ON THE SOLUTIONS OF THREE ORDER DIFFERENTIAL EQUATION WITH NON-NEGATIVE COEFFICIENTS

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1. Introduction

We consider the third order linear homogeneous differential equation

\[ L_3(y) = y''' + P(x)y' + Q(x)y = 0 \]  \hspace{1cm} (E)

\[ P(x) \geq 0, \; Q(x) > 0 \text{ and } P(x)/Q(x) \text{ is nondecreasing on } [a, \infty) \text{ for some real number } a. \]  \hspace{1cm} (1)

In this paper we discuss the distribution of zeros of solutions and a condition of oscillatory for equation (E).

(E) is said to be disconjugate on \([a, \infty)\) if no nontrivial solution of (E) has more than two zeros on \([a, \infty)\).

A nontrivial solution of (E) is said to be oscillatory on \([a, \infty)\), if it has an infinite number of zeros on \([a, \infty)\). The nontrivial solution of (E) is non-oscillatory if it is not oscillatory.

If (E) has an oscillatory solution, it is said to be oscillatory. And if all solutions of (E) are non-oscillatory then (E) is said to be non-oscillatory.

We give some basic definitions.
2. Preliminaries

**Definition 2.1.** $L^*_3(z) = (z'' + P(x)z)' - Q(x)z = 0$ is adjoint of \((E)\). \((E^*)\)

**Definition 2.2.** $c \in [a, \infty)$ and $U_i(x, c), i = 1, 2$ be pair of solutions determined by the initial conditions at $x = c$.

(a) $U_1(x, c); \ y(c) = 0, \ y'(c) = 1, \ y''(c) = 0$ ; first principal solution.

(b) $U_2(x, c); \ y(c) = 0, \ y'(c) = 0, \ y''(c) = 1$ ; second principal solution.

**Definition 2.3.** Let $D_2(y) = y'' + P(x)y$, be second order differential operator and $c \in [a, \infty)$.

(a) $U_1^*(x, c); \ Z(c) = 0, \ Z'(c) = 1, \ D_2Z(c) = 0$ ; first principal solution of \((E^*)\) at $x = c$.

(b) $U_2^*(x, c); \ Z(c) = 0, \ Z'(c) = 0, \ D_2Z(c) = 1$ ; second principal solution of \((E^*)\) at $x = c$.

The wronskian of any two solutions of \((E)\) is a solution of \((E^*)\) and converse holds. Thus,

$$U_2^*(x, c) = W(U_1, U_2) = U_1U_2' - U_2U_1'$$

$$U_2(x, c) = W(U_1^*, U_2^*) = U_1^*U_2^* - U_2^*U_1^*$$

Differentiating these identities yields followings.

$$U_2^{*'}(x, c) = U_1U_2'' - U_2U_1''$$

$$D_2U_2^*(x, c) = U_1^*U_2''' - U_2^*U_1'''$$

$$U_2'(x, c) = U_1^*D_2U_2^* - U_2^*D_2U_1^*$$

$$U_2''(x, c) = U_1^{*'}D_2U_2^* - U_2^{*'}D_2U_1^*$$

$$D_2U_2(x, c) = U_1^{*'}U_2''' - U_2^{*'}U_1'''$$
3. Main Theorem

Lemma 3.1. Let \( (E) \) be disconjugate on \([a, \infty)\) and let its coefficients satisfy (1). If \( U_2''(x, a) \) has a zero on \((a, \infty)\) with \( x = t \), being the first zero of \( U_2''(x, a) \) then

(a) \( U_2''(x, a) \) has a second zero \( t_2 \in (t_1, \infty) \).

(b) \( U_2'(x, a) \) has exactly one zero \( s_1 \in (t_1, t_2) \) and \( U_2'(x, a) < 0 \) on \((s_1, \infty)\).

Proof. Assume \( U_2''(x, a) \) has a zero at \( x = t_1 \).

Suppose \( U_2'(x, a) > 0 \) on \((a, \infty)\). Then \( U_2'''(x, a) < 0 \) which implies \( U_2''(x, a) \) is decreasing on \((a, \infty)\). Therefore, \( U_2''(x, a) < 0 \) on \((t_1, \infty)\). Let the first such zero of \( U_2'(x, a) \) be \( s_1 \) and assume \( U_2'(x, a) \) has a second zero \( s_2 \). Then \( U_2'(x, a) < 0 \) on \((s_1, s_2)\).

From the identity \( U_2^*(x, a) = W(U_1, U_2) \), we find \( U'(s_2, a) < 0 \). Let

\[
\lambda_1(x) = \frac{U_1'(x, a)}{U_2'(x, a)}.
\]

We find \( \lambda_1(x) \to \infty \) as \( x \to s_2 \) on \((s_1, s_2)\).

And

\[
\lambda_1'(x) = \frac{U_2'(x, a)U_1''(x, a) - U_2''(x, a)U_1'(x, a)}{(U_2'(x, a))^2}
= -\frac{D_2U_2^*(x, a)}{(U_2'(x, a))^2}, \text{ on } (s_1, s_2).
\]

since \( D_2U_2^*(x, a) = 1 + \int_a^x Q(t)U_2^*(t, a)dt > 0 \), \( \lambda_1'(x) < 0 \) on \((s_1, s_2)\) and this contradicts \( \lambda_1(x) \to \infty \). Therefore, \( U_2'(x, a) \) has exactly one zero \( s_1 \in (t_1, \infty) \). If \( U_2''(x, a) \) does not have a zero on \((t_1, \infty)\), then \( U_2''(x, a) < 0 \) and \( U_2'(x, a) < 0 \) on some interval and we conclude that \( U_2(x, a) \) has a zero, contradicting the fact \( U_2(x, a) > 0 \) on \((a, \infty)\). Thus \( U_2'(x, a) \) has a second zero \( t_2 \in (t_1, \infty) \) and the Lemma follows.
Lemma 3.2. Let \((E)\) be disconjugate on \([a, \infty)\) and let its coefficients satisfy (1). Then \(P(x)D_2U_2^*(x, a) + Q(X)U_2^*(x, a) > 0\) on \((a, \infty)\).

