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퍼지 분리 공리에 관하여

조 진 선*

ON FUZZY SEPARATION AXIOMS

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요 약

퍼지위상의 개념이 도입된 이후로 퍼지분리공간에 관한 몇 가지 정의가 소개되었다. 본 연구에서는 Ganguly와 Saha의 정의와 Hutton과 Reilly의 정의를 비교하였다.

ABSTRACT

Several fuzzy separation axioms have been defined and investigated by many authors. The purpose of this note is to compare fuzzy T_i-axioms due to Ganguly and Saha with ones due to Hutton and Reilly.

Ⅰ. Introduction

In this section, we shall recall some definitions.

Let X be a set. A fuzzy set A in X is a function from X into $[0,1]$. The fuzzy set which always takes value 1 on X is denoted by 1 and the fuzzy set which always takes value 0 on X is denoted by ϕ. We denote by $A(x)$ the membership function of a fuzzy set A in X.

[Definition 1.1] For fuzzy sets A, B and $A_\alpha (\alpha \in I)$ we define

(1) $A = B$ whenever $A(x) = B(x)$ for all $x \in X$.

(2) $A \leq B$ whenever $A(x) \leq B(x)$ for all $x \in X$.

(3) $(\bigcup_{\alpha \in I} A_\alpha)(x) = lub\{A_\alpha(x) \mid \alpha \in I\}$ for all $x \in X$.

(4) $(\bigcap_{\alpha \in I} A_\alpha)(x) = glb\{A_\alpha(x) \mid \alpha \in I\}$ for all $x \in X$.

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[Definition 1.2] A fuzzy point x_α in X is a fuzzy set in X defined by

$$x_\alpha = \begin{cases} \alpha\ (\alpha \in [0,1]) & \text{for } y = x, \\ 0 & \text{for } y \neq x\ (y \in X). \end{cases}$$

Let A be a fuzzy set and let x_α be a fuzzy point in X. If $x_\alpha \leq A$ then we shall write $x_\alpha \in A$.

[Definition 1.3] A collection τ of fuzzy sets in X is called a fuzzy topology on X if

(1) $1, \phi \in \tau$

(2) $\bigcap_{i=1}^{n} A_i \in \tau$ for any finite subcollection $\{A_i \mid i = 1, \cdots, n\}$ of τ.

(3) $\bigcup_{\alpha \in I} A_\alpha \in \tau$ for any subcollection $\{A_\alpha \mid \alpha \in I\}$ of τ.

Every member of a fuzzy topology τ on X is called a fuzzy open set in X. A fuzzy set C is said to be fuzzy closed in X if $1 - C$ is fuzzy open in X. A set X equipped with a fuzzy toplogy on X is called a fuzzy topological space. Throughout this paper, we write a fuzzy topological space in short as an fts.

[Definition 1.4] A fuzzy set A in X is said to be q-coincident with a fuzzy set B in X, denoted by $A_q B$, if there exists $x \in X$ such that $A(x) + B(x) > 1$. When two fuzzy sets A and B in X are not q-coincident, we shall write $A_q B$.

[Definition 1.5] Let A be a fuzzy set in X and let x_α be a fuzzy point in X.

(1) If A is fuzzy open (resp. closed) and $x_\alpha \in A$, then A is called a fuzzy open (resp. closed) neighborhood. of x_α.

(2) If there exists a fuzzy open set U in X such that $x_{\alpha q} U \leq A$, then A is called a q-neighborhood of x_α.

[Definition 1.6] Let A be a fuzzy set in an fts X.

(1) The set $\{x \in X \mid A(x) > 0\}$, denoted by A_0, is called the support of A.

(2) The intersection of all fuzzy closed sets containing A is called the fuzzy closure of A and denoted by $Cl(A)$.

For definitions and notations which are not explained in this paper, we refer to [3] and [4].

Ⅱ. Fuzzy T_0-spaces

[Definition 2.1]([3]) An fts is said to be fuzzy T_0 in the sense of Ganguly and Saha if for every pair of distinct fuzzy points x_α and y_β, the following conditions are satisfied:

(1) If $x \neq y$, then either x_α has a fuzzy open neighborhood which is not q-coincident with y_β or y_β has a fuzzy open neighborhood which is not q-coincident with x_α.

(2) If $x = y$ and $\alpha < \beta$ (say), then y_β has a fuzzy q-neighborhood which is not q-coincident with x_α.

