of X if and only if χ_I is a fuzzy weakly implicative ideal of X.*

Proof: Since I is an ideal of X, $0 \in I$. $\chi_I(0)=1$. So $\chi_I(0) \geq \chi_I(x)$ for all $x \in X$. It remains to show that $\chi_I((x*z)*z) \geq \chi_I((x*y)*z) \wedge \chi_I(y*z)$. If $(x*y)*z$, $y*z \in I$, then $\chi_I((x*y)*z)=\chi_I(y*z)=1$ and $(x*z)*z$ is also in I. Thus $\chi_I((x*z)*z) \geq \chi_I((x*y)*z) \wedge \chi_I(y*z)$. If $(x*y)*z \notin I$ or $y*z \notin I$, then $\chi_I((x*y)*z) \wedge \chi_I((y*z) = 0 \leq \chi_I((x*z)*z)$. Hence χ_I is a fuzzy weakly implicative ideal of X. Conversely, for all $x \in X$, $\chi_I(x)=0$ or 1. Since $\chi_I(0) \geq \chi_I(x)$, $\chi_I(0)=1$. Thus $0 \in I$. Suppose that $(x*y)*z$ $y*z \in I$. Then $\chi_I((x*y)*z)=\chi_I(y*z)=1$. $\chi_I((x*z)*z) \geq \chi_I((x*y)*z) \wedge \chi_I(y*z)=1$. $\chi_I((x*z)*z)=1$. Therefore $((x*z)*z) \in I$. Hence I is a weakly implicative ideal of X.

Ⅱ. FUZZY WEAKLY IMPLICATIVE IDEALS OF BOUNDED COMMUTATIVE BCK-ALGEBRAS

In this section, X will denote a bounded commutative BCK-algebra and μ a general fuzzy ideal of X which may have other properties as specified.

Definition 2.1. The *set of nilpotent element* of X is

$N(X)=\{x \in X \mid x^n=0$ for some $n \geq 1\}$.

Proposition 2.2. [3] *If μ is a fuzzy positive implicative ideal of X, then $\mu(x^n)=\mu(x)$ for all $x \in X$ and $n \geq 1$.*

Example 1. For a fuzzy weakly implicative ideal, Proposition 2.2 fails. Let $X=\{0, 1, 2\}$ which binary operation $*$ is given by the table then $(X, *, 0)$ is a bounded commutative BCK-algebra with unit 2. Define a function $\mu: X \to [0, 1]$ by for $a \geq b \geq c$, $\mu(0)=a$, $\mu(1)=b$, $\mu(2)=c$. Then μ is a fuzzy weakly implicative ideal of X. $\mu(1)=b$ but $\mu(1^2)=\mu(1*\bar{1})=1\mu(1*1)=\mu(0)=a \neq \mu(1)$.

*	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Remark. *For x, y in a bounded comutative BCK-algebra X,* define $x \wedge y = y*(y*x) = x*(x*y)$, $\bar{x}=1*x$ *(denoted by N_x)*, $\bar{y}=1*y$, $x \vee y = (\bar{x}*\bar{y})^- = N(N_x \wedge N_y)$.

Theorem 2.3. *If μ is a fuzzy weakly implicative ideal of X and $x*(x*y)=x*y$ for all $x, y \in X$ then $\mu(x^n)=\mu(x)$, for all $x \in X$ and $n \geq 1$.*

Proof: This is true for $n=1$. In case of $n=2$, we have $\mu(x)=\mu(1*\bar{x})=\mu((1*\bar{x})*\bar{x}) \geq \mu((1*\bar{x})*\bar{x}) \wedge \mu(\bar{x}*\bar{x}) = \mu(x^2)$. But $x^2 \leq x$ and hence $\mu(x^2) \geq \mu(x)$. This means that $\mu(x^2)=\mu(x)$. Suppose that $n \geq 3$ and $\mu(x^{n-1}) = \mu(x)$ Then $\mu(x^{n-1})=\mu(x^{n-2}*\bar{x})=\mu(x^{n-2}*\bar{x})*\bar{x}) \geq \mu((x^{n-2}*\bar{x})*\bar{x}) \wedge \mu(\bar{x}*\bar{x})=\mu(x^n)$. But $x^n \leq x^{n-1}$ and hence $\mu(x^n)=\mu(x^{n-1})$. This proves that $\mu(x^n)=\mu(x^{n-1}) = \mu(x)$.