Proof. Since \(U_2^*(x, a)\) is a solution of \((E^*)\), we have \([U_2^{**}(x, a) + P(x)U_2^*(x, a)]' = Q(x)U_2^*(x, a)\).

Integrating from \(a\) to \(x\),

\[U_2^{**}(x, a) + P(x)U_2^*(x, a) = 1 + \int_a^x Q(t)U_2^*(t, a)dt.\]

Therefore

\[U_2^{**}(x, a) = (x - a) + \int_a^x \int_a^t Q(s)U_2^*(s, a)dsdt - \int_a^x P(t)U_2^*(t, a)dt\]

\[= (x - a) + \int_a^x (x - t)Q(t)U_2^*(t, a)dt - \int_a^x P(t)U_2^*(t, a)dt.\]

\[P(x)D_2U_2^*(x, a) + Q(x)U_2^{**}(x, a) = P(x) + P(x)\int_a^x Q(t)U_2^*(t, a)dt\]

\[+ Q(x)(x - a) + Q(x)\int_a^x (x - t)Q(t)U_2^*(t, a)dt - Q(x)\int_a^x P(t)U_2^*(t, a)dt\]

\[= P(x) + Q(x)(x - a) + Q(x)\int_a^x (x - t)Q(t)U_2^*(t, a)dt\]

\[+ \int_a^x [P(x)Q(t) - Q(x)P(t)]U_2^*(t, a)dt.\]

Since \(P(x)/Q(x)\) is nondecreasing and \(U_2^*(x, a) > 0\), it follows that \(P(x)D_2U_2^*(x, a) + Q(x)U_2^{**}(x, a) > 0\) on \((a, \infty)\).

Theorem 3.1. Let \((E)\) be disconjugate on \([a, \infty)\) and let its coefficients satisfy (1). Assume \(U_2''(x, a)\) has a zero at \(t_1\). Then \(U_2''(x, a)\) has a second zero at \(t_2\), and \(U_2''(x, a) > 0\) on \((t_2, \infty)\), \(a < t_1 < t_2\).

Proof. Suppose \(U_2''(x, a)\) has a zero on \((t_2, \infty)\). Let \(t_3\) be the first zero of \(U_2''(x, a)\) on this interval. Then the identity \(D_2U_2^*(x, a)\) implies that \(U_3''(t_3, a) > 0\).
Let $\lambda_2(x) = \frac{U_1''(x, a)}{U_2''(x, a)}$. Then $\lambda_2(x) \to \infty$ as $t \to t_3$ on $(t_1, t_3)$.

$$
\lambda_2'(x) = \frac{U_1'''(x, a)U_2''(x, a) - U_2'''(x, a)U_1''(x, a)}{(U_2''(x, a))^2}.
$$

Since $U_1(x, a), U_2(x, a)$ are solution of (E) and from the identity of (2), we have

$$
\lambda_2'(x) = -\frac{P(x)D_2U_2^*(x, a) + Q(x)U_2^{*'}(x, a)}{(U_2''(x, a))^2}.
$$

By Lemma 3.2, the numerator is positive. Thus $\lambda_2'(x) < 0$ on $(t_2, t_3)$. This is a contradiction and Lemma 3.3 follows.

In next, we give a criterion for the oscillation of (E)

**Lemma 3.3 [4].** If $2Q(x) - P'(x) \leq 0$ and not identically zero in any interval then (E) has a solution $U(x)$ for which

$$
F[U(x)] = U'(x)^2 - 2U(x)U''(x) - P(x)U^2(x)
$$

$$
= F[U(a)] + \int_a^x (2Q(t) - P'(t))U^2(t)dt
$$

is always negative. Consequently $U(x)$ is nonoscillatory.

**Definition 3.1.** If (E) has a non-trivial solution with three zeros on $[t, \infty), t \in [a, \infty)$, then the first conjugate point $\eta_1(t)$ of $x = t$ is defined by $\eta_1(t) = \inf\{x_3; t \leq x_1 \leq x_2 \leq x_3, y(x_i) = 0, i = 1, 2, 3, y \neq 0, L_3(y) = 0\}$

**Lemma 3.4 [2].** If (E) is non-oscillatory then either

(i) for each $t \in [a, \infty), (E)$ has $\eta_1(t) < \infty$ or

(ii) $(E^*)$ is oscillatory.
**Theorem 3.2.** Let the coefficients of (E) satisfy \( P(x) \geq 0, \ Q(x) \geq 0 \) and \( P(x) + Q(x) \neq 0 \) on \([a, \infty)\). If \( \eta_1(t) < \infty \) for each \( t \in [a, \infty) \) and \( 2Q(x) - P'(x) \leq 0 \), then (E) is oscillatory.

**Proof.** Assume (E) is non-oscillatory. By Lemma 3.4, \((E^*)\) is oscillatory. Since \( P'(x) - Q(x) \geq P'(x) - 2Q(x) \geq 0 \) and \( 2(P'(x) - Q(x)) - P'(x) = P'(x) - 2Q(x) \geq 0 \), a result of Lemma 3.3 implies \((E^*)\) has a non-oscillatory solution. This is a contradiction and (E) is oscillatory.

**References**


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