[Definition 2.2]([5]) An fts X is said to be fuzzy T_0 in the sense of Hutton and Reilly if for every fuzzy set A in X, there exists a collection $\{U_{ij} \mid i \in I, j \in J_i\}$ of fuzzy open or fuzzy closed sets in X such that $A = \bigcup_{i \in I} (\bigcap_{i \in J_i} U_{ij})$.

[Theorem 2.3]([3]) An fts X is fuzzy T_0 in the sense of Ganguly and Saha if and only if for every pair of distinct fuzzy points x_α and y_β, either $x_\alpha \notin Cl(y_\beta)$ or $y_\beta \notin Cl(x_\alpha)$.

[Theorem 2.4] An fts X is fuzzy T_0 in the sense of Hutton and Reilly if for every fuzzy point x_α, $x_\alpha = \bigcap_{U \in \mathcal{U}} U$, where $\mathcal{U}$ is the collection of fuzzy open or fuzzy closed neighborhoods of x_α.

Proof. Let A be a fuzzy set in X and let x be a point in A_0. By hypothesis, $x_{A(x)} = \bigcap_{U \in \mathcal{U}_x} U$, where $\mathcal{U}_x$ is the collection of fuzzy open or fuzzy closed sets in X which contains $x_{A(x)}$. It is easy to show that $A = \bigcup_{x \in A_0} (\bigcap_{U \in \mathcal{U}_x} U)$. This completes the proof.

[Theorem 2.5] If an fts X is fuzzy T_0 in the sense of Ganguly and Saha, then it is fuzzy T_0 in the sense of Hutton and Reilly.

Proof. Let x_α be a fuzzy point in X and let $\mathcal{U}$ be the collection of fuzzy open or fuzzy closed sets in X which contain x_α. Clearly, $\alpha \leq (\bigcap_{U \in \mathcal{U}} U)(x) = \beta$. Assume $\beta > \alpha$. By hypothesis, $x_\beta \notin Cl(x_\alpha)$. That is, $\beta > Cl(x_\alpha)(x)$. But, since $Cl(x_\alpha) \in \mathcal{U}$, we have an obvious contradiction that $\beta = (\bigcap_{U \in \mathcal{U}} U)(x) \leq (Cl(x_\alpha))(x) < \beta$. Thus $\alpha = (\bigcap_{U \in \mathcal{U}} U)(x)$.

Now, assume that there exists $y \in X - \{x\}$ such that $(\bigcap_{U \in \mathcal{U}} U)(y) = \gamma > 0$. Then for all $U \in \mathcal{U}$, $U(y) \geq \gamma$. Being $Cl(x_\alpha) \in \mathcal{U}$, $y_\gamma \in Cl(x_\alpha)$. Thus, by hypothesis, $x_\alpha \notin Cl(y_\gamma)$. This leads to a contradiction that $y_\gamma \in Cl(x_\alpha) \nleq Cl(y_\gamma)$, and hence $(\bigcap_{U \in \mathcal{U}} U)(y) = 0$ for all $y \in X - \{x\}$.

[Remark 2.6] The converse of Theorem 2.5 is not necessarily true. (Example 6.1)

Ⅲ. Fuzzy T_1-spaces

[Definition 3.1]([3]) An fts is said to be fuzzy T_1 in the sense of Ganguly and Saha if for every pair of distinct fuzzy points x_α and y_β, the following conditions are satisfied:

(1) If $x \neq y$, then x_α has a fuzzy open neighborhood which is not q-coincident with y_β and y_β has a fuzzy open neighborhood which is not q-coincident with x_α

(2) If $x = y$ and $\alpha < \beta$ (say), then y_β has a fuzzy q-neighborhood V such that $x_\alpha q V$.

[Definition 3.2]([5]) An fts X is said to be fuzzy T_1 in the sense of Hutton and Reilly if for every fuzzy set A in X, there exists a collection $\{C_i \mid i \in I\}$ of fuzzy closed sets in X such that $A = \bigcup_{i \in I} C_i$.

[Theorem 3.3]([3]) An fts X is fuzzy T_1 in the sense of Ganguly and Saha if and only if every fuzzy point of X is fuzzy closed in X.

[Corollary 3.4] If an fts X is fuzzy T_1 in the sense of Ganguly and Saha, then it is fuzzy T_1 in the sense of Hutton and Reilly.

Proof. Let an fts X be T_1 in the sense of Ganguly and Saha and let A be a fuzzy set in X. Since $A = \bigcup_{x \in A_0} x_{A(x)}$ and, by Theorem 3.3, every fuzzy point in X is fuzzy closed, X is fuzzy T_1 in the sense of Hutton and Reilly.