Corollary 2.4. *Let I be a positive (weakly) implicative ideal of X. If $x^n \in I$ for some $n \geq 1$ then $x \in I$. Hence $N(X) \subset I$.*

Proof: For a positive (weakly) implicative ideal I, consider the fuzzy positive (weakly) implicative ideal χ_I of X. Note that $X_{\chi_I}=\{x \in X \mid \chi_I(0)=\chi_I(x)\}=I$. If $x^n \in I$ for some $n \geq 1$ then $\chi_I(x^n)=1$. Since $0 \in I$ and by Theorem 2.3, $\chi_I(x^n)=\chi_I(x)=1$. Hence $x \in I$. i.e. $N(X) \subset I$.

Proposition 2.5. *If μ is a fuzzy positive implicative ideal of X, then $\mu(x \vee y)=\mu(x+y)=\mu(x) \wedge \mu(y)$ for all x, $y \in X$.*

Proof: We have $x, y \leq x \vee y \leq x+y$. Hence $\mu(x) \wedge \mu(y) \geq \mu(x \vee y) \geq \mu(x+y)$. Now $(x+y)*x=(x+y)\bar{x} = \bar{x}*(\bar{x}\,\bar{y}) = \bar{x}*(\bar{x}*y) = \bar{x} \wedge y$. Hence $\mu((x+y)*x)=\mu(\bar{x} \wedge y) \geq \mu(\bar{x}) \vee \mu(y)$, since $\bar{x} \wedge y \leq \bar{x}, y$. This means that $\mu(x+y) \geq \mu((x+y)*x) \wedge \mu(x) \geq (\overline{\mu(x)} \vee \mu(y)) \wedge \mu(x) \geq (\mu(\bar{x}) \vee \mu(y)) \wedge \mu(x) = (\mu(\bar{x}) \wedge \mu(x)) \vee (\mu(y) \wedge \mu(x)) = \mu(x)) = \mu(1) \vee (\mu(y) \wedge \mu(x)) = \mu(x) \wedge \mu(y)$. Using (5) of Proposition 1.2, this Proposition is proved.

Proposition 2.6. *If μ is a fuzzy positive implicative ideal of X, then $\mu(x)=\mu(xy)\wedge\mu(x\bar{y})$ for all $x, y\in X$.*

Proof: $\mu(x)=\mu(1*\bar{x})\geq\mu((1*(1*y))*\bar{x})\wedge\mu(1*y)*\bar{x})$ $=\mu(yx)\wedge\mu(x\bar{y})$. But $xy\leq x$ and $x\bar{y}\leq x$ and hence $\mu(xy)\wedge\mu(x\bar{y})\geq\mu(x)$.

Example 2. Let $X=\{0, 1, 2\}$ in which the binary operation $*$ is given by the table then $(X, *, 0)$ is bounded commutative BCK-algebra with unit 2. Define $\mu: X\rightarrow[0, 1]$ by $\mu(0)=a$ $\mu(1)=b$ $\mu(2)=c$ with $a\geq b\geq c$. Routine calculation gives that μ is a fuzzy weakly implicative ideal of X. If $x=2$, $y=1$ then $\mu(2)=c$. But $\mu(2\cdot 1)\wedge(2\cdot\bar{1})=\mu(1)\wedge\mu(1)=b$. In fact, $(2*1)*1\neq 1$. Hence above proposition may not be true for a fuzzy weakly implicative ideal of X.

