Relation between Multidimensional Linear Interpolation and Fuzzy Reasoning

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ABSTRACT

This paper examines the relation between multidimensional linear interpolation (MDI) and fuzzy reasoning, and shows that an MDI is a special form of Tsukamoto's fuzzy reasoning. From this result, we found new possibility of defuzzification strategy.

notation

MDI: Multidimensional Linear Interpolation

STM system: Special Tsukamoto's Membership system

I. Introduction

Interpolation technique is used in the application of signal processing widely [1], and there is also a study of the application to fuzzy learning [2]. Multidimensional linear interpolation (MDI) is a useful method for nonlinear function problem. One of application of this method is the estimation of pump output of artificial heart, and showed good performance [3]. This paper examines the relation between multidimensional linear interpolation and fuzzy reasoning, and shows that an MDI is a special form of Tsukamoto's fuzzy reasoning [4]. From this result, we found a new possibility of defuzzification strategy.

This paper is organized as follows. We state an MDI and Tsukamoto's fuzzy reasoning in section II and III, respectively. In section IV, we derive the MDI from fuzzy reasoning. In section V, we summarize and discuss about our study. Finally, in section VI, conclusions are stated.

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II. Multidimensional Linear Interpolation

Before we proceed, it is necessary to comprehend that what we mean the MDI is the problem of interpolating on a mesh that is Cartesian, i.e., has not tabulated function values at 'random' points in n-dimensional space rather than at the vertices of a rectangular array. For simplicity, we consider only the case of three dimensions, the cases of two and four or more dimensions being analogous in every way. If the input variable arrays are $x_{1a}[\ ]$, $x_{2a}[\ ]$, and $x_{3a}[\ ]$, the output $y\ (x_1, x_2, x_3)$ has following relation [5].

$$y_a[m][n][r] = y(x_{1a}[m], x_{2a}[n], x_{3a}[r]). \qquad (1)$$

The goal is to estimate, by interpolation, the function y at some untabulated point (x_1, x_2, x_3). If x_1, x_2, x_3 satisfy

$$\begin{cases} x_{1a}[m] \leq x_1 \leq x_{1a}[m+1] \\ x_{2a}[n] \leq x_2 \leq x_{2a}[n+1] \\ x_{3a}[r] \leq x_3 \leq x_{3a}[r+1], \end{cases} \tag{2}$$

the grid points are

$$\begin{aligned} y_1 &= y_a[m][n][r] \\ y_2 &= y_a[m][n][r+1] \\ y_3 &= y_a[m][n+1][r] \\ y_4 &= y_a[m][n+1][r+1] \\ y_5 &= y_a[m+1][n][r] \\ y_6 &= y_a[m+1][n][r+1] \\ y_7 &= y_a[m+1][n+1][r] \\ y_8 &= y_a[m+1][n+1][r+1]. \end{aligned} \tag{3}$$

The final 3-dimensional linear interpolation is

$$\begin{aligned} y(x_1, x_2, x_3) = &(1-u)(1-v)(1-w)\,y_1 \\ &+(1-u)(1-v)(w)\,y_2 \\ &+(1-u)(v)(1-w)\,y_3 \\ &+(1-u)(v)(w)\,y_4 \\ &+(u)(1-v)(1-w)\,y_5 \\ &+(u)(1-v)(w)\,y_6 \\ &+(u)(v)(1-w)\,y_7 \\ &+(u)(v)(w)\,y_8, \end{aligned} \tag{4}$$

where

$$\begin{aligned} u &= \frac{x_1 - x_{1a}[m]}{x_{1a}[m+1]-x_{1a}[m]}, \\ v &= \frac{x_2 - x_{2a}[n]}{x_{2a}[n+1]-x_{2a}[n]}, \\ w &= \frac{x_3 - x_{3a}[r]}{x_{3a}[r+1]-x_{3a}[r]}. \end{aligned} \tag{5}$$

(u, v, and w each lie between 0 and 1.)

We can see the estimated y uses 2^n table terms if n -dimensions, and it satisfies 8 terms in case of three dimensions as above.

Ⅲ. Tsukamoto's Fuzzy Reasoning

There are four popular fuzzy reasoning methods. They are Mamdani's minimum operation rule, Larsen's product operation rule, Tsukamoto's method with linguistic terms as monotonic membership functions, and finally Takagi and Sugeno's method [6][7]. Here, what we are interested in is Tsukamoto's method. Tsukamoto used monotonic membership functions for linguistic terms [4]. As an example, consider the case of two input variables and one output variable.

$$\begin{aligned} &R1: \textit{If } x_1 = A_{11} \text{ and } x_2 = A_{21}, \textit{ then } y_1 = B_1, \\ &R2: \textit{If } x_1 = A_{12} \text{ and } x_2 = A_{22}, \textit{ then } y_2 = B_2, \\ &R3: \textit{If } x_1 = A_{13} \text{ and } x_2 = A_{23}, \textit{ then } y_3 = B_3, \\ &R4: \textit{If } x_1 = A_{14} \text{ and } x_2 = A_{24}, \textit{ then } y_4 = B_4. \end{aligned} \tag{6}$$

where y_i: Inferred variable of the consequence.
x_1, x_2: Variables of the premise.
A_{1i}, A_{2i}: Normalized fuzzy sets over the input domain U and V respectively.
B_i: Normalized fuzzy sets over the output domain W.