[Remark 3.5] The converse of Corollary 3.4 is not necessarily true. (Example 6.1)

Ⅳ. Fuzzy T_2-spaces

[Definition 4.1]([3]) An fts is said to be fuzzy T_2 in the sense of Ganguly and Saha if for every pair of distinct fuzzy points x_α and y_β, the following conditions are satisfied:

(1) If $x \neq y$, then x_α and y_β have fuzzy open neighborhoods which are not q-coincident.

(2) If $x = y$ and $\alpha < \beta$ (say), then y_β has a fuzzy q-neighborhood V and x_α has a fuzzy open neighborhood U such that $V_q U$.

[Definition 4.2]([5]) An fts X is said to be fuzzy T_2 in the sense of Hutton and Reilly if for every fuzzy set A in X, there exists a collection $\{U_{ij} \mid i \in I,\ j \in J_i\}$ of fuzzy open sets in X such that $A = \bigcup_{i \in I} (\bigcap_{j \in J_i} U_{ij}) = \bigcup_{i \in I} (\bigcap_{j \in J_i} Cl(U_{ij}))$ or, equivalently, $A = \bigcup_{i \in I} (\bigcap_{j \in J_i} U_{ij}) = \bigcup_{i \in I} (\bigcap_{j \in J_i} Cl(U_{ij}))$.

[Theorem 4.3]([2]) For an fts X, the following are equivalent:

1. X is a fuzzy T_2-space in the sense of Ganguly and Saha.
2. For any fuzzy point x_α, $x_\alpha = \bigcap_{U \in \mathcal{U}} Cl(U)$, where $\mathcal{U}$ is the collection of fuzzy open neighborhoods of x_α.
3. For any two distinct fuzzy points x_α and y_β:

   (1) if $x \neq y$, then there exist fuzzy open sets U and V in X such that $x_\alpha \in U$, $y_\beta \in V$, $x_\alpha \notin Cl(V)$ and $y_\beta \notin Cl(U)$.

   (2) if $x = y$ and $\alpha < \beta$ (say), then there exists a fuzzy open set U in X such that $x_\alpha \in U$ and $y_\beta \notin Cl(U)$.

[Theorem 4.4]([2]) If X is fuzzy T_2-space in the sense of Ganguly and Saha, then it is fuzzy T_2 in the sense of Hutton and Reilly.

[Remark 4.5] The converse of Theorem 4.4 is not necessarily true. (Example 6.1)

[Theorem 4.6]([2]) Let x_α be a fuzzy point and let $\mathcal{U}$ be the collection of fuzzy open neighborhoods of x_α. If X is fuzzy T_2 in the sense of Hutton and Reilly, then

(1) $(\bigcap_{U \in \mathcal{U}} U)(x) = (\bigcap_{U \in \mathcal{U}} Cl(U))(x) = \alpha$ and

(2) $x_\alpha = \bigcap_{U \in \mathcal{U}} U = \bigcap_{U \in \mathcal{U}} Cl(U)$ whenever $\alpha < 1$.

Ⅴ. Fuzzy T_3-spaces

[Definition 5.1]([3]) An fts X is fuzzy regular in the sense of Ganguly and Saha if for any fuzzy point x_α and any fuzzy open set G in X with $x_{\alpha q} G$, there exists a fuzzy open set U in X such that $x_{\alpha q} U \leq$

$Cl(U) \leq G$. An fts X is fuzzy T_3 in the sense of Ganguly and Saha if it is both fuzzy T_1 and fuzzy regular in the sense of Ganguly and Saha.

[Definition 5.2]([5]) An fts X is fuzzy regular in the sense of Hutton and Reilly if for each fuzzy open set G, there exists a collection $\{U_i \mid i \in I\}$ of fuzzy open sets such that $G = \cup U_i$ and $Cl(U_i) \leq G$ for every $i \in I$. An fts X is fuzzy T_3 in the sense of Hutton and Reilly if it is both fuzzy T_0 and fuzzy regular in the sense of Hutton and Reilly.

[Theorem 5.3] For an fts X, the folowing statements are equivalent:

(1) X is fuzzy regular in the sense of Gangular and Saha.

(2) X is fuzzy regular in the sense of Hutton and Reilly.

(3) for each $x \in X$, each $\alpha \in (0,1)$ and each fuzzy open set G in X with $\alpha < G(x)$, there exists a fuzzy open set U in X such that $\alpha < U(x)$ and $Cl(U) \leq G$.

(4) for each $x \in X$, each $\alpha \in (0,1)$ and each fuzzy closed set C in X with $\alpha < 1 - C(x)$ there exist fuzzy open sets U and V in X such that $\alpha < U(x)$ $C \leq V$ and $U \leq 1 - V$.