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Theorem 2.7. *If μ is a fuzzy weakly implicative ideal of X and $(x*y)*y=x*y$ for all $x, y\in X$, then $\mu(x)=\mu(xy)\wedge\mu(x\bar{y})$ for all x and y in X.*

Proof: In general $xy\leq x$, $x\bar{y}\leq x$. So $\mu(xy)\geq\mu(x)$ and $\mu(x\bar{y})\geq\mu(x)$. Hence $\mu(xy)\wedge\mu(x\bar{y})\geq\mu(x)$. We claim that $\mu(x)\geq\mu(xy)\wedge\mu(x\bar{y})$. $\mu(x)=\mu(1*\bar{x})=\mu((1*\bar{x})*\bar{x})\geq\mu((1*(1*y))*\bar{x})\wedge\mu((1*y)*\bar{x})=\mu(yx)\wedge\mu(x\bar{y})=\mu(xy)\wedge\mu(x\bar{y})$.

Corollary 2.8. *Let I be a positive implicative ideal of X and let $x, y\in X$. Then $x\in I$ if and only if $xy\in I$ and $x\bar{y}\in I$.*

Example 3. Let $X=\{0, 1, 2, 3\}$ be given and the binary operation $*$ of X is defined by the table. Then $(X, *, 0)$ is a bounded commutative BCK-algebra with unit 3. It is easy to show that a nonempty subset $\{0, 2\}$ of X is a weakly implicative ideal of X.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	2	1	9

Theorem 2.9. *Let I be a weakly implicative ideal of X and $(x*y)*y=x*y$ for all x, y in X. Then $x\in I$ if and only if $xy\in I$ and $x\bar{y}\in I$.*

Proof: Suppose that $x\in X$. Since I is a weakly implicative ideal of X, χ_I is also a fuzzy weakly implicative ideal. By Theorem 2.7, $\chi_I(x)=\chi_I(xy)\wedge\chi_I(x\bar{y})$. $\chi_I(xy)=\chi_I(x\bar{y})=1$. Hence $xy\in I$ and $x\bar{y}\in I$. Conversely, suppose that $xy\in I$ and $x\bar{y}\in I$, then $\chi_I(xy)=\chi_I(x\bar{y})=1$. Since χ_I is a fuzzy weakly implicative ideal, $\chi_I(x)=\chi_I(xy)\wedge\chi_I(x\bar{y})=1$. Therefore $x\in I$.

Corollary 2.10. *Let I be an ideal of X which is both weakly implicative and prime ideal of X, then for each $x\in X$, either $x\in I$ or $\bar{x}\in I$.*

Proof: $(x*y)\wedge(y*x)=0$ imply $(x*\bar{x})=0\in I$. Since I is a prime ideal, either $x*\bar{x}\in I$ or $\bar{x}*x\in I$. So either $x^2\in I$ or $\bar{x}^2\in I$ or $\bar{x}^2\in I$. By Corollary 2.4, $x\in I$ or $\bar{x}\in I$.

Corollary 2.11. *Suppose that μ is both fuzzy weakly implicative and fuzzy prime ideal of X. Then the followings hold for all $x, y\in X$,*

(1) $\mu(x\vee y)=\mu(x)\wedge\mu(y)$,
(2) $\mu(x\wedge y)=\mu(x)\vee\mu(y)$,
(3) $\mu(x)\wedge\mu(\bar{x})=\mu(1)$,
(4) $\mu(x)\vee\mu(\bar{x})=\mu(0)$.

Proof: (1) It is clear by Proposition 2.5.

(2) Since $x\wedge y\leq x, y$, $\mu(x\wedge y)\geq\mu(x)\vee\mu(y)$. We claim that $\mu(x\wedge y)\leq\mu(x)\vee\mu(y)$. Note that $\mu(x(x\wedge y))=\mu(x*y)$ and similarly $\mu(y*(x\wedge y))=\mu(y*x)$. But $(x*y)\wedge(y*x)=0$. Hence $(x*y)\wedge(y*x)\in X_\mu$ which is prime by Proposition 1.4. Thus either $x*y\in$

X_μ or $y * x \in X_\mu$ i.e. $\mu(x * (x \wedge y)) = 0$ or $\mu(y * (x \wedge y)) = 0$. By (3) of Proposition 1.2, $\mu(x) \geq \mu(x \wedge y)$ or $\mu(y) \geq \mu(x \wedge y)$. So $\mu(x) \vee \mu(y) \geq \mu(x \wedge y)$.