If we define the fuzzified value A_1' and A_2' for input $x_1 = x_1^0$, $x_2 = x_2^0$, as fuzzy singletons as follows,

$$\begin{aligned} A_1' &= \begin{cases} 1, & \text{if } x_1 = x_1^0, \\ 0, & \text{otherwise,} \end{cases} \\ A_2' &= \begin{cases} 1, & \text{if } x_2 = x_2^0, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{7}$$

the compatibility w_i for ith rule is,

$$w_i = A_{1i}(x_1^0) \wedge A_{2i}(x_2^0), \tag{8}$$

or

$$w_i = A_{1i}(x_1^0)\; A_{2i}(x_2^0). \tag{9}$$

Here, Eq. (8) means logical product ($\wedge$), and Eq. (9) algebraic product ($\bullet$). The result $y_i^{\bullet}$ inferred from R_i is defined as follow:

$$w_i = B(y_i^*) \rightarrow y_i^* = B^{-1}(w_i). \tag{10}$$

The final inferred value y^* from all rules usually calculated by weighted combination method as follows:

$$y^* = \frac{w_1 B_1^{-1}(w_1) + w_2 B_2^{-1}(w_2) + w_3 B_3^{-1}(w_3) + w_4 B_4^{-1}(w_4)}{w_1 + w_2 + w_3 + w_4} = \frac{w_1 y_1^* + w_2 y_2^* + w_3 y_3^* + w_4 y_4^*}{w_1 + w_2 + w_3 + w_4}. \tag{11}$$

B_i must be monotonic, whereas A_{1i}, and A_{2i} have no restriction of shape.

Ⅳ. Expression of Multidimensional Linear Interpolation from Fuzzy Reasoning

Triangular membership functions are widely used in the application of fuzzy theory for their simple form and easiness to handle [8]. From now, we derive the MDI from fuzzy reasoning. Triangular membership functions are used to subdivide the input universe. A fuzzy set A_i defined by triangular membership functions has the form

$$\mu(x) = \begin{cases} \dfrac{x - a_{i-1}}{a_i - a_{i-1}}, & \text{if } a_{i-1} \le x \le a_i, \\ \dfrac{-x + a_{i+1}}{a_{i+1} - a_i}, & \text{if } a_i \le x \le a_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

The point a_i will be referred to as the midpoint of A_i. The point a_{i-1} and a_{i+1} are midpoints of nearby A_i. Thus the fuzzy sets satisfy

$$\sum_{i=1}^{n} \mu_{A_i}(u) = 1, \tag{13}$$

for every $u \in U$. The leftmost and rightmost fuzzy regions are truncated with the midpoint as the leftmost and rightmost position, respectively. And the straight line membership functions are used for the output universe W. Fig. 1 and Fig. 2 help to understand the above relations.

From now, previous membership system will be called a STM (Special Tsukamoto's Membership) system. As an example, we consider three input variables. If we use fuzzy singletons as inputs x_1^0, x_2^0 and x_3^0,

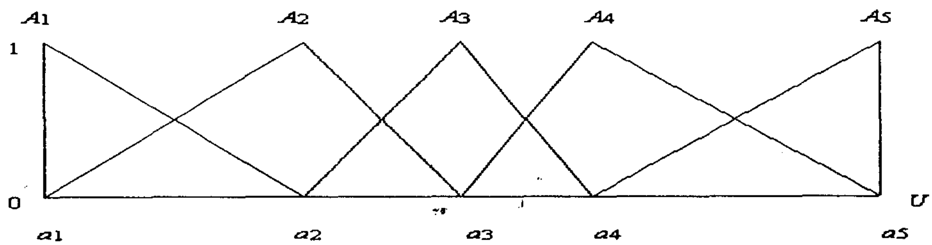


Fig. 1 Triangular decomposition of input domain.

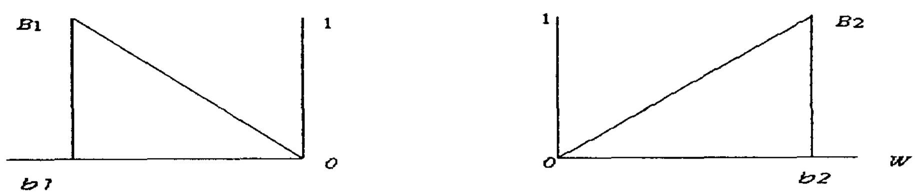


Fig. 2. Representation of output domain using straight line monotonic membership function.