Proof. The equivalence between (1), (2) and (4) have been shown by Ali in [1]. Thus, the proof is completed by showing the equivalence between (1) and (3).

(1) $\Rightarrow$ (3). Since $(1-\alpha) + G(x) > (1-\alpha) + \alpha = 1$, we have $x_{(1-\alpha)q}G$. By (1), there exists a fuzzy open set U in X such that $x_{(1-\alpha)q}U \leq Cl(U) \leq G$. Being $(1-\alpha) + U(x) > 1$, we obtain $U(x) > \alpha$.

(3) $\Rightarrow$ (1). Assume $x_{\alpha q}G$. Then $\alpha + G(x) > 1$. Let $\varepsilon = (\alpha + G(x) - 1)/2$. Then $1/2 < G(x) - \varepsilon < 1$. By (3), there exists a fuzzy open set U in X such that $G(x) - \varepsilon < U(x)$ and $Cl(U) \leq G$. Since $\alpha + U(x) > \alpha + G(x) - \varepsilon > 1$, we have $x_{\alpha q}U$.

[Corollary 5.4] If an fts X is fuzzy T_3 in the sense of Ganguly and Saha, then it is fuzzy T_3 in the sense of Hutton and Reilly.

[Remark 5.5] The converse of Corollary 5.4 is not necessarily true. (Example 6.1)

Ⅵ. Example

[Example 6.1] Let $X = \{x,y\}$ and let $A_{\lambda\mu}$ be the fuzzy set in X defined by

$$A_{\lambda\mu}(z) = \begin{cases} \lambda & \text{for } z = x \\ \mu & \text{for } z = y \end{cases}$$

Clearly, the collection $\tau = \{A_{\lambda\mu} \mid 0 < \lambda \leq 1, 0 < \mu \leq 1\} \cup \{A_{00}\}$ is a fuzzy topology on X.

(Claim 1) X is fuzzy regular in the sense of Hutton and Reilly: let z_α be a fuzzy point in X and let G be a fuzzy open set in X with $x_{\alpha q}G$. Then $0 < G(x) \leq 1$, $0 < G(y) \leq 1$ and $1 - \alpha < G(x)$. Choose $\beta \in (1-\alpha, G(x))$ and $\gamma \in (0, G(y))$. Clearly, $x_{\alpha q}A_{\beta\gamma}$. Since $\beta, \gamma \in (0,1)$, the fuzzy set $A_{\beta\gamma}$ is both fuzzy open and fuzzy closed, and hence $Cl(A_{\beta\gamma}) = A_{\beta\gamma} \leq G$. Consequently, X is fuzzy regular in the sense of Ganguly and Saha. By Theorem 5.3, X is fuzzy regular in the sense of Hutton and Reilly.

(Claim 2) X is fuzzy T_0 in the sense of Hutton and Reilly: note that for any $\alpha, \beta \in (0,1]$, $x_\alpha = \bigcap_{0<\mu<1} A_{\alpha\mu}$ and $y_\beta = \bigcap_{0<\lambda<1} A_{\lambda\beta}$. Since the collection $\{A_{\alpha\mu} \mid 0 < \alpha < 1\}$ and $\{A_{\lambda\beta} \mid 0 < \lambda < 1\}$ consist of fuzzy open sets in X we have, by Theorem 2.4, X is fuzzy T_0 in the sense of Hutton and Reilly

(Claim 3) X is not fuzzy T_0 in the sense of Ganguly and Saha.: consider the fuzzy points x_1 and y_1. Since for all $\lambda, \mu \in [0,1)$, $A_{1\mu}$ and $A_{\lambda 1}$ are not fuzzy closed in X, we have $Cl(x_1) = Cl(y_1) = A_{11}$. Thus $x_1 \in Cl(y_1)$ and $y_1 \in Cl(x_1)$. By Theorem 2.3, X is not fuzzy T_0 in the sense of Ganguly and Saha.

From the definitions, we obtain, in both cases, the following implications:

(*) fuzzy $T_3 \Rightarrow$ fuzzy $T_2 \Rightarrow$ fuzzy $T_1 \Rightarrow$ fuzzy T_0.

Combining (*) with (Claim 1) and (Claim 2), we reach the conclusion that X satisfies all fuzzy T_i axioms in the sense of Hutton and Reilly. On the other hand, (*) and (Claim 3) implies that X does not satisfies all fuzzy T_i axioms in the sense of Ganguly and Saha.

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