(3) It follows by (5) of Proposition 1.2 Since X is bounded, $\mu(x) \wedge \mu(\bar{x}) = \mu(x) \wedge \mu(1 * x) = \mu(1)$.

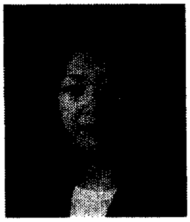
(4) Let $I = X_\mu$. Since μ is a fuzzy weakly implicative and prime ideal of X, by Theorem 2.10, either $x \in I$ or $\bar{x} \in I$ then $\mu(x) = \mu(0)$ or $\mu(\bar{x}) = \mu(0)$. Thus $\mu(x) \vee \mu(\bar{x}) = \mu(0)$.

Corollary 2.12. *If μ is both fuzzy weakly implicative and fuzzy prime ideal of X, then $Im\mu = \{\mu(0), \mu(1)\}$.*

Proof: By above theorem, $\mu(x) \wedge \mu(\bar{x}) = \mu(1)$, $\mu(x) \vee \mu(\bar{x}) = \mu(0)$. Then for all $x \in X$, either $\mu(x) = \mu(0)$ or $\mu(x) = \mu(1)$. So $Im\mu = \{\mu(0), \mu(1)\}$.

REFERENCES

1. Shaban. A. Bhatti, M. Anwar Chaudhry, B. Ahmad, "On classification of BCI-algebras", Math. Japon, Vol. 34, No. 6, pp. 865-876, 1989.
2. C. S. Hoo, "Fuzzy implicative and Boolean ideals of MV-algebras", Fuzzy set and systems, Vol. 66, pp. 315-327, 1994.
3. C. S. Hoo, "Fuzzy ideals of BCI and MV-algebras", Fuzzy sets and systems, Vol. 62, pp. 111-114, 1994.
4. K. Iseki, S. Tanaka, "An introduction to the theory of BCK-algebras", Math. Japon, No 23, pp. 1-26, 1978-79.
5. K. Iseki, S. Tanaka, "Ideal theory of BCK-algebras", Math. Japon, Vol. 21, pp. 351-366, 1976.
6. Y. Imai, K. Iseki, "On axiom of systems of proportional calculi XIV", Proc. Japan Academy, Vol. 42, pp 19-22, 1966.
7. Y. B. Jun, "Characterization of fuzzy ideals by their level ideals in BCK (BCI)-algebras", Math. Japon, Vol. 38, No. 1, pp. 67-71, 1993.
8. Y. B. Jun, E. H. Roh, "Fuzzy commutative ideals of BCK-algebras", Fuzzy sets and systems, Vol. 64, pp. 401-405, 1994.
9. J. Meng, "Commutative ideals in BCK-algebras", Pure Appl. Math.(in China), Vol. 9, pp. 49-53, 1991.
10. Daniel Mundici, "MV-algebras are categorically equivalent to bounded commutative BCK-algebras", Math. Japon, Vol. 31, No 6, pp. 889-942, 1986.
11. Xi Ougen, "Fuzzy BCK-algebras", Math. Japon, Vol. 36, No. 5, pp. 935-942, 1991.
12. A. Rosenfeld, "Fuzzy groups", J. Math. Anal, Appl Vol. 35, pp. 512-517, 1971.
13. Lei Tiande, Xi Changchang, "P-radical in BCI-algebras", Math. Japon, Vol. 30, No. 4, pp. 511-517, 1985.
14. T. Traczyk, "On the variety of bounded commutative BCK-algebras", Math. Japon, Vol. 24, pp. 283-292, 1979.
15. L. A. Zadeh, "Fuzzy sets", Inform. and Control, Vol. 8, pp. 338-353, 1965.



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