the compatibility w_i is $A_{1i}(x_1^0) \nabla A_{2i}(x_2^0) \nabla A_{3i}(x_3^0)$ where ∇ denotes triangular norms (T-norm). And $w_i = 0$ if $A_{1i}(x_1^0) = 0$ or $A_{2i}(x_2^0) = 0$ or $A_{3i}(x_3^0) = 0$. If the number of input variables are n, there are at most 2^n cases of possible maximum rules that have non-zero w_i, and the possible number of non-zero rules is eight for the case of $n = 3$. Possible rules are smaller than 2^n when there are variables of premise which lies on midpoints of A_i. This fact coincides with the number of table terms which are used in MDI. If we use algebraic product for w_i,

$$w_i = A_{1i}(x_1^0) A_{2i}(x_2^0) A_{3i}(x_3^0). \tag{14}$$

The defuzzified ith value is

$$\begin{aligned} y_i^* &= \mu_{B_i}^{-1}(w_i) \\ &= \mu_{B_i}^{-1}(A_{1i}(x_1^0) A_{2i}(x_2^0) A_{3i}(x_3^0)). \end{aligned} \tag{15}$$

We define the overall defuzzified value y^* as Eq. (16) (Note that weighted combination method is usually used in Tsukamoto's defuzzification.),

$$y^* = \sum_{i=1}^{N} y_i^*. \tag{16}$$

Here, N is the all rules and there are at most eight rules that have non-zero value. This results is equal to that of Eq. (4). We can easily verify that the cases of n-dimensions (one, two, four or more) in an MDI produce the same results of previous special Tsukamoto's fuzzy reasoning in which the number of input variable is n.

V. Discussion

We showed two interesting results in this paper. First, multidimensional linear interpolation (MDI) is a special form of Tsukamoto's fuzzy reasoning. Second, if compatibility w_i is well defined, defuzzification strategy can be achieved simply by summing of each defuzzified ith value y_i^*. If we use the followings in Tsukamoto's method, the result is equal to that of an MDI.

① input variable: fuzzy singleton.

② membership system: STM system as described in section IV.

③ algebraic product for compatibility w_i.

④ final overall defuzzified value $y^* = \sum_{i=1}^{N} y_i^*$.

At this point, we need to compare both methods. The MDI uses valid data whereas the fuzzy reasoning calculates all possible rules even if they produce zero value. So even if we can get the same output, the MDI is efficient than fuzzy reasoning in the perspective of operation cost. If we think input data are contaminated by noise, we can regard input value as fuzzy number when we use fuzzy reasoning, but an MDI has no flexibility.

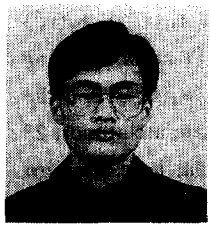
VI. Conclusion

We found a multidimensional linear interpolation (MDI) is a special form of Tsukamoto's fuzzy reasoning. From this result, we found the overall defuzzification strategy can be accomplished by adding only each rule's defuzzified value if compatibility w_i is well defined. We compared both MDI and fuzzy reasoning in section V. Further researches are necessary to find the strategies for compatibility w_i which can be used in simple defuzzification strategies that the overall defuzzification can be accomplished by adding each defuzzified values as stated before. And there remains the problem of finding relation between MDI of tabulated function values at 'random' points in n-dimensional space and fuzzy reasoning.

References

1. A. K. Jain, *Fundamentals of Digital Image Processing*, Prentice-Hall, 1989.
2. T. Sudkamp and R. J. Hammell II, "Interpolation,

Completion, and Learning Fuzzy Rules," *IEEE Trans. on Syst. Man, and Cybern.*, vol. SMC-24, no. 2, pp. 332-342, Feb. 1994.
3. K. S. Om, W. W. Choi, J. M. Ahn, S. K. Park, Y. H. Jo, J. S. Choi, J. J. Lee, H. C. Kim, and B. G. Min, "Experimental Approach for the Estimation of Cardiac Output of Left Ventricular Assist Device Using Multi-dimensional Interpolation Technique," *KOSOMBE Spring Conference*, pp. 232-234, 1996.
4. Y. Tsukamoto, "An Approach to Fuzzy Reasoning Method," in *Advances in Fuzzy Set Theory and Applications*, M. M. Gupta, R. K. Ragada, and R. R. Yager, Eds, Amsterdam: North-Holland, 1979.
5. W. H. Press, *Numerical Recipes in C*, Cambridge University Press, 1992.
6. C. C. Lee, "Fuzzy Logic in Control Systems: Fuzzy Logic Controller-Part I," *IEEE Trans. on Syst. Man, and Cybern.*, vol. SMC-20, no. 2, pp. 404-418, Mar./Apr. 1990.
7. C. C. Lee, "Fuzzy Logic in Control Systems: Fuzzy Logic Controller-Part II," *IEEE Trans. on Syst. Man, and Cybern.*, vol. SMC-20, no. 2, pp. 419-435, Mar./Apr. 1990.
8. G. G. Lee and J. G. Lee, "Properties of Triangle-Shaped Fuzzy Membership Function," *J. of Fuzzy Logic and Intelligent Systems*, vol. 5, no. 1, pp. 15-20, 1995.



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